

ON THE DOMAIN OF TRACES AND SEQUENTIAL COMPOSITION

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Abstract

An enhancement of Mazurkiewicz's trace theory with infinite traces is presented. Infinite traces have been obtained by introducing the trace preorder relation on possibly infinite strings. It is shown that the extension gives rise to the domain of traces in the sense of Scott and a complete metric space. Sequential composition (concatenation) of possibly infinite traces is also considered. The difficulty of finding an appropriate concatenation of infinite traces is a consequence of the concatenation of finite traces being non-uniformly continuous wrt the metric for traces. A natural extension of the concatenation operation for finite traces is proposed; the extended operation is total, yields a generalization of Levi's lemma for infinite traces, but is non-associative.

1. INTRODUCTION

Trace theory, originating from Mazurkiewicz [Maz77], has recently gained popularity as a semantic model of non-interleaving concurrency. Although the properties of finite traces are relatively well-known, see e.g. [Maz84, Aa88], it seems that infinite traces have not yet been fully investigated. This is undesirable as infinite behaviours are required in order to establish certain properties of reactive systems, for example *liveness* [Pnu86], and also has impact on *fairness* issues [Kwi89t, Kwi90t]. The corresponding enhancement of finite strings with infinite ones has been known for some time, see e.g. [BoN80]; it gives rise to an infinitary monoid, a domain in the sense of Scott and a complete metric trace. An interesting question arises as to what properties of the algebra of finite and infinite strings generalize onto the algebra of traces.

In this paper, we investigate the monoid and order-theoretic properties of the algebra of finite and infinite traces. First, trace equivalence over infinite strings and the notion of trace prefix ordering for possibly infinite traces are introduced. It is shown that this ordering gives rise to a Scott domain and a complete metric space. Then, the difficulty of finding an appropriate extension of the concatenation of finite traces to infinite ones is discussed, and a total operation of sequential composition is proposed. The operation allows to generalize Levi's lemma to possibly infinite traces, but unfortunately is non-associative. Finally, possible ways of introducing an acceptable operation are discussed. This paper is a

contribution to the search for an adequate semantic framework for non-interleaving concurrency, which uses topological theory originating from Nivat, see e.g. [BoN80].

An alternative approach has been independently developed in [Gas90]. While preparing the final version of this paper, it has come to our knowledge that Levi's lemma has also been extended to infinite traces in [Die91].

2. PRELIMINARIES

This section introduces notation conventions and recalls standard definitions, which can be found in e.g. [GHK80, BoN80, Law87].

Let (P, \leq) be a poset. For $X \subseteq P$ and $x \in P$ write: $\downarrow X = \{y \in P \mid y \leq x \text{ for some } x \in X\}$, $\uparrow X = \{y \in P \mid x \leq y \text{ for some } x \in X\}$. $X \subseteq P$ is a *lower set* (also *prefix-closed*) iff $X = \downarrow X$. $X \subseteq P$ is an *upper set* (or *upward-closed*) iff $X = \uparrow X$. $X \subseteq P$ is *directed* iff it is non-empty and every pair of elements $x, y \in X$ have a bound z also in X . $X \subseteq P$ is an *ideal* iff it is a directed lower set.

To distinguish between orderings, we shall use \leq and \sqsubseteq ; the least upper bounds and greatest lower bounds are denoted by \sqcup and \sqcap respectively.

Let (D, \sqsubseteq) be a complete partial order (cpo) and $x, y \in D$. We say x is *essentially below* (or *way below*) y , denoted $x \sqsubseteq^{\text{fin}} y$, iff given a directed set $M \subseteq D$ such that $y \sqsubseteq \sqcup M$, then there exists $z \in M$ such that $x \sqsubseteq z$. Intuitively, x is essentially below y if x is some finite approximation of y . For $X \subseteq D$ and $x \in D$ write: $\downarrow^{\text{fin}} X = \{y \in D \mid y \sqsubseteq^{\text{fin}} x \text{ for some } x \in X\}$, $\uparrow^{\text{fin}} X = \{y \in D \mid x \sqsubseteq^{\text{fin}} y \text{ for some } x \in X\}$.

Let (D, \sqsubseteq) be a cpo. D is *consistently complete* iff every subset $X \subseteq D$ bounded in D has a least upper bound. $x \in D$ is a *finite element* (also called *compact*) if, whenever $M \subseteq D$ is directed and $x \sqsubseteq \sqcup M$, then there exists $y \in M$ such that $x \sqsubseteq y$. Equivalently, x is finite iff $x \sqsubseteq^{\text{fin}} x$. The set of all finite elements of D is denoted B_D . D is *algebraic* iff, for every $x \in D$, the set $M = \{y \in B_D \mid y \sqsubseteq x\} = \downarrow^{\text{fin}} x$ is directed and $\sqcup M = x$. D is a *Scott domain* iff D is algebraic, consistently complete and B_D is countable.

Example 2.1. Consider the poset $P = \{0,1\}^*$ of all finite strings over $\{0,1\}$ (including the empty string ϵ) with the usual prefix order denoted by \leq . The ideal completion $\text{Id}(P)$ can be identified with all finite and infinite strings. $\text{Id}(P)$ is a Scott domain. $\{0,1\}^*$ are finite elements. The essentially below relation \leq^{fin} is the inherited order.

We use A^{ω} to represent the set of all infinite sequences over A , and A^{∞} is the union of A^* and A^{ω} . The *prefix order* over A^{∞} will be denoted by \leq and is defined as follows:

$$\forall x, y \in A^{\infty}: x \leq y \Leftrightarrow \forall i \in \mathbb{N} - \{0\}: (i \leq \text{len}(x) \Rightarrow x(i) = y(i))$$

where $x(i)$ denotes the i th symbol of the sequence x if it exists, and ϵ otherwise.

Concatenation is extended onto A^{∞} [BoN80] by taking as the concatenation of sequences x, y :

$$\begin{aligned} \forall x \in A^*, y \in A^{\omega}: & \text{ the infinite sequence } xy \\ \forall x \in A^{\omega}, y \in A^{\infty}: & \text{ the infinite sequence } x. \end{aligned}$$

Concatenation of strings will be denoted as juxtaposition. A^{∞} , together with the above concatenation, forms an infinitary monoid [BoN80]. This is a natural extension of the concatenation over finite strings. A^{∞} is a complete metric space wrt to the ultrametric:

$$d_{\text{str}}(x,y) = \begin{cases} 2^{-k} & \text{if } \exists n \in \mathbb{N} - \{0\}: x(n) \neq y(n) \\ 0 & \text{iff } \forall n \in \mathbb{N} - \{0\}: x(n) = y(n) \end{cases}$$

where $k = \min\{n \in \mathbb{N} - \{0\} \mid x(n) \neq y(n)\}$. The concatenation in A^* is uniformly continuous wrt the metric d_{str} , (A^*, d_{str}) is dense in $(A^\infty, d_{\text{str}})$, and hence the concatenation has a canonical continuous extension on A^∞ .

