Parameter Estimates for a Pencil of Lines: Bounds and Estimators

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Abstract. Estimating the parameters of a pencil of lines is addressed. A statistical model for the measurements is developed, from which the Cramer Rao lower bound is determined. An estimator is derived, and its performance is simulated and compared to the bound. The estimator is shown to be asymptotically efficient, and superior to the classical least squares algorithm.

1 Introduction

Identifying straight lines and estimating their common point of intersection is a frequent task in image processing applications. The particular problem of estimating the parameters of a single line from two dimensional measurements has been studied in [1], [4], [8]. When multiple lines are known to intersect at a common point, however, parameter estimates for a given line can be improved owing to the common information available from the other lines. Moreover, the structure of the estimator changes from that presented in [1] and [8]. It is the purpose of this paper to present the line parameter estimation problem by providing a parameterized statistical model of the measurements, analyzing the limitations imposed by this model on the line parameter estimates, and finally by proposing an estimator for the line parameters.

The analysis begins in Section 2, where the measurements of points on a line are statistically modelled by the parameters of interest. In Section 3, parameter estimation is addressed by using the parameterized statistical model of Section 2 to determine the Fisher information matrix for the line parameters. From the Fisher information matrix, the Cramer-Rao lower bounds for line parameter estimates are determined explicitly in terms of the line parameters and the statistical parameters influencing the measurements. Section 4 addresses the problem of estimating line parameters from a pencil of lines when each of these lines is modelled according to Section 2. The methods of Section 3 are used to find the Fisher information matrix for the totality of line parameters and their point of intersection. The performance benefit attained by using a mutual point of intersection is reflected in the Cramer Rao lower bound, which is compared with the results of Section 3. An estimator for the point of intersection and the respective line parameters is then developed in Section 5. In Section 6 the performance of this estimator is simulated and compared with the Cramer Rao lower

A. Heyden et al. (Eds.): ECCV 2002, LNCS 2350, pp. 432–446, 2002.

bounds from Sections 3 and 4. In addition, the proposed estimator is compared with the least squares estimator for the point of intersection when the lines are parameterized independently of each other.

2 The Line Data Model

One method for fitting two dimensional point measurements to a line is proposed by Ponce and Forsyth in [1]. The N line measurements (x_n, y_n), $n = 1, \ldots, N$ in the coordinate system (x, y) are modelled as a rotation by ϕ of points (χ_n, γ_n), $n = 1, \ldots, N$ in an initial coordinate system (χ, γ). In (χ, γ) coordinates, a line is assumed to be described by $\gamma = A$, although the measurements γ_n are perturbed from A by zero mean noise ν_n. Thus, line modelling begins with the transformation by coordinate rotation

$$\begin{aligned} \begin{bmatrix} x_n \\ y_n \end{bmatrix} &= \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \chi_n \\ \gamma_n \end{bmatrix} \\ &= \begin{bmatrix} b & a \\ -a & b \end{bmatrix} \begin{bmatrix} \chi_n \\ A + \nu_n \end{bmatrix} \\ &= -\mathbf{S} \begin{bmatrix} \chi_n \\ c - \nu_n \end{bmatrix} \end{aligned} \tag{1}$$

where the rotation matrix $\mathbf{S}$ is given by

$$\mathbf{S} = \begin{bmatrix} -b & a \\ a & b \end{bmatrix} \tag{2}$$

and

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} \sin\phi \\ \cos\phi \\ -A \end{bmatrix} \tag{3}$$

In this notation, a line is described in (x, y) coordinates by $ax + by + c = 0$.

In order to model the points (x_n, y_n) statistically, it is sufficient to characterize χ_n and ν_n, as (x_n, y_n) are related to these by the linear transformation (1). For simplicity of analysis, assume the noise samples ν_n to be zero mean, independent, and identically distributed Gaussian random variables with variance σ_ν^2, and denote this distribution by $\nu_n \sim N(0, \sigma_\nu^2)$, $n = 1, \ldots, N$. Similarly, the coordinates χ_n are assumed to be independent and identically distributed samples from a Gaussian distribution having mean μ_χ and variance σ_χ^2, such that $\chi_n \sim N(\mu_\chi, \sigma_\chi^2)$, $n = 1, \ldots, N$. As σ_ν^2 is the measurement noise and σ_χ^2 the spread of the points on the line, it must be that $\sigma_\chi^2 >> \sigma_\nu^2$.

As the two random variables ν_n and χ_n are Gaussian and independent, they are jointly Gaussian [3]. Thus, the vector $\boldsymbol{z}_n = \begin{bmatrix} x_n \; y_n \end{bmatrix}^T$ has Gaussian distribution $\boldsymbol{z}_n \sim N(\boldsymbol{\mu_z}, \mathbf{C_z})$ with mean vector $\boldsymbol{\mu_z} = E\{\boldsymbol{z}_n\}$ found from (1) as

$$\boldsymbol{\mu_z} = -\mathbf{S} \begin{bmatrix} \mu_\chi \\ c \end{bmatrix} \tag{4}$$

The correlation matrix $\mathbf{C}_{\boldsymbol{z}}$ is found from equations (1) and (4), and by noting that by the independence of χ_n and ν_n, $E\{(\chi_n - \mu_\chi)\,\nu_n\} = E\{\chi_n - \mu_\chi\}\,E\{\nu_n\} = 0$.

$$\begin{aligned}\mathbf{C}_{\boldsymbol{z}} &= E\left\{\mathbf{S}\begin{bmatrix}\chi_n - \mu_\chi \\ -\nu_n\end{bmatrix}\begin{bmatrix}\chi_n - \mu_\chi \\ -\nu_n\end{bmatrix}^T \mathbf{S}^T\right\} \\ &= \mathbf{S}\begin{bmatrix}\sigma_\chi^2 & 0 \\ 0 & \sigma_\nu^2\end{bmatrix}\mathbf{S}^T \\ &= \mathbf{S}\boldsymbol{\Lambda}\mathbf{S}^T \qquad (5)\end{aligned}$$

Since the matrix $\mathbf{S}$ is unitary and diagonalizes $\mathbf{C}_{\boldsymbol{z}}$, it holds the eigenvectors of $\mathbf{C}_{\boldsymbol{z}}$. From (2) and (5), the eigenvector $[a\ b]^T$ of $\mathbf{S}$ is associated with the eigenvalue σ_ν^2, the variance of the measurement noise.

The joint statistics of $\boldsymbol{z}_n$ can be expressed succinctly in terms of the concatenation vector $\boldsymbol{Z}$ given by

$$\boldsymbol{Z} = \left[\,\boldsymbol{z}_1^T\ \boldsymbol{z}_2^T\ \cdots\ \boldsymbol{z}_N^T\,\right]^T \qquad (6)$$

The measurements $\boldsymbol{z}_n$ are jointly Gaussian, so the vector $\boldsymbol{Z}$ has Gaussian distribution $\boldsymbol{Z} \sim N(\boldsymbol{\mu}_{\boldsymbol{Z}},\, \mathbf{C}_{\boldsymbol{Z}})$. The mean $\boldsymbol{\mu}_{\boldsymbol{Z}} = E\{\boldsymbol{Z}\}$ is given by the $2N \times 1$ vector

$$\mu_{\boldsymbol{Z}} = \left[\,\boldsymbol{\mu}_{\boldsymbol{z}}^T\ \ldots\ \boldsymbol{\mu}_{\boldsymbol{z}}^T\,\right]^T \qquad (7)$$

