The Relevance of Non-generic Events in Scale Space Models

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Abstract. In order to investigate the deep structure of Gaussian scale space images, one needs to understand the behaviour of spatial critical points under the influence of blurring. We show how the mathematical framework of catastrophe theory can be used to describe the behaviour of critical point trajectories when various different types of generic events, *viz.* annihilations and creations of pairs of spatial critical points, (almost) coincide. Although such events are non-generic in mathematical sense, they are not unlikely to be encountered in practice. Furthermore the behaviour leads to the observation that fine-to-coarse tracking of critical points doesn't suffice. We apply the theory to an artificial image and a simulated MR image and show the occurrence of the described behaviour.

1 Introduction

The concept of scale space has been introduced in the English image literature by Witkin [16] and Koenderink [8]. They showed that the natural way to represent an image at finite resolution is by convolving it with a Gaussian , thus obtaining a smoothened image at a scale determined by the bandwidth. Consequently, each scale level only requires the choice of an appropriate scale and the image intensity at that level follows linearly from any previous level. It is therefore possible to trace the evolution of certain image entities, e.g. critical points, over scale. The exploitation of various scales simultaneously has been referred to as *deep structure* by Koenderink [8]. It pertains to the dynamic change of the image from highly detailed –including noise – to highly smoothened. Furthermore, it may be expected that large structures "live" longer than small structures (a reason that Gaussian blur is used to suppress noise). Since multi-scale information can be ordered, one obtains a hierarchy representing the subsequent simplification of the image with increasing scale [10].

In one dimensional images critical points can only vanish. Investigation of these locations has been done by several authors [7,15]. Higher dimensional images are more complicated since extrema can be created in scale space This phenomenon has been studied in detail by Damon, [2], proving that creations are generic in images of dimension larger than one. That means that they are not some kind of artifact, introduced by noise or numerical errors, but that they are to be expected in any typical case. This was somewhat counterintuitive, since blurring seemed to imply that structure could only disappear, thus suggesting that only annihilations could occur. Damon, however, showed that both

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annihilations and creations are generic catastrophes and gave a complete list of local perturbations of these generic events. Whereas Damons results were stated theoretically, application of these results were reported in *e.g.* [6,9,12]. The main outcome of the investigation of the generic results is that in order to be able to use the topological approach one necessarily needs to take into account both the annihilation and creation events.

In images the location of critical points can be found up to the numerical precision of the image. The same holds for the location of catastrophe points in scale space. So although the appearance of catastrophe events can be uniquely separated in annihilations or creations of pairs of critical points, due to *e.g.* numerical limitations, (almost) symmetries in the image, or coarse sampling also indistinguishable compounds of these annihilation and creation events can be found in practise. In this way a couple of nearby generic events may well look like a single, non-generic one. In this paper we describe these so-called non-generic catastrophes in scale space. The investigation is based on the description of the evolution of critical points in scale space, called (scale space) critical curves, in the neighbourhood of the catastrophe point(s). The compounds of generic events can be modelled using descriptions of "Catastrophe Theory". Obviously, the models obey the property that assuming infinite precision, in non-generic compounds the generic events can be distinguished.

Furthermore we investigate the appearance of creations as described by these models in more detail and explain why they are, albeit generic, rarely found, probably the reason for current applications to simply ignore them.

2 Theory

Gaussian Scale Space. Let $L(\mathbf{x})$ denote an arbitrary n-dimensional image, the initial image. Let $L(\mathbf{x};t)$ denotes the $(n+1)$-dimensional *Gaussian scale space image* of $L(\mathbf{x})$, obtained by convolution of an initial image with a normalised Gaussian kernel of zero mean and standard deviation $\sqrt{2t}$. Differentiation is now well-defined, since an arbitrary derivative of the image is obtained by the convolution of the initial image with the corresponding derivative of a Gaussian. Consequently, $L(\mathbf{x};t)$ satisfies the diffusion equation:

$$\partial_t L(\mathbf{x};t) = \Delta L(\mathbf{x};t)$$

Here $\Delta L(\mathbf{x};t)$ denotes the Laplacean. The type of a spatial critical point ($\nabla L(\mathbf{x};t) = 0$) is given by the eigenvalues of the Hessian H, the matrix with the second order spatial derivatives, evaluated at its location. The trace of the Hessian equals the Laplacean. For maxima (minima) all eigenvalues of the Hessian are negative (positive). At a spatial saddle point H has both negative and positive eigenvalues. Since $L(\mathbf{x};t)$ is a smooth function in $(\mathbf{x};t)$-space, spatial critical points are part of a one dimensional manifold in scale space, called the *critical curve*, by virtue of the implicit function theorem. Consequently, the intersection of all critical curves in scale space with a plane of certain fixed scale t_0 yields the spatial critical points of the image at that scale.