Applications of the topological theory of complete metric spaces in semantics originate from Nivat, see e.g. [BoN80]. An alternative approach is due to Scott. Given cpo's D, E , a function $f: D \rightarrow E$ is Scott-continuous iff f is monotone, i.e. $x \leq y$ implies $f(x) \leq f(y)$, and $f(\bigsqcup M) = \bigsqcup f(M)$ for every directed set M . The concatenation on A^* is not Scott-continuous in both arguments (it is not even monotone, consider e.g. $(a, b) \leq (aa, bb)$, but $ab \not\leq aabb$). However, the following simple modification to the definitions results in concatenation becoming Scott continuous. Let A be a (finite) alphabet and $\surd \notin A$ denote the *termination* symbol. Consider the set:

$$A^{\surd\infty} = A^* \cup A^* \surd \cup A^{\omega}$$

of strings with the prefix ordering \leq as usual. It is a Scott domain, and the concatenation ; defined by:

$$x;y = \begin{cases} x'y & \text{if } x = x'\surd \\ x & \text{otherwise} \end{cases}$$

is Scott continuous.

3. INFINITE TRACES

Trace Equivalence and Finite Traces

In this section we recall basic definitions concerning trace theory; for complete presentations see e.g. [Maz77, AaR88, Maz84, Maz89]. Trace languages constitute an abstraction of concurrent behaviour derivable from the notion of *causal independency*. Define the *independency* to be an irreflexive and symmetric relation $\iota \subseteq A \times A$. Two actions are independent if they can happen concurrently without affecting the result, and dependent otherwise. A *concurrent alphabet* (A, ι) is formed from a finite set of action symbols A and an independency $\iota \subseteq A \times A$. Given a concurrent alphabet (A, ι) one may obtain *trace equivalence* [Maz77] as the least congruence \equiv^{fin} in the monoid of finite strings (A^*, \cdot, ϵ) such that $a \iota b \Rightarrow ab \equiv ba$. It then follows that for all $v, w \in A^*$, $v \equiv^{\text{fin}} w$ iff w can be obtained from v through a finite number of transformations involving only permutations of consecutive independent symbols.

The equivalence class of a string is called a *trace*. Every trace should be thought of as containing the set of all possible *sequentializations* of a given (non-sequential) execution; two sequentializations are the same up to permutations of consecutive independent action symbols. For example, if $A = \{a, b, c\}$ and $\iota = \{(a,b), (b,a)\}$ then:

$$\begin{aligned} [ab] &= \{ab, ba\} \\ [ac] &= \{ac\} \\ [acb] &= \{acb\}. \end{aligned}$$

The monoid structure induces an order relation over finite traces. A trace σ is a *prefix* of τ , denoted $\sigma \sqsubseteq^{\text{fin}} \tau$, iff there exists γ such that $\tau = \sigma\gamma$. $(\Theta^*, \sqsubseteq^{\text{fin}})$ is a poset. Trace prefix order can be viewed as *approximation* ordering on executions, where each trace corresponds to a (global) state. $[\epsilon]$ is the initial state, incomparable prefixes of the same trace represent states arising during concurrent execution, and $\tau_1 \sqsubseteq^{\text{fin}} \tau_2$ means that τ_1 is a partial execution leading to τ_2 . Fig. 1 shows a sample concurrent execution together with its prefixes.

The following is a summary of properties of finite traces:

- (i) $(\Theta^*, \cdot, [\epsilon])$ is a cancellative monoid;
- (ii) $(\Theta^*, \sqsubseteq^{\text{fin}})$ is a consistently complete countable poset [Kwi89t].

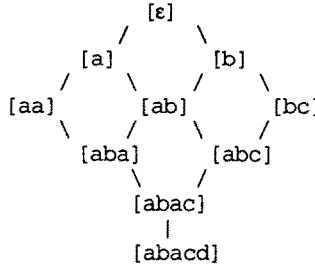


Fig. 1. The poset of all trace prefixes of $[abcd]$ with $a \uparrow b, a \uparrow c$.

Extending Trace Equivalence onto Infinite Strings

We extend trace equivalence onto possibly infinite strings by introducing *trace preorder* \leq_{tr} for finite and infinite strings as follows:

$$u \leq_{\text{tr}} v \Leftrightarrow (\forall x \leq^{\text{fin}} u \exists y \leq^{\text{fin}} v : (\exists z \in A^* : x \leq z \text{ and } z \equiv^{\text{fin}} y)).$$

Observe that $u \leq v$ implies $u \leq_{\text{tr}} v$, that is, the preorder \leq_{tr} is weaker than the string prefix order \leq , in the sense that it ignores permutations of consecutive independent symbols. We now define *trace equivalence* in A^ω , denoted \equiv , as the equivalence relation determined by the trace preorder \leq_{tr} ; more precisely, for all $x, y \in A^\omega$:

$$x \equiv y \Leftrightarrow (x \leq_{\text{tr}} y \text{ and } y \leq_{\text{tr}} x).$$

It can be shown that this definition coincides with trace equivalence \equiv^{fin} over A^* .

The set of all infinite traces over the concurrent alphabet (A, \uparrow) is denoted by Θ_{\uparrow}^ω , and $\Theta_{\uparrow}^\infty = \Theta_{\uparrow}^* \cup \Theta_{\uparrow}^\omega$. The subscript \uparrow will often be omitted.

Example 3.1. Consider $A = \{a, b, c\}$ with $a \uparrow b$. It is easy to see that $a \leq_{\text{tr}} ab$, $a \leq_{\text{tr}} ba$, $a \leq_{\text{tr}} ac$, $a \not\leq_{\text{tr}} ca$, $ab \leq_{\text{tr}} ba$, $ba \leq_{\text{tr}} ab$, $a \leq_{\text{tr}} a^\omega$, $a^\omega \leq_{\text{tr}} (ab)^\omega$, and $a^\omega \leq_{\text{tr}} (ba)^\omega$.

It should be emphasised that, unlike the definition of trace equivalence in the monoid of finite strings, our definition of \equiv allows for an *infinite* number of permutations of two consecutive independent symbols. For example, $(ab)^\omega \equiv (ba)^\omega$ for $a \uparrow b$.