By virtue of the independent and identically distributed nature of the random variables ν_n and χ_n, $E\{\nu_n\nu_m\} = \sigma_\nu^2\delta_{n-m}$ and $E\{(\chi_n - \mu_\chi)(\chi_m - \mu_\chi)\} = \sigma_\nu^2\delta_{n-m}$, where δ_n is the Dirac function. The $2N \times 2N$ covariance matrix $\mathbf{C}_{\boldsymbol{Z}}$ is then given by

$$\mathbf{C}_{\boldsymbol{Z}} = \begin{bmatrix}\mathbf{C}_{\boldsymbol{z}} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\boldsymbol{z}} & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_{\boldsymbol{z}}\end{bmatrix} \qquad (8)$$

3 Cramer Rao Bounds for Line Parameter Estimates Given a Single Line

Having statistically characterized the joint distribution of the samples (x_n, y_n), $n = 1, \ldots, N$, the influence of this statistical model on parameter estimates can be determined. First, note from equations (2), (4) and (5) that the mean vector $\boldsymbol{\mu}_{\boldsymbol{z}}$ and covariance matrix $\mathbf{C}_{\boldsymbol{z}}$ are seen to be functions of the parameters ϕ, c, μ_χ, σ_χ^2, and σ_ν^2, denoted by the vector $\boldsymbol{\theta}$ as

$$\boldsymbol{\theta} = \left[\,\phi\ c\ \mu_\chi\ \sigma_\chi^2\ \sigma_\nu^2\,\right]^T \qquad (9)$$

Since the random vectors $\boldsymbol{z}_n$ have the Gaussian distribution $\boldsymbol{z}_n \sim N(\boldsymbol{\mu}_{\boldsymbol{z}}(\boldsymbol{\theta}), \mathbf{C}_{\boldsymbol{z}}(\boldsymbol{\theta}))$ for any choice of $\boldsymbol{\theta}$, the random vector $\boldsymbol{Z}$, by (7) and (8), is also statistically parameterized as $\boldsymbol{Z} \sim N(\boldsymbol{\mu}_{\boldsymbol{Z}}(\boldsymbol{\theta}), \mathbf{C}_{\boldsymbol{Z}}(\boldsymbol{\theta}))$. The random vectors $\boldsymbol{z}_n$, $n = 1, \ldots, N$ and $\boldsymbol{Z}$ are therefore called Generalized Gaussian random vectors [2]. The significance of this characterization lies in the fact that the parameters $\boldsymbol{\theta}$ are assumed deterministic and unknown, like the line parameters in (3), for example, which are being estimated. Estimates of fixed parameters, like the components of $\boldsymbol{\theta}$, have their minimum variance bounded by the Cramer Rao Lower Bound (CRLB), which is determined by relating the $K \times K$ covariance matrix $\mathbf{C}_{\hat{\boldsymbol{\theta}}}$ of the K unbiased parameter estimates $\hat{\boldsymbol{\theta}}$ to the inverse Fisher information matrix $\mathbf{I}^{-1}(\boldsymbol{\theta})$ as [2]

$$\mathbf{C}_{\hat{\boldsymbol{\theta}}} \geq \mathbf{I}^{-1}(\boldsymbol{\theta}) \tag{10}$$

So that for each estimate $\hat{\theta}_i$ of the true parameter θ_i, the variance $\sigma^2_{\hat{\theta}_i}$ is given by $\mathbf{C}_{\hat{\boldsymbol{\theta}}}(i,i)$, and the CRLB by $\mathbf{I}^{-1}(\boldsymbol{\theta})(i,i)$. For the case of a Generalized Gaussian random vector $\boldsymbol{X} \sim N(\boldsymbol{\mu}_{\boldsymbol{X}}(\boldsymbol{\theta}), \mathbf{C}_{\boldsymbol{X}}(\boldsymbol{\theta}))$, the elements of $\mathbf{I}(\boldsymbol{\theta})$ are given by [2], equation (3.31)

$$\begin{aligned}[\mathbf{I}(\theta)]_{ij} &= \left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial \theta_i}\right]^T \mathbf{C}_{\boldsymbol{X}}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial \theta_j}\right] \\ &\quad + \frac{1}{2}\mathrm{tr}\left[\mathbf{C}_{\boldsymbol{X}}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial \theta_i} \mathbf{C}_{\boldsymbol{X}}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}_{\boldsymbol{X}}(\boldsymbol{\theta})}{\partial \theta_j}\right]\end{aligned} \tag{11}$$

where $\mathrm{tr}[\mathbf{D}]$ denotes the trace of the matrix $\mathbf{D}$, and $\partial \boldsymbol{\mu}_{\boldsymbol{X}}(\boldsymbol{\theta}) / \partial \theta_j$ is the partial derivative of every element in the mean vector $\boldsymbol{\mu}_{\boldsymbol{X}}(\boldsymbol{\theta})$ with respect to the j^{th} element of the parameter vector $\boldsymbol{\theta}$, just as $\partial \mathbf{C}_{\boldsymbol{X}}(\boldsymbol{\theta}) / \partial \theta_i$ is the partial derivative of every component of the covariance matrix $\mathbf{C}_{\boldsymbol{X}}(\boldsymbol{\theta})$ with respect to the i^{th} element of $\boldsymbol{\theta}$.

Applying (11) to the measurements $\boldsymbol{Z} \sim N(\boldsymbol{\mu}_{\boldsymbol{Z}}(\boldsymbol{\theta}), \mathbf{C}_{\boldsymbol{Z}}(\boldsymbol{\theta}))$ with (7) and (8), then the elements $[\mathbf{I}(\theta)]_{ij}$ are found as

$$\begin{aligned}[\mathbf{I}(\theta)]_{ij} &= N \left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{z}}(\boldsymbol{\theta})}{\partial \theta_i}\right]^T \mathbf{C}_{\boldsymbol{z}}^{-1}(\boldsymbol{\theta}) \left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{z}}(\boldsymbol{\theta})}{\partial \theta_j}\right] \\ &\quad + \frac{N}{2}\mathrm{tr}\left[\mathbf{C}_{\boldsymbol{z}}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}_{\boldsymbol{z}}(\boldsymbol{\theta})}{\partial \theta_i} \mathbf{C}_{\boldsymbol{z}}^{-1}(\boldsymbol{\theta}) \frac{\partial \mathbf{C}_{\boldsymbol{z}}(\boldsymbol{\theta})}{\partial \theta_j}\right]\end{aligned} \tag{12}$$

Equation (12) states that the Fisher information matrix for N measurements $\boldsymbol{z}_n$ is that for a single measurement scaled by N.