Catastrophe Theory. The spatial critical points of a function with non-zero eigenvalues of the Hessian are called *Morse critical points*. The *Morse Lemma* states that at these points the qualitative properties of the function are determined by the quadratic part of the Taylor expansion of this function. This part can be reduced to the *Morse canonical form* by a slick choice of coordinates. If at a spatial critical point the Hessian degenerates, so that at least one of the eigenvalues is zero (and consequently the determinant is zero), the type of the spatial critical point cannot be determined. Such a point is called a *catastrophe points*.

The term catastrophe was introduced by Thom [14]. It denotes a (sudden) qualitative change in an object as the parameters on which this object depends change smoothly. A thorough mathematical treatment on this topic can be found in the work of Arnol'd, see e.g. [1]. More pragmatic introductions and applications are widely published, e.g. [5]. The catastrophe points are also called *non-Morse critical points*, since a higher order Taylor expansion is essentially needed to describe the qualitative properties. Although the dimension of the variables is arbitrary, the *Thom Splitting Lemma* states that one can split up the function in a Morse and a non-Morse part. The latter consists of variables representing the k "bad" eigenvalues of the Hessian that become zero. The Morse part contains the $n-k$ remaining variables. Consequently, the Hessian contains a $(n-k) \times (n-k)$ sub-matrix representing a Morse function. It therefore suffices to study the part of k variables. The canonical form of the function at the non-Morse critical point thus contains two parts: a Morse canonical form of $n-k$ variables, in terms of the quadratic part of the Taylor series, and a non-Morse part. The latter can by put into canonical form called the *catastrophe germ*, which is obviously a polynomial of degree 3 or higher.

Since the Morse part does not change qualitatively under small perturbations, it is not necessary to further investigate this part. The non-Morse part, however, does change. Generally the non-Morse critical point will split into a non-Morse critical point, described by a polynomial of lower degree, and Morse critical points, or even exclusively into Morse critical points. This event is called a *morsification*. So the non-Morse part contains the catastrophe germ and a perturbation that controls the morsifications. Then the general form of a Taylor expansion $f(\mathbf{x})$ at a non-Morse critical point of an n dimensional function can be written as (*Thom's Theorem*) $f(\mathbf{x}; \lambda) = CG + PT + Q$, where $CG(x_1, \ldots, x_k)$ denotes the catastrophe germ, $PT(x_1, \ldots, x_k; \lambda_1, \ldots, \lambda_l)$ the perturbation germ with an l-dimensional space of parameters, and $Q = \sum_{i=k+1}^{n} \epsilon_i x_i^2$ with $\epsilon_i = \pm 1$, the Morse part. Of the so-called simple real singularities we will discuss the catastrophe germs given by the two infinite series $A_k^{\pm} \stackrel{\text{def}}{=} \pm x^{k+1}, k \geq 1$, and $D_k^{\pm} \stackrel{\text{def}}{=} x^2 y \pm y^{k-1}, k \geq 4$. For notational convenience we will rewrite the latter to $x^{k-1} \pm xy^2$.

Catastrophes and Scale Space. The number of equations defining the catastrophe point equals $n+1$ and therefore it is over-determined with respect to the n spatial variables. Consequently, catastrophe points are generically not found in typical images. In scale space, however, the number of variables equals $n+1$ and catastrophes occur as isolated points.

Although the list of catastrophes starts very simple, it is not trivial to apply it directly to scale space by assuming that scale is just one of the perturbation parameters. As Damon [2] points out: *"There are significant problems in trying to directly apply Morse theory to solutions of to the heat equation. First, it is not clear that generic solutions to the* *heat equation must be generic in the Morse sense. Second, standard models for Morse critical points and their annihilation and creation do not satisfy the heat equation. How must these models be modified? Third, there is the question of what constitutes generic behaviour. This depends on what notion of local equivalence one uses between solutions to the heat equation."* For example, in one-dimensional images the A_2 catastrophe reduces to $x^3 + \lambda x$. It describes the change from a situation with two critical points (a maximum and a minimum) for $\lambda < 0$ to a situation without critical points for $\lambda > 0$. This event can occur in two ways. The extrema are annihilated for increasing λ, but the opposite – creation of two extrema for decreasing λ – is also possible.