An equivalent trace preorder definition has been independently introduced in [Gas90]. It is worth mentioning that there are other ways of defining trace equivalence, see e.g. [Shi85, Gas90], and other representations of traces, e.g. dependency graphs [AaR88] and vector languages [Shi85].

Trace Prefix Ordering for Possibly Infinite Traces

Trace prefix ordering on possibly infinite traces is determined by the preorder \leq_{tr} . A (finite or infinite) trace σ is a *prefix* of a (finite or infinite) trace τ over $C = (A, \iota)$, denoted by $\sigma \sqsubseteq \tau$, iff:

$$\forall x \in \sigma \exists y \in \tau: x \leq_{\text{tr}} y.$$

The following property follows from the definition:

Proposition 3.2.

- (i) If $\sigma \sqsubseteq \tau$ then for all $y \in \tau$ there exists $x \in \sigma$ such that $x \leq_{\text{tr}} y$.
- (ii) If $\sigma \sqsubseteq \tau$ then for all $x \in \sigma, y \in \tau: x \leq_{\text{tr}} y$.

The restriction of the relation \sqsubseteq to the finite elements coincides with the ordering \sqsubseteq^{fin} over Θ^* . The following may also be easily shown:

Proposition 3.3. Let $\sigma \in \Theta^*, \tau \in \Theta^\infty$ such that $\sigma \sqsubseteq \tau$. Then:

- (i) For all $x \in \sigma$ there exists $y \in \tau$ such that $x \leq y$ (i.e. x is a string prefix of y).
- (ii) For all $y \in \tau$ there exists $x \in \sigma$ such that $x \leq y$.

Proposition 3.3(ii) does *not* extend onto infinite traces; for example $[a^\omega] \sqsubseteq [ba^\omega]$ when $a \iota b$, but for no $y \in [ba^\omega] = \{a^*ba^\omega\}$ do we have $a^\omega \leq y$.

The set of prefixes of a given trace is directed, but not necessarily a total order like in the algebra of strings. As a consequence, in contrast to prefix ordering for strings, it is possible to show two *distinct* infinite traces, of which one dominates the other one. For example, the following holds for $a \iota b$:

$$[(a)^\omega] \sqsubseteq [b(a)^\omega] \sqsubseteq [bb(a)^\omega] \sqsubseteq \dots \sqsubseteq [(ab)^\omega].$$

If there exists a trace that dominates both τ_1 and τ_2 we say that τ_1 and τ_2 are *consistent*; otherwise they are *inconsistent*; in other words, two consistent traces correspond to consistent observations of a non-sequential behaviour, while inconsistent ones to observations that could not possibly have occurred during the same execution. If $\tau_1 \sqsubseteq \tau_2$ or $\tau_2 \sqsubseteq \tau_1$ we say that τ_1 and τ_2 are *comparable*; otherwise they are *incomparable*; two incomparable but consistent traces arise from concurrent execution. Unlike in the algebra of strings, two incomparable traces are not necessarily inconsistent.

A trace τ is *maximal* in $T \subseteq \Theta^\infty$ iff there does not exist a trace $\sigma \in T$ such that $\tau \sqsubseteq \sigma$ and $\sigma \neq \tau$. Maximal traces correspond to *complete* executions, whereas non-maximal traces are partial. The notion of maximality has impact on fairness issues [Kwi89t].

4. PROPERTIES OF TRACES

The Domain of Traces

In this section we investigate order-theoretic properties of $(\Theta^\infty, \varepsilon)$. We show that $(\Theta^\infty, \varepsilon)$ is a generalization of (A^∞, \leq) in the sense that it forms a Scott domain which is isomorphic to (A^∞, \leq) in the case of $t = \emptyset$.

Lemma 4.1. *Let (P, \leq) be a poset. Then:*

- (i) $X \subseteq P$ is directed $\Leftrightarrow \downarrow X$ is directed.
- (ii) If X is a finite directed subset of P then $\sqcup X$ exists and belongs to X .

Proof.

(i) Can be found in [GHK80].

(ii) By induction on cardinality of the set. □

Theorem 4.2. *Let (A, ι) be a concurrent alphabet. Then $(\Theta^\infty, \varepsilon)$ is a cpo.*

Proof.

1) It is clear that $\{\varepsilon\}$ is the least element.

2) We prove that every directed subset X of Θ^∞ has a least upper bound. By Lemma 4.1 it is sufficient to show the existence of least upper bounds of infinite directed sets $X \subseteq \Theta^\infty$ such that $X = \downarrow X$. Let $X \subseteq \Theta^\infty$ be such a set. Since $X \cap \Theta^*$ is countable, its elements can be ordered into a sequence $\{\sigma_k \mid k \in \mathbb{N}\}$. We construct $\sqcup X$ as a limit of a monotonically increasing sequence in $X \cap \Theta^*$. Define the monotonically increasing family M_k of directed subsets of $X \cap \Theta^*$ by:

$$M_0 = \emptyset$$

$$M_k = \begin{cases} M_{k-1} \cup \{\sigma_k\} & \text{if } M_{k-1} \cup \{\sigma_k\} \text{ is directed} \\ M_{k-1} \cup \{\sigma_k, \beta\} & \text{if } M_{k-1} \cup \{\sigma_k\} \text{ is not directed and } \beta \text{ is a bound for } M_{k-1} \cup \{\sigma_k\}. \end{cases}$$

Note that β always exists by the assumption of directedness of X . Also note that $X \cap \Theta^* = \bigcup \{M_k \mid k \in \mathbb{N}\}$. Since each M_k is finite and directed, $\sqcup M_k$ exists and belongs to M_k by Lemma 4.1(ii). By construction of the family M_k we have $M_k \subseteq M_{k+1}$, and hence:

$$\sqcup M_k \in \sqcup M_{k+1} \text{ for all } k \in \mathbb{N}.$$

Let $x_k \in \sqcup M_k$, $x_{k+1} \in \sqcup M_{k+1}$; then $x_k \leq_{\text{tr}} x_{k+1}$ by Proposition 3.2(ii). Since each $\sqcup M_k \in \Theta^*$ we have $x_k \in A^*$ for all $k \in \mathbb{N}$ and by definition of trace preorder \leq_{tr} :

$$\exists z_{k+1}: x_k \leq z_{k+1} \text{ and } x_{k+1} \equiv^{\text{fin}} z_{k+1};$$

it follows that $z_{k+1} \in \sqcup M_{k+1}$. We have thus shown:

$$\forall k \forall z_k \in \sqcup M_k \exists z_{k+1} \in \sqcup M_{k+1}: z_k \leq z_{k+1},$$

and hence it is possible to construct inductively a monotonic sequence $\{z_k \mid k \in \mathbb{N}\} \subseteq A^*$ starting from $\sqcup M_0$. Define $x = \sqcup \{z_k \mid k \in \mathbb{N}\}$ (exists because A^∞ is a cpo), and take $\gamma = [x]_t$. It is easy to see that γ is the least upper bound of X .