Noting from (2) that $\mathbf{S}^T = \mathbf{S}$, the following identities from (5) make determining the components of (11) straightforward

$$\frac{\partial \mathbf{C}_{\boldsymbol{z}}}{\partial \phi} = \frac{\partial \mathbf{S}}{\partial \phi} \boldsymbol{\Lambda} \mathbf{S} + \mathbf{S} \boldsymbol{\Lambda} \frac{\partial \mathbf{S}}{\partial \phi} \tag{13}$$

$$\mathbf{S}\mathbf{S}^T = \mathbf{S}^T\mathbf{S} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{14}$$

$$\mathbf{S}\frac{\partial \mathbf{S}}{\partial \phi} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{15}$$

$$\frac{\partial \mathbf{S}}{\partial \phi}\mathbf{S} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{16}$$

$$\mathbf{C}_{\boldsymbol{z}}^{-1} = \mathbf{S}\boldsymbol{\Lambda}^{-1}\mathbf{S} \tag{17}$$

By applying the identities (13) through (17) in (12) for all combinations of the parameter elements in (9), it can be shown that the Fisher information matrix takes the form

$$\mathbf{I}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{I}(\phi, c, \mu_\chi) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}\left(\sigma_\chi^2, \sigma_\nu^2\right) \end{bmatrix} \tag{18}$$

The block diagonal structure of (18) implies that the inverse of (18) is similarly block diagonal, so that to find the bounds on estimating the line parameters ϕ, c, and μ_χ, one need only consider the Fisher information matrix $\mathbf{I}(\phi, c, \mu_\chi)$. This matrix is found as

$$\mathbf{I}(\phi, \mathbf{c}, \mu_\chi) = \frac{N}{\sigma_\nu^2\sigma_\chi^2} \begin{bmatrix} \sigma_\chi^2\mu_\chi^2 + c^2\sigma_\nu^2 + \left(\sigma_\nu^2 - \sigma_\chi^2\right)^2 & \mu_\chi\sigma_\chi^2 & -c\sigma_\nu^2 \\ \mu_\chi\sigma_\chi^2 & \sigma_\chi^2 & 0 \\ -c\sigma_\nu^2 & 0 & \sigma_\nu^2 \end{bmatrix} \tag{19}$$

and the inverse is given by

$$\mathbf{I}(\phi, \mathbf{c}, \mu_\chi)^{-1} = \frac{1}{N\left(\sigma_\nu^2 - \sigma_\chi^2\right)^2} \begin{bmatrix} \sigma_\chi^2\sigma_\nu^2 & -\mu_\chi\sigma_\chi^2\sigma_\nu^2 & c\sigma_\chi^2\sigma_\nu^2 \\ -\mu_\chi\sigma_\chi^2\sigma_\nu^2 & \kappa_1 & -c\mu_\chi\sigma_\chi^2\sigma_\nu^2 \\ c\sigma_\chi^2\sigma_\nu^2 & -c\mu_\chi\sigma_\chi^2\sigma_\nu^2 & \kappa_2 \end{bmatrix} \tag{20}$$

where $\kappa_1 = \sigma_\nu^2\left[\sigma_\chi^2\mu_\chi^2 + \left(\sigma_\nu^2 - \sigma_\chi^2\right)^2\right]$ and $\kappa_2 = \sigma_\chi^2\left[c^2\sigma_\nu^2 + \left(\sigma_\nu^2 - \sigma_\chi^2\right)^2\right]$. From (20), the CRLB for any estimate $\hat{\phi}$ of the rotation angle ϕ is given by the first diagonal entry

$$\mathrm{CRLB}\left(\hat{\phi}\right) = \frac{\sigma_\nu^2\sigma_\chi^2}{N\left(\sigma_\chi^2 - \sigma_\nu^2\right)^2} \tag{21}$$

To find the CRLB for the parameter estimates of $a = \sin\phi$, $b = \cos\phi$, and $c = c$ from (3), we use [2] p45 (3.30)

$$\mathbf{I}^{-1}(\mathbf{g}(\boldsymbol{\theta})) = \frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}\mathbf{I}^{-1}(\boldsymbol{\theta})\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}^T \tag{22}$$

which determines the inverse of the Fisher information matrix for estimating the functions $\mathbf{g}(\boldsymbol{\theta}) = \left[g_1(\boldsymbol{\theta})\ g_2(\boldsymbol{\theta}) \cdots g_r(\boldsymbol{\theta})\right]$ of the parameter vector $\boldsymbol{\theta}$ from the inverse of the Fisher information matrix for $\boldsymbol{\theta}$ itself. The matrix $\partial \mathbf{g}(\boldsymbol{\theta})/\partial \boldsymbol{\theta}$ in (22) is the $r \times p$ Jacobian matrix

$$\frac{\partial \mathbf{g}(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}} = \begin{bmatrix} \frac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_1(\boldsymbol{\theta})}{\partial \theta_p} \\ \frac{\partial g_2(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g_2(\boldsymbol{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_2(\boldsymbol{\theta})}{\partial \theta_p} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_r(\boldsymbol{\theta})}{\partial \theta_1} & \frac{\partial g_r(\boldsymbol{\theta})}{\partial \theta_2} & \cdots & \frac{\partial g_r(\boldsymbol{\theta})}{\partial \theta_p} \end{bmatrix} \tag{23}$$

Letting $\mathbf{g}(\boldsymbol{\theta}) = \left[a\ b\ c\right]$ as in (3), then substitution into (23) yields

$$\frac{\partial\left[a\ b\ c\right]}{\partial\boldsymbol{\theta}} = \begin{bmatrix} \cos\phi & 0\ 0 \\ -\sin\phi & 0\ 0 \\ 0 & 1\ 0 \end{bmatrix} \tag{24}$$

Applying (22) with (20) and (24) and noting (21) yields

$$\mathbf{I}^{-1}(a,b,c) = \mathrm{CRLB}\left(\hat{\phi}\right) \begin{bmatrix} \cos^2\phi & -\sin\phi\cos\phi & -\mu_\chi\cos\phi \\ -\sin\phi\cos\phi & \sin^2\phi & \mu_\chi\sin\phi \\ -\mu_\chi\cos\phi & \mu_\chi\sin\phi & \mu_\chi^2 + \frac{\sigma_\nu^2}{N\mathrm{CRLB}(\hat{\phi})} \end{bmatrix} \tag{25}$$

The terms along the main diagonal in (25) are the Cramer Rao lower bounds on the variance of the line parameter estimates $\hat{a}$, $\hat{b}$, and $\hat{c}$, respectively.

$$\begin{bmatrix} \sigma_{\hat{a}}^2 \\ \sigma_{\hat{b}}^2 \\ \sigma_{\hat{c}}^2 \end{bmatrix} \geq \begin{bmatrix} b^2\mathrm{CRLB}\left(\hat{\phi}\right) \\ a^2\mathrm{CRLB}\left(\hat{\phi}\right) \\ \mu_\chi^2\mathrm{CRLB}\left(\hat{\phi}\right) + \frac{\sigma_\nu^2}{N} \end{bmatrix} \tag{26}$$

4 Cramer Rao Bounds for Line Parameter Estimates Given a Pencil of Lines

The results of the previous section can be extended to find the influence of a common point of intersection (x_0, y_0) on line parameter estimates for a pencil of L lines. In this case, each line ℓ, $\ell = 1, \ldots, L$ has associated with it a distinct group of data $\boldsymbol{Z}_\ell$ of the form (6), each having N_ℓ two dimensional data points $\boldsymbol{z}_{n_\ell}$. The fact that the data families $\boldsymbol{Z}_\ell$, $\ell = 1, \ldots, L$ are all distinct means that the noise models that generate them, as in (1), are all independent. The vectors $\boldsymbol{Z}_\ell$, $\ell = 1, \ldots, L$ are jointly Gaussian with

$$E\left\{\left(\boldsymbol{Z}_\ell - \boldsymbol{\mu}_{\boldsymbol{Z}_\ell}\right)\left(\boldsymbol{Z}_k - \boldsymbol{\mu}_{\boldsymbol{Z}_k}\right)^T\right\} = \delta_{\ell-k}\mathbf{C}_{\boldsymbol{Z}_\ell} \tag{27}$$

where $\mathbf{C}_{\boldsymbol{Z}_\ell}$, $\ell = 1, \ldots, L$ are of the form (8). A vector $\boldsymbol{Q}$ defined as the concatenation of the measurements $\boldsymbol{Z}_\ell$, $\ell = 1, \ldots, L$ such that