In scale space, however, there is an extra constraint: the germ has to satisfy the diffusion equation. Thus the catastrophe germ x^3 implies an extra term $6xt$. On the other hand, the perturbation term is given by λx, so by taking $\lambda = 6t$ scale plays the *role* of the perturbing parameter. This gives a directionality to the perturbation parameter, in the sense that the only remaining possibility for this A_2-catastrophe in one-dimensional images is an annihilation. So the Fold catastrophe is adjusted such that it satisfies the heat equation, but this adjustment only allows annihilations. However, it does not imply that only annihilations are generic in scale space. In higher dimensional images also the opposite – *i.e.* a A_2 catastrophe describing the creation of a pair of critical points – is possible. Then the perturbation $\lambda = -6t$ with increasing t requires an additional term of the form $-6xy^2$ in order to satisfy the diffusion equation as we will see.

The transfer of the catastrophe germs to scale space, taking into account the special role of scale, has been made by many authors, [2,3,7,9,12], among whom Damon's account – answering his questions – is probably the most rigorous. He showed that the only generic morsifications in scale space are the aforementioned A_2 catastrophes describing *annihilations* and *creations* of pairs of critical points. These two points have opposite sign of the determinant of the Hessian before annihilation and after creation.

Definition 1. *The generic scale space catastrophe germs are given [2] by*

$$f^{\mathrm{A}}(\mathbf{x};t) \stackrel{\mathrm{def}}{=} x_1^3 + 6x_1 t + Q(\mathbf{x};t)\,,$$
$$f^{\mathrm{C}}(\mathbf{x};t) \stackrel{\mathrm{def}}{=} x_1^3 - 6x_1 t - 6x_1 x_2^2 + Q(\mathbf{x};t)\,.$$

where $Q(\mathbf{x};t) \stackrel{\mathrm{def}}{=} \sum_{i=2}^{n} \epsilon_i (x_i^2 + 2t)$, $\sum_{i=2}^{n} \epsilon_i \neq 0$, *and* $\epsilon_i \neq 0\ \forall i$.

Note that the scale space catastrophe germs f^{A} and f^{C}, and the quadratic term Q satisfy the diffusion equation. The germs f^{A} and f^{C} correspond to the two qualitatively different A_2 catastrophes at the origin, an annihilation and a creation respectively. From Definition 1 it is obvious that annihilations occur in any dimension, but creations require at least 2 dimensions. Consequently, in 1D signals only annihilations occur. Furthermore, for images of arbitrary dimension and less than three vanishing eigenvalues of the Hessian at a degenerated point, it suffices to investigate the 2D case due to the Splitting Lemma. All other catastrophe events in scale space are compounds of Fold catastrophes. It is however possible that one may not be able to distinguish these generic events, *e.g.* due to numerical limitations, coarse sampling, or (almost) symmetries in the image. For instance, one may find at some scale three nearby critical points, e.g. two extrema and a saddle, and at the subsequent scale only one extremum. Obviously, one pair of critical points is annihilated, but one may not be able to identify the annihilating extremum at the former scale. This is illustrated in Figure 1.

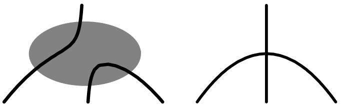

Fig. 1. Left: Annihilation of two critical points in the neighbourhood of a third critical point. The grey area represents the uncertainty in determining the catastrophe. Right: Non-generic representation and model of this event.

3 Scale Space Catastrophe Models

In this section we describe how catastrophes can be used to model events in $(2+1)$-dimensional scale space. The catastrophes describe in canonical coordinates how critical curves pass the origin yielding compounds of annihilations and / or creations of pairs of critical points. We will see that although most of these catastrophes are *non-generic*, they may still be relevant for modelling *compounds of generic events* that one is not capable of, or willing to, segregate as such. Recall, for example, Figure 1. The catastrophe germs are adjusted such that they satisfy the heat equation. Furthermore, by choosing the perturbation terms non-zero and adjusting them in the same way, descriptions of critical curves in scale space are obtained. These critical curves only contain the generic Fold annihilation(s) and/or creation(s).

***A_2* Fold catastrophe.** The Fold catastrophe in scale space is given by

$$L(x,y;t) = x^3 + 6xt + \delta(y^2 + 2t) \ ,$$

where $\delta = \pm 1$. One can verify that at the origin a saddle and an extremum (a minimum if $\delta = 1$, a maximum if $\delta = -1$) moving in the $y = 0$ plane meet and annihilate while increasing the scale parameter t.

***A_3* Cusp catastrophe.** The Cusp catastrophe germ is given by x^4. Its scale space addition is $12x^2t + 12t^2$. The perturbation term contains two terms: $\lambda_1 x + \lambda_2 x^2$. Obviously, scale takes the role of λ_2. Taking the dual Cusp gives the same geometry by changing the sign of λ_1, or by setting $x = -x$. The scale space Cusp catastrophe germ with perturbation is thus defined by

$$L(x,y;t) = x^4 + 12x^2t + 12t^2 + \lambda_1 x + \delta(y^2 + 2t) \ ,$$

with $\delta = \pm 1$. Morsification by the perturbation $\lambda_1 \neq 0$ yields one Fold catastrophe and one regular critical curve. One can verify that the $A_k, k > 3$ catastrophes describes the (non-generic) simultaneous annihilations of critical points in one dimension under the influence of blurring, albeit in more complicated appearances.