This concludes the proof. □

We now show $(\Theta^\infty, \sqsubseteq)$ is consistently complete. The following binary *left cancellation* operator, denoted $/: \Theta^* \times \Theta^* \rightarrow \Theta^*$, will be required:

$$\forall \sigma, \gamma \in \Theta^*: \sigma/\gamma = \tau \Leftrightarrow (\gamma \sqsubseteq^{\text{fin}} \sigma \text{ and } \gamma\tau = \sigma).$$

σ/γ is pronounced σ *after* γ ; it denotes the *continuation* of σ after its prefix γ has been completed.

Lemma 4.3. *Let (P, \sqsubseteq) be a poset, and let $X \subseteq P$. Then the following conditions are equivalent:*

- (i) $\sqcup X$ exists
- (ii) $\sqcup \downarrow X$ exists.

And if these conditions are satisfied, then $\sqcup X = \sqcup \downarrow X$. Moreover, if every finite subset of X has a least upper bound and if F denotes the set of all those finite least upper bounds, then F is directed and (i) and (ii) are equivalent to:

- (iii) $\sqcup F$ exists

Under these circumstances, $\sqcup X = \sqcup F$.

Proof. Can be found in [GHK80]. □

Lemma 4.4. *Let (D, \sqsubseteq) be a cpo. Then the following are equivalent:*

- (i) (D, \sqsubseteq) is consistently complete,
- (ii) Every non-empty subset of D has a greatest lower bound.

Proof.

\Rightarrow) Observe $\prod X$ equals $\sqcup L_X$ where L_X denotes the set of lower bounds of X .

\Leftarrow) Observe $\sqcup X$ equals $\prod U_X$ where U_X denotes the set of upper bounds of X . □

Lemma 4.5. *Let $\sigma, \gamma \in \Theta^*$, $\beta \in \Theta^\infty$.*

If $\sigma, \gamma \sqsubseteq \beta$ then there exists $\delta \in \Theta^$ such that $\delta = \sigma \sqcup \gamma$.*

Proof.

Let $\sigma, \gamma \in \Theta^*$, $\beta \in \Theta^\infty$ such that $\sigma, \gamma \sqsubseteq \beta$.

1) We show there exists $\beta' \in \Theta^*$ such that $\sigma, \gamma \sqsubseteq \beta' \sqsubseteq \beta$. Let $x \in \sigma, y \in \gamma, z \in \beta$; then $x \sqsubseteq_{\text{tr}} z$ and $y \sqsubseteq_{\text{tr}} z$ by Proposition 3.2. Since x and y are finite, we have by definition of \sqsubseteq_{tr} that there exist finite prefixes z_1, z_2 of z such that $x \sqsubseteq u_1$ and $y \sqsubseteq u_2$ for some $u_1 \sqsubseteq z_1, u_2 \sqsubseteq z_2$. Let $\beta' = [\max\{z_1, z_2\}]$. It is clear that $\sigma, \gamma \sqsubseteq \beta' \sqsubseteq \beta$ and $\beta' \in \Theta^*$.

2) We can now assume $\sigma, \gamma, \beta \in \Theta^*$ and $\sigma, \gamma \sqsubseteq \beta$. We prove by induction on the length of σ that $\delta = \sigma \sqcup \gamma$ exists. If $\sigma = [e]$ then take $\delta = \gamma$. It is clear that $\delta = \sigma \sqcup \gamma$. Suppose $\sigma \neq [e]$ and $[a] \sqsubseteq \sigma$ for some $a \in A$. Since $\sigma \sqsubseteq \beta$ we have $[a] \sqsubseteq \beta$. There are two cases:

2a) $[a] \sqsubseteq \gamma$, and thus $\gamma = [a](\gamma/[a]) \sqsubseteq \beta = [a](\beta/[a])$, which implies, by left cancellation law, that $\gamma/[a] \sqsubseteq \beta/[a]$. By symmetry, $\sigma/[a] \sqsubseteq \beta/[a]$. By induction on the length of σ it follows that $\delta' = (\sigma/[a]) \sqcup (\gamma/[a])$ exists. It is easy to see that:

$$\delta = \sigma \sqcup \gamma = [a]\delta'.$$

2b) $[a] \not\sqsubseteq \gamma$, from which it follows by a fairly straightforward argument (omitted, see [Kwi89t]) that $\gamma \sqsubseteq \beta/[a]$. By induction on the length of σ we have $\delta' = (\sigma/[a]) \sqcup \gamma$ exists. Finally, it is clear that:

$$\delta = \sigma \sqcup \gamma = [a]\delta'$$

This concludes the proof. □

Theorem 4.6.

- (i) $(\Theta^\infty, \sqsubseteq)$ is consistently complete.
- (ii) Every subset X of Θ^∞ has a greatest lower bound $\sqcap X$ wrt \sqsubseteq .

Proof.

- (i) Suppose $X \subseteq \Theta^\infty$ bounded. We show $\sqcap X$ exists. Let Y be a finite subset of X , then the existence of $\sqcup Y$ can be proved using induction and Lemma 4.5. Let F denote the set of $\sqcup Y$ for all finite subsets Y of X . Then F is directed by Lemma 4.3, and hence $\sqcup F$ exists because $(\Theta^\infty, \sqsubseteq)$ is a cpo (Theorem 4.2). Thus, since $\sqcup F = \sqcup X$ by Lemma 4.3, we have shown that $\sqcup X$ exists.
- (ii) Direct from Lemma 4.4. □

Finally, we prove that the set of all traces forms a domain in the sense of Scott.

Lemma 4.7. *Let (D, \sqsubseteq) be a cpo. If D is consistently complete, then $\downarrow^{\text{fin}} x$ is directed for all $x \in D$.*

Proof. See e.g. [Law87]. □

Theorem 4.8. *Let $C = (A, \tau)$ be a concurrent alphabet. Then $(\Theta^\infty, \sqsubseteq)$ is a Scott domain.*

Proof.

1) Since $(\Theta^\infty, \sqsubseteq)$ is consistently complete (Proposition 4.6), it follows by Lemma 4.7 that $\downarrow^{\text{fin}} \sigma$ is directed for all $\sigma \in \Theta^\infty$, and the least upper bound of $\downarrow^{\text{fin}} \sigma$ exists. It is clear that $\sqcup \downarrow^{\text{fin}} \sigma = \sigma$. Hence, $(\Theta^\infty, \sqsubseteq)$ is algebraic.