$$\boldsymbol{Q} = \left[\boldsymbol{Z}_1^T \cdots \boldsymbol{Z}_L^T\right]^T \tag{28}$$

is therefore distributed as $\boldsymbol{Q} \sim N(\boldsymbol{\mu}_{\boldsymbol{Q}}, \mathbf{C}_{\boldsymbol{Q}})$ with mean vector $\boldsymbol{\mu}_{\boldsymbol{Q}} = E\left\{\boldsymbol{Q}\right\}$

$$\boldsymbol{\mu}_{\boldsymbol{Q}} = \left[\boldsymbol{\mu}_{\boldsymbol{Z}_1}^T \cdots \boldsymbol{\mu}_{\boldsymbol{Z}_L}^T\right]^T \tag{29}$$

and, using (27), the covariance matrix $\mathbf{C}_{\boldsymbol{Q}}$ is

$$\mathbf{C}_{\boldsymbol{Q}} = \begin{bmatrix} \mathbf{C}_{\boldsymbol{Z}_1} & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{\boldsymbol{Z}_2} & & \mathbf{0} \\ \vdots & & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{C}_{\boldsymbol{Z}_L} \end{bmatrix} \tag{30}$$

By substitution of $\boldsymbol{Q}$ in (11), and using (29) and (30), its not hard to show that the elements of the Fisher Information matrix are given by

$$[\mathbf{I}(\boldsymbol{\vartheta})]_{ij} = \sum_{\ell=0}^{L} N_\ell \left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{z}_\ell}(\boldsymbol{\vartheta})}{\partial \vartheta_i}\right]^T \mathbf{C}_{\boldsymbol{z}_\ell}^{-1}(\boldsymbol{\vartheta}) \left[\frac{\partial \boldsymbol{\mu}_{\boldsymbol{z}_\ell}(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right] + \frac{N_\ell}{2} \mathrm{tr}\left[\mathbf{C}_{\boldsymbol{z}_\ell}^{-1}(\boldsymbol{\vartheta}) \frac{\partial \mathbf{C}_{\boldsymbol{z}_\ell}(\boldsymbol{\vartheta})}{\partial \vartheta_i} \mathbf{C}_{\boldsymbol{z}_\ell}^{-1}(\boldsymbol{\vartheta}) \frac{\partial \mathbf{C}_{\boldsymbol{z}_\ell}(\boldsymbol{\vartheta})}{\partial \vartheta_j}\right] \tag{31}$$

where $\boldsymbol{\mu}_{\boldsymbol{z}_\ell}$ and $\mathbf{C}_{\boldsymbol{z}_\ell}$ are from (4) and (5). The parameter vector $\boldsymbol{\vartheta}$ holds the parameter vectors $\boldsymbol{\theta}_\ell$ defined in (9) for each family of line data $\ell = 1, \ldots, L$. This parameter space is reduced for the pencil of lines by noting that the common point of intersection (x_0, y_0) lies on each of the L lines, and thus satisfies $a_\ell x_0 + b_\ell y_0 + c_\ell = 0$, $\ell = 1, \ldots, L$. Thus, $c_\ell = -a_\ell x_0 - b_\ell y_0$ and the parameter vector $\boldsymbol{\vartheta}$ is given by

$$\boldsymbol{\vartheta} = \left[\phi_1\ \mu_{\chi_1}\ \sigma_{\nu_1}^2\ \sigma_{\chi_1}^2 \cdots \phi_L\ \mu_{\chi_L}\ \sigma_{\nu_L}^2\ \sigma_{\chi_L}^2\ x_0\ y_0\right] \tag{32}$$

It should be clear from (31) and the definitions (4) and (5) that $\partial \boldsymbol{\mu}_{\boldsymbol{z}_i}/\partial \vartheta_j = 0$ and $\partial \boldsymbol{C}_{\boldsymbol{z}_i}/\partial \vartheta_j = 0$ for $i \neq j$ where ϑ_j are the line parameters of the j^{th} line, excluding x_0 and y_0. By applying the identities (13) through (17) in (31), it's straightforward to show that parameter estimates of ϕ_ℓ, μ_{χ_ℓ}, x_0 and y_0 are independent of those for $\sigma_{\nu_\ell}^2$ and $\sigma_{\chi_\ell}^2$, $\ell = 1, \ldots, L$, exactly as for (18). The Fisher information matrix for the reduced parameter vector $\boldsymbol{\vartheta}'$

$$\boldsymbol{\vartheta}' = \left[\phi_1\ \mu_{\chi_1} \cdots \phi_L\ \mu_{\chi_L}\ x_0\ y_0\right] \tag{33}$$

is then given by

$$\mathbf{I}(\boldsymbol{\vartheta}') = \begin{bmatrix} I_{\phi_1,\phi_1} & I_{\phi_1,\mu_{\chi_1}} & 0 & \cdots & 0 & 0 & I_{\phi_1,x_0} & I_{\phi_1,y_0} \\ I_{\mu_{\chi_1},\phi_1} & I_{\mu_{\chi_1},\mu_{\chi_1}} & 0 & \cdots & 0 & 0 & I_{\mu_{\chi_1},x_0} & I_{\mu_{\chi_1},y_0} \\ 0 & 0 & \ddots & & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & I_{\phi_L,\phi_L} & I_{\phi_L,\mu_{\chi_L}} & I_{\phi_L,x_0} & I_{\phi_L,y_0} \\ 0 & 0 & 0 & \cdots & I_{\mu_{\chi_L},\phi_L} & I_{\mu_{\chi_L},\mu_{\chi_L}} & I_{\mu_L,x_0} & I_{\mu_{\chi_L},y_0} \\ I_{x_0,\phi_1} & I_{x_0,\mu_{\chi_1}} & I_{x_0,\phi_2} & \cdots & I_{x_0,\phi_L} & I_{x_0,\mu_{\chi_L}} & I_{x_0,x_0} & I_{x_0,y_0} \\ I_{y_0,\phi_1} & I_{y_0,\mu_{\chi_1}} & I_{y_0,\phi_2} & \cdots & I_{y_0,\phi_L} & I_{y_0,\mu_{\chi_L}} & I_{y_0,x_0} & I_{y_0,y_0} \end{bmatrix} \tag{34}$$

where the elements of (34) are given by $I_{\phi_\ell,\phi_\ell} = N_\ell\big(\frac{(\mu_{\chi_\ell} - b_\ell x_0 + a_\ell y_0)^2}{\sigma_{\nu_\ell}^2} + \frac{(a_\ell x_0 + b_\ell y_0)^2}{\sigma_{\chi_\ell}^2} + \frac{(\sigma_{\nu_\ell}^2 - \sigma_{\chi_\ell}^2)^2}{\sigma_{\nu_\ell}^2 \sigma_{\chi_\ell}^2}\big)$, $I_{\phi_\ell,\mu_{\chi_\ell}} = I_{\mu_{\chi_\ell},\phi_\ell} = N_\ell \frac{a_\ell x_0 + b_\ell y_0}{\sigma_{\chi_\ell}^2}$, $I_{\phi_\ell,x_0} = I_{x_0,\phi_\ell} = N_\ell \frac{a_\ell (b_\ell x_0 - a_\ell y_0 - \mu_{\chi_\ell})}{\sigma_{\nu_\ell}^2}$, $I_{\phi_\ell,y_0} = I_{y_0,\phi_\ell} = N_\ell \frac{b_\ell (b_\ell x_0 - a_\ell y_0 - \mu_{\chi_\ell})}{\sigma_{\nu_\ell}^2}$, $I_{\mu_{\chi_\ell},\mu_{\chi_\ell}} = \frac{1}{\sigma_{\chi_\ell}^2}$, $I_{\mu_{\chi_\ell},x_0} = I_{x_0,\mu_{\chi_\ell}} = I_{\mu_{\chi_\ell},y_0} = I_{y_0,\mu_{\chi_\ell}} = 0$, $I_{x_0,x_0} = \sum_{\ell=1}^{L} \frac{a_\ell^2}{\sigma_{\nu_\ell}^2}$, $I_{x_0,y_0} = I_{y_0,x_0} = \sum_{\ell=1}^{L} \frac{a_\ell b_\ell}{\sigma_{\nu_\ell}^2}$, and $I_{y_0,y_0} = \sum_{\ell=1}^{L} \frac{b_\ell^2}{\sigma_{\nu_\ell}^2}$.