2) We now show that Θ^* are the finite elements. Suppose $\sigma \in \Theta^*$ and $M \subseteq \Theta^\infty$ is directed. Then $\sqcup M$ exists because $(\Theta^\infty, \sqsubseteq)$ is a cpo (Theorem 4.2). We need to show the existence of $\tau \in M$ such that $\sigma \sqsubseteq \tau$ if $\sigma \sqsubseteq \sqcup M$. Suppose $\sigma \sqsubseteq \sqcup M$. If $\sqcup M \in M$, then take $\tau = \sqcup M$. Otherwise, if $\sigma \sqsubseteq \sqcup M$ and $\sqcup M \notin M$, then M must be infinite by Lemma 4.1(ii); hence, $\sqcup M$ must be an infinite trace while σ is its finite prefix, and thus $\sigma \neq \sqcup M$. Let γ be an arbitrary member of M , then $\sigma, \gamma \sqsubseteq \sqcup M$. It follows that $M \cup \{\sigma, \sqcup M\}$ is directed. Since $\sigma \neq \sqcup M$, we conclude $M \cup \{\sigma\}$ is directed. Thus, there must exist $\tau \in M$ such that $\sigma, \gamma \sqsubseteq \tau$.

To show that elements of Θ^ω are not the finite elements, take any $\sigma \in \Theta^\omega$ and the directed set $M = \downarrow^{\text{fin}} \sigma$. Then $\sigma = \sqcup M \notin M$ and for no $\tau \in M$ do we have $\sigma \sqsubseteq \tau$.

This concludes the proof. □

Thus, finite trace prefixes can be given the usual domain-theoretic interpretation of being a 'finite approximation' of a possibly infinite trace.

The following result provides us with an alternative construction for the domain of traces.

Lemma 4.9. *Let (D, \sqsubseteq) be a cpo. Then the following are equivalent:*

- (i) (D, \sqsubseteq) is algebraic,
- (ii) D is isomorphic to the ideal-completion of a poset.

Proof. See e.g. [Law87]. □

Proposition 4.10. $(\Theta^\infty, \sqsubseteq)$ is isomorphic to the ideal completion of $(\Theta^*, \sqsubseteq^{\text{fin}})$.

Proof. Direct from Lemma 4.9. □

Next, we summarize the remaining properties of the domain of traces.

Lemma 4.11. $\forall \sigma, \tau \in \Theta^\infty: \sigma \sqsubseteq \tau \Leftrightarrow \downarrow^{\text{fin}} \sigma \subseteq \downarrow^{\text{fin}} \tau$.

Proof.

\Rightarrow) Follows from transitivity of \sqsubseteq .

\Leftarrow) Direct from algebraicity of traces (Theorem 4.8). □

Finally, the domain of traces can be shown to specialise to the domain of strings.

Proposition 4.12. *Let $C = (A, \iota)$ be a concurrent alphabet such that $\iota = \emptyset$. Then: $(\Theta^\infty, \sqsubseteq)$ is isomorphic to (A^∞, \leq) .*

Proof. Easy; observe that if $\iota = \emptyset$ then traces are singleton sets. □

The Metric Space of Traces

There exists a natural metric for traces defined as follows. Let $\sigma, \tau \in \Theta^\infty$ and put:

$$d_{\text{tr}}(\sigma, \tau) = 2^{-\inf\{\text{len}(\gamma) \mid \gamma \in \sigma \Delta \tau\}}$$

where $\text{len}(\gamma)$ denotes the length of some string $x \in \gamma$, $\inf \emptyset = +\infty$, and $\sigma \Delta \tau$ is the symmetric difference of the sets of finite prefixes below σ and τ respectively. Formally:

$$\sigma \Delta \tau = (\downarrow^{\text{fin}} \sigma \setminus \downarrow^{\text{fin}} \tau) \cup (\downarrow^{\text{fin}} \tau \setminus \downarrow^{\text{fin}} \sigma).$$

Note that the value of $\inf\{\text{len}(\gamma) \mid \gamma \in \sigma \Delta \tau\}$ is a measure of the 'depth' of approximation necessary to distinguish between traces σ and τ .

It can be shown [Kwi90m] that d_{tr} is an ultrametric and that the following holds.

Proposition 4.13. ([Kwi89m])

(i) $(\Theta^\infty, d_{\text{tr}})$ is a complete metric space.

(iii) If A is finite then the d_{tr} -topology in Θ^∞ is the Lawson topology.

Also, the metric d_{tr} specializes to the metric d_{str} of [BoN80] in case of $\iota = \emptyset$. Thus, infinite traces have two characterizations: as metric limits of Cauchy sequences of finite traces, and as order-theoretic limits of directed sets of finite traces.

It is an interesting question to investigate the continuity properties of the concatenation of finite and traces. Unfortunately, unlike in the case of strings, concatenation is *not* uniformly continuous on Θ^* wrt the metric d_{tr} . To demonstrate this fact, we need to show that:

$$\exists M \forall N \exists \sigma, \tau, \sigma', \tau': [d_{\text{tr}}(\langle \sigma, \tau \rangle, \langle \sigma', \tau' \rangle) < 2^{-N} \text{ and } d_{\text{tr}}(\sigma\tau, \sigma'\tau') \geq 2^{-M}].$$

Let $A = \{a, b, c\}$ with $a \iota b$, and take $\sigma = [a^n]$, $\sigma' = [a^n c]$, $\tau = \tau' = [b]$. Then:

$$d_{\text{tr}}(\sigma\tau, \sigma'\tau') = d_{\text{tr}}([a^n b], [a^n c b]) = 2^{-1}$$

because $[b]$ is the shortest prefix in $[a^n b] \Delta [a^n c b] = \{[b], [a^n c b], [a^n c b]\}$, and yet:

$$\begin{aligned} d_{tr}(\langle \sigma, \tau \rangle, \langle \sigma', \tau' \rangle) &= \max\{d_{tr}(\sigma, \sigma'), d_{tr}(\tau, \tau')\} \\ &= \max\{d_{tr}([a^n], [a^n c]), d_{tr}([b], [b])\} = \max\{2^{-(n+1)}, 0\} = 2^{-(n+1)} \end{aligned}$$

because $[a^n c]$ is the shortest prefix in $[a^n] \Delta [a^n c]$. Thus, unlike in the case of strings, the concatenation over Θ^* does *not* have a canonical extension to Θ^∞ .

5. SEQUENTIAL COMPOSITION OF TRACES

We now investigate the monoid properties of Θ^∞ . We believe that concatenation in Θ^∞ should be a suitable generalization of the concatenation in A^∞ , as the latter corresponds to our intuitive understanding of sequential composition of two (sequential) executions x and y : if the behaviour represented by x does not terminate, then the behaviour represented by y should never proceed. What is then the right intuition for defining sequential composition of two *non-sequential* executions represented by traces? Ideally, causal independency should be taken into account. Suppose $.$ denotes the concatenation operation, then for a and b *independent* it would be against reasonable intuition to define $[a^\omega].[b]$ as $[a^\omega]$ because the occurrence of a can in no way affect b ; thus, the occurrence of a should *not* prevent b from proceeding. Observe that traces $[a^\omega]$ and $[b]$ are consistent, i.e. have a common bound, e.g. $[ba^\omega]$. On the other hand, if a and b are *dependent*, $[a^\omega].[b]$ should be $[a^\omega]$. Note that in this case the traces $[a^\omega]$ and $[b]$ are inconsistent.

However, if we accept the above intuition regarding respecting causality, we will lose associativity. Consider $A = \{a, b, c\}$ with $a \perp b$. Then:

$$\begin{aligned} [a^\omega].[c].[b] &= [a^\omega].[cb] = [a^\omega] \neq \\ ([a^\omega].[c]).[b] &= [a^\omega].[b] = [ba^\omega] \end{aligned}$$

and $[a^\omega] \neq [ba^\omega]$. Moreover, continuity wrt the metric d_{tr} fails for very similar reasons. We have:

$$\lim ([a^n c].[b]) = \lim ([a^n c b]) = [a^\omega] \neq (\lim [a^n c]).(\lim [b]) = [a^\omega].[b] = [ba^\omega].$$

Scott continuity will also fail because the concatenation of finite traces is not even monotone. It remains to be seen if techniques similar to those used in Section 2 to show that concatenation of strings is Scott-continuous can be applied in this case.

Defining Sequential Composition

We now introduce an extension of the concatenation of finite traces to Θ^∞ which agrees with the above intuition. We also show that $(\Theta^\infty, ., [\epsilon])$ is a pseudo-monoid, in the sense that $.$ is total and associativity holds on a subset. Although non-associativity may be viewed as undesirable, our definition allows us to generalize Levi's lemma [Maz84, CoP85] to possibly infinite traces.

Definition 5.1.

- (i) We say traces σ, τ are *independent* iff $\text{Act}(\sigma) \times \text{Act}(\tau) \subseteq \perp$.
- (ii) A trace $\tau \in \Theta_t^\infty$ is *tail-independent* with the trace $\sigma \in \Theta^\infty$ iff:
 - $\exists \sigma' \sqsubseteq^{\text{fin}} \sigma, \forall \sigma'' \in \Theta^*: \sigma' \sigma'' \sqsubseteq \sigma \Rightarrow \sigma'', \tau$ are independent.
- (iii) A trace $\tau \in \Theta^\infty$ is *tail-independent* with the trace $\sigma \in \Theta^\infty$ *after* σ' iff

$$\sigma' \sqsubseteq^{\text{fin}} \sigma \text{ and} \\ \forall \sigma'' \in \Theta^*: \sigma' \sigma'' \sqsubseteq \sigma \Rightarrow \sigma'', \tau \text{ are independent.}$$

The following may be shown.

Proposition 5.2.

- (i) For every $\sigma \in \Theta^\infty$, σ is tail-independent with $[\varepsilon]$.
- (ii) For all $\sigma, \tau \in \Theta^\infty$, if σ, τ independent then σ, τ tail-independent.
- (iii) For all $\sigma, \tau \in \Theta^*$:
if σ, τ tail-independent after σ' , then $\sigma' \tau$ and σ are consistent.

Define an auxiliary operator $\Lambda(\sigma, \tau)$ for $\sigma, \tau \in \Theta^\infty$ as follows.

$$\Lambda(\sigma, \tau) = \{ \sigma' \tau' \mid \sigma' \sqsubseteq^{\text{fin}} \sigma, \tau' \sqsubseteq^{\text{fin}} \tau \text{ and } \tau' \text{ tail-independent with } \sigma \text{ after } \sigma' \}.$$

Note that $\downarrow^{\text{fin}} \sigma \subseteq \Lambda(\sigma, \tau)$. Intuitively, we break the 'past history' of the execution σ into finite segments $\sigma' \sqsubseteq^{\text{fin}} \sigma$. Similarly, the 'potential future' execution τ is broken into finite segments $\tau' \sqsubseteq^{\text{fin}} \tau$. We then concatenate the finite past σ' with the finite future τ' and choose only those segments $\sigma' \tau'$ which represent a viable *continuation* of the past history given by σ , in the sense of $\sigma' \tau'$ and σ being consistent, i.e. having a common bound. Thus, $\Lambda(\sigma, \tau)$ represents all *finite* pieces of the new history. It is clear that for finite traces σ and τ the whole of the trace $\sigma \tau$ becomes the new history; however, when traces σ and τ are infinite, part of the future τ may be delayed indefinitely.

$\Lambda(\sigma, \tau)$ is the maximal directed subset of ΣT where $\sigma = \downarrow^{\text{fin}} \sigma$ and $T = \downarrow^{\text{fin}} \tau$ that is consistent with σ . We can now define sequential composition in Θ^∞ .

Definition 5.3. For $\sigma, \tau \in \Theta^\infty$ define:

$$\sigma \cdot \tau = \sqcup \Lambda(\sigma, \tau)$$

Proposition 5.4.

- (i) For all $\sigma, \tau \in \Theta^\infty$, $\sigma \cdot \tau$ is well defined.
- (ii) If $\sigma, \tau \in \Theta^*$, then $\Lambda(\sigma, \tau) = \downarrow^{\text{fin}} [xy]$, for some $x \in \sigma, y \in \tau$.

Proof.

(i) We need to show $\Lambda(\sigma, \tau)$ is a directed set. Note that $\Lambda(\sigma, \tau)$ is non-empty because $\downarrow^{\text{fin}} \sigma \subseteq \Lambda(\sigma, \tau)$ and $\downarrow^{\text{fin}} \sigma$ contains $[\varepsilon]$. Let $\sigma_1 \tau_1, \sigma_2 \tau_2 \in \Lambda(\sigma, \tau)$, then by definition:

- (a) $\sigma_1 \sqsubseteq^{\text{fin}} \sigma \wedge \tau_1 \sqsubseteq^{\text{fin}} \tau \wedge \tau_1$ tail-independent with σ after σ_1
- (b) $\sigma_2 \sqsubseteq^{\text{fin}} \sigma \wedge \tau_2 \sqsubseteq^{\text{fin}} \tau \wedge \tau_2$ tail-independent with σ after σ_2 .