The Fisher information matrix for line parameter estimates of a_ℓ, b_ℓ, $\ell = 1, \ldots, L$ and (x_0, y_0) is found from (22) using the transformation $\boldsymbol{g}(\boldsymbol{\vartheta}')$ of the parameter vector $\boldsymbol{\vartheta}'$ given by

$$\boldsymbol{g}(\boldsymbol{\vartheta}') = \left[\sin\phi_1 \; \cos\phi_1 \; \sin\phi_2 \; \cos\phi_2 \; \cdots \; \sin\phi_L \; \cos\phi_L \; x_0 \; y_0\right] \tag{35}$$

Then

$$\mathbf{I}^{-1}(\mathbf{g}(\boldsymbol{\vartheta}')) = \frac{\partial \mathbf{g}(\boldsymbol{\vartheta}')}{\partial \boldsymbol{\vartheta}'} \mathbf{I}^{-1}(\boldsymbol{\vartheta}') \frac{\partial \mathbf{g}(\boldsymbol{\vartheta}')}{\partial \boldsymbol{\vartheta}'}^T \tag{36}$$

where $\mathbf{I}^{-1}(\boldsymbol{\vartheta}')$ is from (34), and the matrix $\partial \mathbf{g}(\boldsymbol{\vartheta}')/\partial \boldsymbol{\vartheta}'$ is determined by applying (23) to (35) in the same manner as was done for (24). The bounds afforded by (36) can be compared to the bounds of (26) for unassisted line parameter estimation. Figure 1 illustrates the improved performance of joint estimation by plotting the minimum variance $\sigma^2_{\hat{a}_1}$ as a function of the number of lines L using (36) and (26). The simulation parameters are $x_0 = 1000.0$, $y_0 = 0$, $N = 64$, $\mu_\chi = 0$, $\sigma^2_\nu = 1$, $\sigma^2_\chi = 16$, and $\Delta_\phi = \pi/6$.

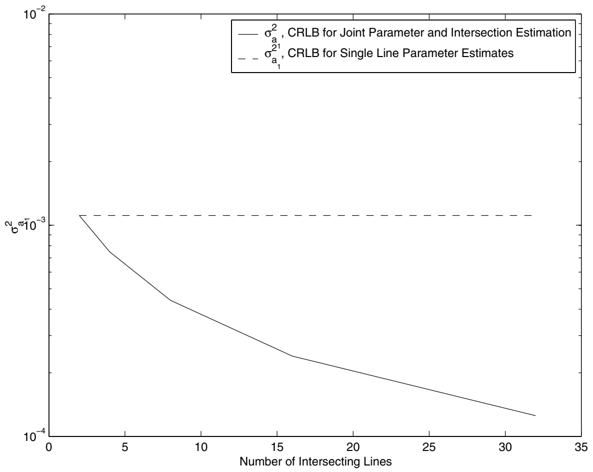

Fig. 1. Comparison of CRLBs from (36) and (26) for estimating parameter a_1 from a pencil of L lines.

5 A Point of Intersection Estimator

To motivate an estimator for the point of intersection $(\hat{x}_0, \hat{y}_0)$ of L lines, consider a particular line ℓ, whose measurements are of the form (1), such that

$$\begin{bmatrix} x_{n,\ell} \\ y_{n,\ell} \end{bmatrix} = \begin{bmatrix} \cos\phi_\ell & \sin\phi_\ell \\ -\sin\phi_\ell & \cos\phi_\ell \end{bmatrix} \begin{bmatrix} \chi_{n,\ell} \\ \gamma_{n,\ell} \end{bmatrix} \tag{37}$$

Multiplying both sides of (37) by a test vector $\left[\sin\hat{\phi}_\ell \cos\hat{\phi}_\ell\right]$ yields

$$\sin\hat{\phi}_\ell x_{n,\ell} + \cos\hat{\phi}_\ell y_{n,\ell} = \sin\left(\hat{\phi}_\ell - \phi_\ell\right)\chi_{n,\ell} + \cos\left(\hat{\phi}_\ell - \phi_\ell\right)\gamma_{n,\ell} \tag{38}$$

The variance of the left hand side of (38) is the same as the variance of the right hand side of (38), which is given by

$$\sin^2\left(\hat{\phi}_\ell - \phi_\ell\right)\sigma^2_{\chi,\ell} + \cos^2\left(\hat{\phi}_\ell - \phi_\ell\right)\sigma^2_{\nu,\ell} \tag{39}$$

As $\sigma^2_{\nu,\ell} < \sigma^2_{\chi,\ell}$ by assumption, (39) is minimized when $\hat{\phi}_\ell = \phi_\ell$. Thus, defining $[\hat{a}_\ell \ \hat{b}_\ell] = [\sin\hat{\phi}_\ell \ \cos\hat{\phi}_\ell]$ and a constant $\hat{c}_\ell$ for $\ell = 1, \ldots, L$, then these are line parameters as in (3), with the constraint

$$\hat{a}_\ell^2 + \hat{b}_\ell^2 = \left[\hat{a}_\ell \ \hat{b}_\ell\right]\begin{bmatrix}\hat{a}_\ell \\ \hat{b}_\ell\end{bmatrix} = 1 \tag{40}$$

In addition, these lines are constrained to intersect at $(\hat{x}_0, \hat{y}_0)$

$$\hat{a}_\ell\hat{x}_0 + \hat{b}_\ell\hat{y}_0 + \hat{c}_\ell = 0 \tag{41}$$

Now, the likelihood function $p_{\boldsymbol{\Theta}}(\boldsymbol{Q})$, where $\boldsymbol{Q}$ is from (28), can be expressed

$$p_{\boldsymbol{\Theta}}(\boldsymbol{Q}) = C_0 e^{-\sum_{\ell=1}^{L}\sum_{n=1}^{N_\ell}\frac{\left(a_\ell x_{n_\ell} + b_\ell y_{n_\ell} + c_\ell\right)^2}{2\sigma^2_{\nu_\ell}} + \frac{\left(-b_\ell x_{n_\ell} + a_\ell y_{n_\ell} - \mu_{\chi_\ell}\right)^2}{2\sigma^2_{\chi_\ell}}} \tag{42}$$

with $\boldsymbol{\Theta} = [a_1 \ b_1 \ c_1 \ \ldots \ a_L \ b_L \ c_L \ x_0 \ y_0]$. Given the variance (39), the constraint (40), and the assumption $\sigma^2_{\nu_\ell} < \sigma^2_{\chi_\ell}$, choosing $\boldsymbol{\Theta}$ to minimize the double sum over the first squared term in the exponent of (42) will maximize $p_{\boldsymbol{\Theta}}(\boldsymbol{Q})$. Thus, a suitable cost for an estimator of $\boldsymbol{\Theta}$ is

$$\begin{aligned} C(\boldsymbol{Q}) = &\sum_{\ell=1}^{L}\sum_{n=1}^{N_\ell}\frac{\left(\hat{a}_\ell x_{n,\ell} + \hat{b}_\ell y_{n,\ell} + \hat{c}_\ell\right)^2}{2\sigma_\ell^2} \\ &+ \sum_{\ell=1}^{L}\lambda_\ell\left(\hat{a}_\ell\hat{x}_0 + \hat{b}_\ell\hat{y}_0 + \hat{c}_\ell\right) + \sum_{\ell=1}^{L}\rho_\ell\left(\hat{a}_\ell^2 + \hat{b}_\ell^2 - 1\right) \end{aligned} \tag{43}$$