By Lemma 4.5 we have there exists $\sigma' \sqsubseteq^{\text{fin}} \sigma$ such that $\sigma' = \sigma_1 \sqcup \sigma_2$, and hence $\text{Act}(\sigma') = \text{Act}(\sigma_1) \cup \text{Act}(\sigma_2)$. Thus, by definition of tail-independency, τ_1 must be tail-independent with σ after σ' and τ_2 tail-independent with σ after σ' . Using induction we can construct $\tau' \sqsubseteq^{\text{fin}} \tau$ such that $\tau' = \tau_1 \sqcup \tau_2$. Then, $\text{Act}(\tau') = \text{Act}(\tau_1) \cup \text{Act}(\tau_2)$. Let $\sigma'' \in \Theta^*$ such that $\sigma' \sigma'' \sqsubseteq^{\text{fin}} \sigma$. It is easy to see $\text{Act}(\sigma'') \times \text{Act}(\tau') \subseteq \iota$; hence, σ'' and τ' are independent. It follows τ' is tail-independent with σ after σ' . Let $\sigma_i \sigma_i'' = \sigma'$ for $i \in \{1, 2\}$. Then, since τ_i, σ_i'' are independent, we have by Levi's lemma:

$$\sigma_i \tau_i \sqsubseteq \sigma_i \tau_i \sigma_i'' = \sigma_i \sigma_i'' \tau_i = \sigma' \tau_i \sqsubseteq \sigma' \tau'.$$

We have thus constructed $\sigma' \tau' \in \Lambda(\sigma, \tau)$ such that $\sigma_1 \tau_1, \sigma_2 \tau_2 \sqsubseteq \sigma' \tau'$.

Since Θ^∞ is a cpo, $\sqcup \Lambda(\sigma, \tau)$ must exist.

(ii) Straightforward. □

Example 5.5. The following are examples of sequential composition in Θ^∞ . Let $A = \{a,b,c\}$ with $a \perp b$, then:

$$\begin{aligned} [a].[b^{\omega}] &= \sqcup(\{[b^*] \cup [ab^*]\}) = [ab^{\omega}], \\ [ab^{\omega}].[a] &= \sqcup(\{[\varepsilon]\} \cup [ab^*] \cup [aab^*]) = [aab^{\omega}], \\ [a^{\omega}].[b^{\omega}] &= \sqcup(\{[a^*b^*]\}) = \sqcup(\{[(a^*b^*)^{\omega}]\}) = [(ab)^{\omega}] \end{aligned}$$

On the other hand:

$$\begin{aligned} [a].[c^{\omega}] &= \sqcup(\{[\varepsilon]\} \cup [ac^*]) = [ac^{\omega}], \\ [a^{\omega}].[c] &= \sqcup(\{[a^*]\}) = [a^{\omega}], \\ [a^{\omega}].[c^{\omega}] &= \sqcup(\{[a^*]\}) = [a^{\omega}]. \end{aligned}$$

Finally:

$$\begin{aligned} [ca^{\omega}].[bc] &= \sqcup(\{[\varepsilon]\} \cup [ca^*] \cup \{[cb]\}) = [cba^{\omega}], \\ [ca^{\omega}].[cb] &= \sqcup(\{[\varepsilon]\} \cup [ca^*]) = [ca^{\omega}]. \end{aligned}$$

The following summarises the relationship of the sequential composition of possibly infinite strings and traces.

Proposition 5.6. *If $\perp = \emptyset$ then for any $\sigma, \gamma \in \Theta^\infty$ we have:*

$$\sigma.\gamma = [xy]$$

where $x \in \sigma, y \in \tau$.

Proof.

Straightforward. □

The following important property may also be shown.

Lemma 5.7. (Shields) *Suppose $x, y \in \Theta^*$, then for $z \in \Theta^\infty$:*
 $(xy)/(z \sqcap (xy)) \sqsupseteq x/(z \sqcap x)$.

Proof. By induction on the length of y (see [Kwi89t]). □

Proposition 5.8. *Let $\sigma, \gamma \in \Theta^\infty$. Then*

$$\sigma \sqsubseteq \gamma \Leftrightarrow \exists \beta \in \Theta^\infty: \sigma.\beta = \gamma.$$

Proof.

\Rightarrow) Suppose $\sigma \sqsubseteq \gamma$. We construct the least $\beta \in \Theta^\infty$ such that $\sigma.\beta = \gamma$. By Lemma 4.11 we have $\downarrow^{\text{fin}} \sigma \subseteq \downarrow^{\text{fin}} \gamma$. Define:

$$M = \{\tau/\sigma' \mid \tau \in (\downarrow^{\text{fin}} \gamma) \setminus (\downarrow^{\text{fin}} \sigma), \sigma' = \sqcup M_\tau\}$$

where $M_\tau = \downarrow^{\text{fin}} \sigma \cap \downarrow^{\text{fin}} \tau$.

(1) We prove that M_τ is directed and contains least upper bound of every pair of its elements. Observe that M_τ is non-empty because it contains $[\varepsilon]$. Let $\sigma_1, \sigma_2 \in M_\tau$, then:

$$\sigma_1, \sigma_2 \sqsubseteq \sigma \text{ and } \sigma_1, \sigma_2 \sqsubseteq \tau$$

By Lemma 4.5 it follows that $\sigma_1 \sqcup \sigma_2$ exists and is bounded by σ and τ , and thus $\sigma_1 \sqcup \sigma_2 \in M_\tau$.

Hence, M_τ is directed.

(2) Observe that M is well-defined because $\sigma' = \sqcup M_\tau$ always exists by (1) and $\sigma' \sqsubseteq \tau$. Also, $\sqcup M_\tau$ is finite because τ is finite.

(3) We show that M is directed. Note that M is non-empty because it contains $[\varepsilon] = \sqcup \emptyset$. Let $\beta', \beta'' \in M$, then $\beta' = \tau'/\sigma', \beta'' = \tau''/\sigma''$. We need to show there exists $\beta''' \in M$ such that $\beta', \beta'' \sqsubseteq \beta'''$.