To decouple the parameter estimates for the separate lines in the minimization of (43), the point of intersection $(\hat{x}_0, \hat{y}_0)$ is viewed as a known parameter. The minimization of (43) with respect to the parameters $\hat{a}_\ell$, $\hat{b}_\ell$, and $\hat{c}_\ell$ of line ℓ is then

$$\mathbf{A}_\ell\begin{bmatrix}\hat{a}_\ell \\ \hat{b}_\ell\end{bmatrix} = -\frac{2\sigma_\ell^2\rho_\ell}{N_\ell}\begin{bmatrix}\hat{a}_\ell \\ \hat{b}_\ell\end{bmatrix} - \frac{\lambda_\ell\sigma_\ell^2}{N_\ell}\begin{bmatrix}(\hat{x}_0 - \overline{x_\ell}) \\ (\hat{y}_0 - \overline{y_\ell})\end{bmatrix} \tag{44}$$

As in [1], $\mathbf{A}_\ell$ is the modal matrix of the data associated with the ℓ^{th} line.

$$\mathbf{A}_\ell \equiv \begin{bmatrix}\left(\overline{x_\ell^2} - \overline{x_\ell}^2\right) & \left(\overline{y_\ell x_\ell} - \bar{x}_\ell\bar{y}_\ell\right) \\ \left(\overline{y_\ell x_\ell} - \bar{x}_\ell\bar{y}_\ell\right) & \left(\overline{y_\ell^2} - \overline{y_\ell}^2\right)\end{bmatrix} \tag{45}$$

where the notation $\overline{z}$ means

$$\overline{z} = \frac{\sum_{k=1}^{K} z_k}{K} \tag{46}$$

Clearly, $\mathbf{A}_\ell$ tends asymptotically to $\mathbf{C}_{\boldsymbol{z}_\ell}$ from (8). In addition, note the identity

$$\hat{c}_\ell = -\hat{a}_\ell \overline{x_\ell} - \hat{b}_\ell \overline{y_\ell} - \frac{\sigma_\ell^2 \lambda_\ell}{N_\ell} \tag{47}$$

which is the result of minimizing (43) with respect to $\hat{c}_\ell$, and helps to realize the form (44).

To complete the solution for $\hat{a}_\ell, \hat{b}_\ell$, denote the (column) eigenvectors of $\mathbf{A}_\ell$ by ψ_{0_ℓ} and ψ_{1_ℓ}, and the matrix $\boldsymbol{\Psi}_\ell = [\psi_{0_\ell} \ \psi_{1_\ell}]$ such that

$$\mathbf{A}_\ell = \boldsymbol{\Psi}_\ell \boldsymbol{\Lambda}_\ell \boldsymbol{\Psi}_\ell^T \tag{48}$$

where $\boldsymbol{\Lambda}_\ell$ is the associated diagonal matrix whose entries are the eigenvalues β_{0_ℓ} and β_{1_ℓ} of $\mathbf{A}_\ell$. The relevant parameters can all be defined then as

$$\left[(\hat{x}_0 - \overline{x_\ell}) \ (\hat{y}_0 - \overline{y_\ell}) \right]^T = \boldsymbol{\Psi}_\ell \mathbf{d}_\ell \tag{49}$$

$$[\hat{a}_\ell \ \hat{b}_\ell]^T = \boldsymbol{\Psi}_\ell \boldsymbol{f}_\ell \tag{50}$$

Using (50) and (49), equation (44) may be written

$$\boldsymbol{\Psi}_\ell \boldsymbol{\Lambda}_\ell \boldsymbol{f}_\ell = \alpha_\ell \boldsymbol{\Psi}_\ell \boldsymbol{f}_\ell - \frac{\lambda_\ell \sigma_\ell^2}{N_\ell} \boldsymbol{\Psi}_\ell \boldsymbol{d}_\ell \tag{51}$$

where $\alpha_\ell = -2\sigma_\ell^2 \rho_\ell / N_\ell$. By multiplying both sides of (51) by $\boldsymbol{\Psi}_\ell^T$, noting that $\boldsymbol{\Psi}_\ell^T \boldsymbol{\Psi}_\ell = \mathbf{I}_{2\times 2}$ and then rearranging, its not hard to show that the solution for $(\hat{a}_\ell, \hat{b}_\ell)$ is given by

$$\begin{bmatrix} \hat{a}_\ell \\ \hat{b}_\ell \end{bmatrix} = -\frac{\lambda_\ell \sigma_\ell^2}{N_\ell} \boldsymbol{\Psi}_\ell \begin{bmatrix} \frac{1}{\beta_{0,\ell} - \alpha_\ell} & 0 \\ 0 & \frac{1}{\beta_{1,\ell} - \alpha_\ell} \end{bmatrix} \boldsymbol{d}_\ell \tag{52}$$

The Lagrangian multipliers α_ℓ and λ_ℓ in (52) must now be determined. α_ℓ is found from backwards substitution of (52) into (41). Using (47) and after some manipulation, it can be shown that

$$\frac{d_{0,\ell}^2}{(\beta_{0,\ell} - \alpha_\ell)} + \frac{d_{1,\ell}^2}{(\beta_{1,\ell} - \alpha_\ell)} = -1 \tag{53}$$

Equation (53) yields the Lagrangian multiplier α_ℓ

$$\begin{aligned} \alpha_\ell = & \frac{1}{2} \left(\beta_{0,\ell} + \beta_{1,\ell} + d_{0,\ell}^2 + d_{1,\ell}^2 \right) \\ & \pm \frac{1}{2} \sqrt{\left(\beta_{0,\ell} + \beta_{1,\ell} + d_{0,\ell}^2 + d_{1,\ell}^2 \right)^2 - 4 \left(d_{0,\ell}^2 \beta_{1,\ell} + d_{1,\ell}^2 \beta_{0,\ell} + \beta_{0,\ell} \beta_{1,\ell} \right)} \quad (54) \\ = & \frac{\left(\beta_{0,\ell} + \beta_{1,\ell} + d_{0,\ell}^2 + d_{1,\ell}^2 \right)}{2} \pm \frac{\sqrt{\left(\beta_{0,\ell} - \beta_{1,\ell} + d_{0,\ell}^2 - d_{1,\ell}^2 \right)^2 + 4 d_{0,\ell}^2 d_{1,\ell}^2}}{2} \quad (55) \\ = & \frac{1}{2} \varsigma_\ell \pm \frac{1}{2} \xi_\ell \quad (56) \end{aligned}$$

where (55) shows that the discriminant is always positive, so that α_ℓ is always real and positive.