Observe that $\tau' \sqcup \tau''$ exists because τ', τ'' bounded by γ . Also, $\sigma' = \sigma \sqcap \tau'$ and $\sigma'' = \sigma \sqcap \tau''$ (by definition). Let $y = (\tau' \sqcup \tau'')/\sigma$. By Lemma 5.7:

$$(\tau' \sqcup \tau'') / (\sigma \sqcap (\tau' \sqcup \tau'')) = (\tau' \gamma) / (\sigma \sqcap (\tau' \gamma)) \sqsupseteq \tau' / (\sigma \sqcap \tau') = \tau' / \sigma'.$$

By symmetry, $(\tau' \sqcup \tau'') / (\sigma \sqcap (\tau' \sqcup \tau'')) \sqsupseteq \tau'' / (\sigma \sqcap \tau'') = \tau'' / \sigma''$. Note that $(\tau' \sqcup \tau'') / (\sigma \sqcap (\tau' \sqcup \tau'')) \in M$ because $\tau' \sqcup \tau'' \in (\downarrow^{\text{fin}} \gamma) \setminus (\downarrow^{\text{fin}} \sigma)$ and $\sigma \sqcap (\tau' \sqcup \tau'') = \sqcup M_{\tau' \sqcup \tau''}$.

(4) Since M directed by (3), define $\beta = \sqcup M$ (exists because Θ^∞ is a cpo by Theorem 4.2). It follows by construction that this is the least such β .

(5) We show $\Lambda(\sigma, \beta) \subseteq \downarrow^{\text{fin}} \gamma$. Suppose $\sigma' \beta' \in \Lambda(\sigma, \beta)$, then by definition $\sigma' \sqsubseteq^{\text{fin}} \sigma$, $\beta' \sqsubseteq^{\text{fin}} \beta$ and β' is tail-independent with σ after σ' . On the other hand, $\beta' = \tau / \sigma'$ for some $\tau \in (\downarrow^{\text{fin}} \gamma) \setminus (\downarrow^{\text{fin}} \sigma)$, $\sigma' = \sqcup M_\tau$ (by definition of M). Note that $\sigma' \sqcup \sigma'$ exists, and that β' is tail-independent with σ after $\sigma' \sqcup \sigma'$.

Define $\kappa = (\sigma' \sqcup \sigma') \beta'$ and observe that $\kappa \in \downarrow^{\text{fin}} \gamma$.

(6) We show $\downarrow^{\text{fin}} \gamma \subseteq \Lambda(\sigma, \beta)$. As $\downarrow^{\text{fin}} \sigma \subseteq \downarrow^{\text{fin}} \gamma$ it is sufficient to show that $(\downarrow^{\text{fin}} \gamma) \setminus (\downarrow^{\text{fin}} \sigma) \subseteq \Lambda(\sigma, \beta)$. Let $\tau \in (\downarrow^{\text{fin}} \gamma) \setminus (\downarrow^{\text{fin}} \sigma)$ and define $\beta' = \tau / \sigma'$ where $\sigma' = \sqcup M_\tau$. Now $\beta' \in M$ (by definition of M), and thus $\beta' \in \downarrow^{\text{fin}} \beta$. Finally observe that β' is tail-independent with σ after σ' .

\Leftrightarrow Direct from definition of sequential composition and Lemma 4.11. \square

Finally, we remark that associativity and metric continuity hold for pairs $\langle \sigma, \tau \rangle$ such that $\sigma \in \Theta^*$, $\tau \in \Theta^\infty$.

Levi's Lemma for Infinite Traces

The following property follows from the definition of sequential composition. It states that given any two consistent (i.e. bounded) traces, the corresponding continuations from their greatest lower bound commute. This property is referred to as the 'commutativity lemma'; it is a reformulation of the well-known Levi's lemma for traces, which has been proved for the finite case in [Maz84, CoP85], and independently for the infinite case in [Gas90, Die91].

First, we extend the left cancellation operator on to Θ^∞ in the following way:

Definition 5.9. Let $\sigma, \gamma, \beta \in \Theta^\infty$ such that $\gamma \sqsubseteq \sigma$. We say that β is σ *after* γ (denoted σ / γ) iff β is the least trace such that $\gamma \beta = \sigma$.

As an example, consider $A = \{a, b, c\}$ with $a \sqcup b$. Then $[a^\omega] / [a^\omega] = [\varepsilon]$, $[ba^\omega] / [a^\omega] = [b]$, $[ba^\omega] / [a^\omega] = [b]$, $[ca^\omega] / [a^\omega]$ is undefined.

Proposition 5.10. (*Commutativity lemma*)

Let $\sigma, \tau, \beta \in \Theta^\infty$ such that $\sigma, \tau \sqsubseteq \beta$. Then:

$$(\sigma / \gamma) \sqcup (\tau / \gamma) \text{ and } \sigma \sqcup \tau = \gamma.(\sigma / \gamma).(\tau / \gamma) = \gamma.(\tau / \gamma).(\sigma / \gamma)$$

where $\gamma = \sigma \sqcap \tau$.

Sketch of proof. Observe that $\sigma \sqcup \tau$ exists by Lemma 4.5, $\sigma \sqcap \tau$ exists by Proposition 4.6, and both σ / γ and τ / γ are defined. Suppose $\beta \sqsubseteq^{\text{fin}} (\sigma / \gamma)$, $\delta \sqsubseteq^{\text{fin}} (\tau / \gamma)$, then $\beta \sqcup \delta$ can be proved by induction on the length of β ; thus conclude $(\sigma / \gamma) \sqcup (\tau / \gamma)$. The equality $\sigma \sqcup \tau = \gamma.(\sigma / \gamma).(\tau / \gamma) = \gamma.(\tau / \gamma).(\sigma / \gamma)$ follows from the definition of sequential composition. \square

6. CONCLUSION

The approach presented here has concentrated on a basic semantic framework that can be used to model concurrency in the non-interleaving sense, and its mathematical characterizations. We have investigated ways by which adding *structural* information in the form of the relation of causal independency over actions can influence the mathematical space of the infinitary behaviour of concurrent systems. This has been achieved by a suitable extension of trace theory, which has been shown to give rise to a Scott domain, a complete metric space and a pseudo-monoid. It has been demonstrated that there exists a natural approximation ordering on traces, which may be obtained by relaxing the corresponding approximation ordering on strings. Being less discriminating, trace prefix ordering allows to distinguish between action occurrences that represent either 'consistent' (i.e. co-existing, but perhaps remote) or 'inconsistent' (i.e. conflicting) pieces of some non-sequential 'history'.

The detailed investigation of existence of a Scott- and metric-continuous sequential composition, concurrent composition, and hiding operations is in progress. Further work will include techniques for representing process denotations in the domain of traces, deriving appropriate operations on process denotations and solving reflexive domain equations.

The mathematical structure of the space of finite and infinite traces is of considerable interest because it can be shown that using non-interleaving semantics has useful implications on fairness, see e.g. [Kwi90t, Kwi90c]. Maximality with respect to trace prefix ordering determines a certain notion of fairness [Kwi89e], and other notions of fairness, e.g. process fairness, also arise naturally [Kwi90p].

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