The second Lagrangian multiplier λ_ℓ is found from the constraint (40) which can be rewritten using (52) as

$$\lambda_\ell^2 \left[\left(\frac{d_{0,\ell}}{(\beta_{0,\ell} - \alpha_\ell)} \right)^2 + \left(\frac{d_{1,\ell}}{(\beta_{1,\ell} - \alpha_\ell)} \right)^2 \right] = \frac{N_\ell^2}{(\sigma_\ell^2)^2} \tag{57}$$

This yields λ_ℓ as

$$\lambda_\ell = \pm \frac{N_\ell}{\sigma_\ell^2} \frac{(\beta_{0,\ell} - \alpha_\ell)(\beta_{1,\ell} - \alpha_\ell)}{\sqrt{d_{0,\ell}^2 (\beta_{1,\ell} - \alpha_\ell)^2 + d_{1,\ell}^2 (\beta_{0,\ell} - \alpha_\ell)^2}} \tag{58}$$

The denominator of (58) can be put in a more useful form substituting (53) into (57) for each of the ratios in d_0 and d_1. After some algebra, and substituting (58) for λ_ℓ, it can be shown that

$$d_{0,\ell}^2 (\beta_{1,\ell} - \alpha_\ell)^2 + d_{1,\ell}^2 (\beta_{0,\ell} - \alpha_\ell)^2 = \pm (\beta_{0,\ell} - \alpha_\ell)(\beta_{1,\ell} - \alpha_\ell)\, \xi_\ell \tag{59}$$

with ξ_ℓ from (56). The parameters $(\hat{a}_\ell, \hat{b}_\ell)$ in (52) can then be rewritten with (58) and (59) as

$$\begin{bmatrix} \hat{a}_\ell \\ \hat{b}_\ell \end{bmatrix} = \frac{\mathbf{\Psi}_\ell \begin{bmatrix} d_{0,\ell} (\beta_{1,\ell} - \alpha_\ell) \\ d_{1,\ell} (\beta_{0,\ell} - \alpha_\ell) \end{bmatrix}}{\sqrt{(\beta_{0,\ell} - \alpha_\ell)(\beta_{1,\ell} - \alpha_\ell)\, \xi_\ell}} \tag{60}$$

Returning to the cost (43) and observing that $(\hat{a}_\ell, \hat{b}_\ell)$ implicitly satisfy the constraints, its not hard to show that

$$C(\boldsymbol{Q}) = \sum_{\ell=1}^{L} N_\ell \frac{\begin{bmatrix} \hat{a}_\ell \\ \hat{b}_\ell \end{bmatrix}^T \left(\mathbf{A}_\ell + \begin{bmatrix} (\bar{x}_\ell - \hat{x}_0) \\ (\bar{y}_\ell - \hat{y}_0) \end{bmatrix} \begin{bmatrix} (\bar{x}_\ell - \hat{x}_0) \\ (\bar{y}_\ell - \hat{y}_0) \end{bmatrix}^T \right) \begin{bmatrix} \hat{a}_\ell \\ \hat{b}_\ell \end{bmatrix}}{2\sigma_m^2} \tag{61}$$

Substituting (60) into (61), noting (48) and (49) and making prudent use of equation (53), it can be shown that this cost reduces to

$$C(\boldsymbol{Q}) = \sum_{\ell=1}^{L} \frac{N_\ell}{2\sigma_\ell^2} \alpha_\ell \tag{62}$$

The α_ℓ which minimize (62) are found from (54) by subtracting the radical. To find the point of intersection $(\hat{x}_0, \hat{y}_0)$, (62) can be minimized using Newton Raphson, but an initial guess for $(\hat{x}_0, \hat{y}_0)$ is required. Noting from (55) that for any $\boldsymbol{d}_\ell$ given by (49), $\min(\beta_{0,\ell}, \beta_{1,\ell}) \leq \alpha_\ell \leq \max(\beta_{0,\ell}, \beta_{1,\ell})$, the choice of $\boldsymbol{d}_\ell$ such that $\alpha_\ell = \min(\beta_{0,\ell}, \beta_{1,\ell})$ occurs when the vector $[(\hat{x}_0 - \bar{x}_\ell)\ (\hat{y}_0 - \bar{y}_\ell)]$ from (49) projects entirely onto the eigenvector $\psi_{\max}$ corresponding to the maximum eigenvalue $\beta_{\max}$. To minimize $C(\boldsymbol{Q})$, this condition should be satisfied for as many lines as possible, so that if $[\tilde{a}_\ell \; \tilde{b}_\ell]$ are the components of the eigenvector $\psi_{\min}$ corresponding to the minimum eigenvalue $\beta_{\min}$, then $\tilde{a}_\ell(\hat{x}_0 - \bar{x}_\ell) + \tilde{b}_\ell(\hat{y}_0 - \bar{y}_\ell) = 0$. A point $(\hat{x}_0, \hat{y}_0)$ that seeks to minimize every α_ℓ is thus found from

$$\begin{bmatrix} \tilde{a}_1 & \tilde{b}_1 \\ \tilde{a}_2 & \tilde{b}_2 \\ \vdots & \vdots \\ \tilde{a}_N & \tilde{b}_N \end{bmatrix} \begin{bmatrix} \hat{x}_0 \\ \hat{y}_0 \end{bmatrix} = - \begin{bmatrix} \tilde{c}_1 \\ \tilde{c}_2 \\ \vdots \\ \tilde{c}_N \end{bmatrix} \tag{63}$$

where $\tilde{c}_\ell = -\tilde{a}_\ell \bar{x}_\ell - \tilde{b}_\ell \bar{y}_\ell$. Since $\tilde{a}_\ell$, $\tilde{b}_\ell$, and $\tilde{c}_\ell$ minimize α_ℓ and (62) for $L = 1$, the parameter estimators of a single line from [1] are in fact a special case of the current approach.

With the initial guess $(\hat{x}_0, \hat{y}_0)$, the Newton-Raphson method computes [2]

$$\begin{bmatrix} \hat{x}_0[n+1] \\ \hat{y}_0[n+1] \end{bmatrix} = \begin{bmatrix} \hat{x}_0[n] \\ \hat{y}_0[n] \end{bmatrix} - \left\{ \begin{bmatrix} \frac{\partial^2 C(\boldsymbol{Q})}{\partial \hat{x}_0^2} & \frac{\partial^2 C(\boldsymbol{Q})}{\partial \hat{x}_0 \partial \hat{y}_0} \\ \frac{\partial^2 C(\boldsymbol{Q})}{\partial \hat{y}_0 \partial \hat{x}_0} & \frac{\partial^2 C(\boldsymbol{Q})}{\partial \hat{y}_0^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial C(\boldsymbol{Q})}{\partial \hat{x}_0} \\ \frac{\partial C(\boldsymbol{Q})}{\partial \hat{y}_0} \end{bmatrix} \right\}_{\hat{x}_0 = \hat{x}_0[n], \hat{y}_0 = \hat{y}_0[n]} \tag{64}$$

An algorithm for the point of intersection estimator is given by

1. Estimate the initial point of intersection $(\hat{x}_0[0], \hat{y}_0[0])$ from (63)
2. Compute the modal matrices $\mathbf{A}_\ell$ from (45) and the resulting eigenvectors $\boldsymbol{\Psi}_\ell$ and eigenvalues $\beta_{0,\ell}$, $\beta_{1,\ell}$ for each family of line data $\ell = 1, \ldots, L$.
3. Determine $\boldsymbol{d}_\ell$ from (49) using $(\hat{x}_0[n], \hat{y}_0[n])$
4. Compute $C(\boldsymbol{Q})$ from (62) and its partial derivatives in terms of α_ℓ from (54) to construct (64).
5. Repeat from 3. until (62) is minimum

6 Simulation and Discussion

In keeping with the figures of Section 4, simulations are performed with the same parameters: $\sigma_\nu^2 = 1$, $\sigma_\chi^2 = 16$, $\mu_\chi = 0$, $\hat{x}_0 = 1000$, $\hat{y}_0 = 0$, and $10,000$ iterations. When multiple lines intersect at $(\hat{x}_0, \hat{y}_0)$, they do so by equally dividing an angle of $\pi/6$. Figure 2 illustrates the error variance of the proposed intersection estimator (64) for the estimate of coordinate $\hat{x}_0$ when only two lines intersect, one coincident with the x axis and the second with angle of intersection $\pi/6$. As seen from the figure, for N as low as 16, the simulated variance is within an order of magnitude of the CRLB from (36). Moreover, the estimator (64) is seen to asymptotically attain the CRLB, consistent with the expected behavior of maximum likelihood estimators [2].

The line parameter estimate $\hat{a}_1$ from the pencil is found by substituting the point of intersection estimate $(\hat{x}_0, \hat{y}_0)$ from (64) into (60) using (54) for α_ℓ. The

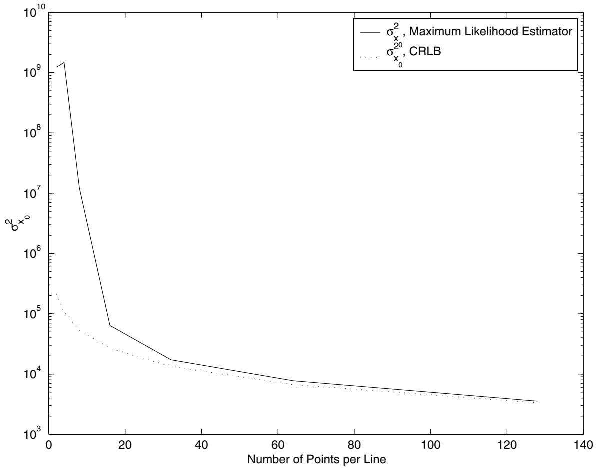

Fig. 2. Simulated $\sigma^2_{\hat{x}_0}$, as estimated from (64) and CRLB from (36) versus number of points N.

curves of figure 3 depict simulations of the estimator variance $\sigma^2_{\hat{a}_1}$, $\sigma^2_{\tilde{a}_1}$ and the CRLB from (36) as a function of the number of intersecting lines L. As can be seen from the figure, $\hat{a}_1$ provides a clear improvement over the alternative of estimating the line parameter $\tilde{a}_1$ from its line data alone, confirming the predictions of figure 1. The number of points per line in the simulation is $N = 64$.

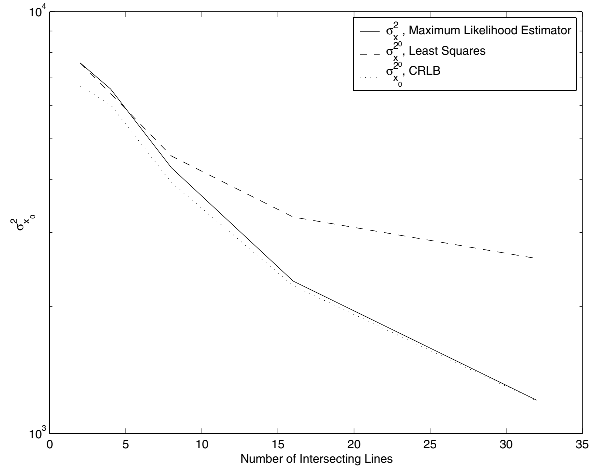

Fig. 3. Simulated $\sigma^2_{\hat{x}_0}$, as estimated from (64), Least Squares, and the CRLB from (36), versus the number of lines L.

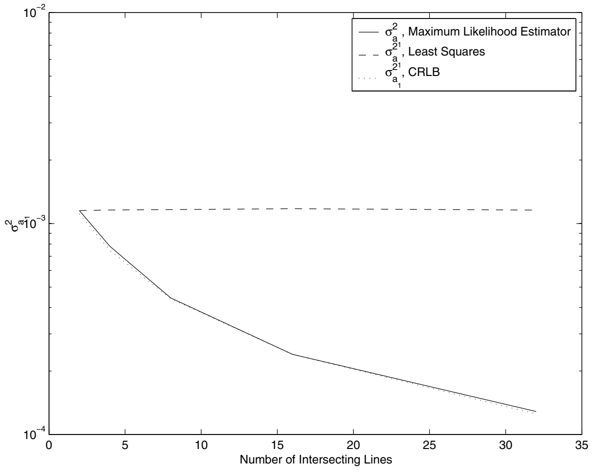

Fig. 4. Simulated $\sigma^2_{\hat{a}_1}$, by applying (60) to estimates $(\hat{x}_0, \hat{y}_0)$ from (64), the Single Line parameter estimator from [1], and the CRLB from (36), versus the number of lines L.

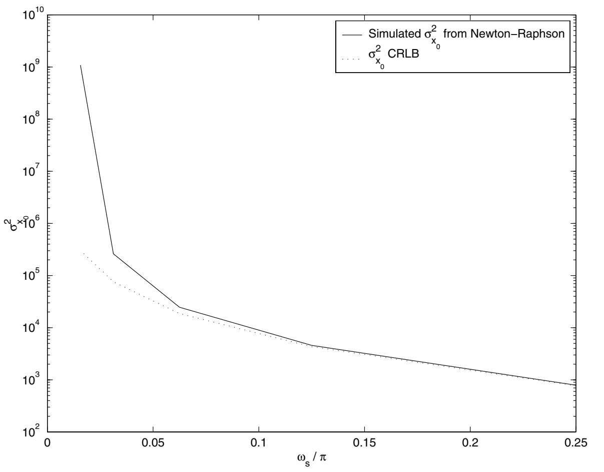

Fig. 5. Simulated $\sigma^2_{\hat{x}_0}$ from (64) and the CRLB from (36) versus ω_s from (65).

The performance of (64) using the least squares estimate of $(\hat{x}_0, \hat{y}_0)$ from (63) is simulated in figure 5 as a function of the subtending angle ω_s

$$\omega_s = \max \left\{ \omega_{\ell,k} = \cos^{-1}\left(a_\ell a_k + b_\ell b_k\right),\ \ell, k = 1, \ldots, L \right\} \tag{65}$$

The simulation parameters are $L = 16$, $N_\ell = 64$, $\sigma^2_{\nu_\ell} = 1$, $\sigma^2_{\chi\ell} = 16$ and ω_s is equally divided by the L intersecting lines. It is apparent from the figure that as ω_s becomes increasingly small, the performance of the method degrades. The large variance $\sigma^2_{\hat{x}_0}$ of the estimator results from a small number of very large outliers. These outliers result both from increasingly poor estimates afforded by the initial guess (63), and the existence of local minima to which (64) converges. It may be possible to improve performance by using several initial points $(\hat{x}_0, \hat{y}_0)$, each obtained from the intersection of different groups of line parameters $\tilde{a}_\ell$, $\tilde{b}_\ell$, and $\tilde{c}_\ell$, but further investigation is required to present a definitive solution. Note that the simulated variance $\sigma^2_{\hat{x}_0}$ is within an order of magnitude of the CRLB for $\omega_s/\pi = 1/32$, or about 1.4 degrees.

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