$D_4^\pm$ Umbilic catastrophes. The $D_4^\pm$ Umbilic catastrophe germs are given by $x^3+\delta xy^2$, where $\delta = \pm 1$. The scale space addition is $(6+2\delta)xt$, yielding x^3+xy^2+8xt for the Hyperbolic Umbilic catastrophe, and x^3-xy^2+4xt for the Elliptic Umbilic catastrophe. The perturbation contains three terms: $\lambda_1 x + \lambda_2 y + \lambda_3 y^2$. Obviously, scale takes the role of λ_1.

$D_4^\pm$ Hyperbolic Umbilic catastrophe. The scale space D_4^+ Hyperbolic Umbilic catastrophe germ with perturbation is thus defined by

$$L(x,y;t) = x^3 + xy^2 + 8xt + \lambda_3(y^2+2t) + \lambda_2 y \,.$$

The critical curves and catastrophe points follow from

$$\begin{cases} L_x = 3x^2 + 8t + y^2 \\ L_y = 2xy + 2\lambda_3 y + \lambda_2 \\ \det(H) = 12x(x+\lambda_3) - 4y^2. \end{cases}$$

In the unperturbed situation four critical points exist for each $t<0$ on the x- and y-axes. At $t=0$ the four critical curves annihilate simultaneously at the origin, see Figure 2a. Taking perturbation into account, the curves are separated into two critical curves each

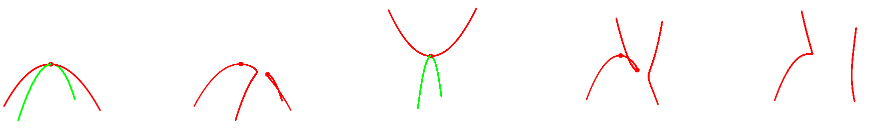

Fig. 2. Critical paths a) D_4^+-Unperturbed. b) D_4^+-Perturbed. c) D_4^--Unperturbed. d) D_4^--Small perturbation. e) D_4^--Large perturbation. Again, if the perturbation is small we may not be able to tell which configuration is the actual one.

containing a Fold catastrophe, see Figure 2b.

D_4^- Elliptic Umbilic catastrophes. The scale space elliptic Umbilic catastrophe germ with perturbation is given by

$$L(x,y;t) = x^3 - xy^2 + 4xt + \lambda_3(y^2+2t) + \lambda_2 y \,. \tag{1}$$

Again, the critical curves and the catastrophe points follow from

$$\begin{cases} L_x = 6x^2 + 4t - y^2 \\ L_y = -2xy + 2\lambda_3 y + \lambda_2 \\ \det(H) = 12x(2\lambda_3 - 2x) - 4y^2. \end{cases}$$

The unperturbed equation gives two critical points for all $t \neq 0$. At the origin a so-called scatter event occurs: the critical curve changes from y-axis to x-axis with increasing t, see Figure 2c. Just as in the hyperbolic case, in fact two Fold catastrophes take place; in this case both an annihilation and a creation. The morsification is shown in Figure 2d. The critical curve on the right does not contain catastrophe points. The critical curve on the left, however, contains two Fold catastrophe points: a creation and an annihilation. So while increasing scale one will find two critical points, suddenly two extra critical points appear, of which one annihilates with one of the already existing ones. Finally, one end up with again two critical points. Clearly, if the samples in scale are taken too large, one could completely miss the subsequent catastrophes, see *e.g.* Figure 2e. The properties of the creations will be discussed in the next section.

Creations. As we showed, a creation event occurs in case of a morsified elliptic Umbilic catastrophe. In most applications, however, creations are rarely found, often giving rise to the (false) opinion that creations are caused by numerical errors and should be disregarded. The reason for their rare appearance lies in the specific requirements for the parameters in the (morsified) Umbilic catastrophe germ. Its general formulation is given by

$$L(x,y;t) = \frac{1}{6}L_{xxx}x^3 + \frac{1}{2}L_{xyy}xy^2 + L_{xt}xt + \frac{1}{2}L_{yy}(y^2+2t) + L_y y \tag{2}$$

In general, the spatial coefficients do *not* equal the derivatives evaluated in the coordinate system of the image. They follow from the alignment of the catastrophe in the plane defined by $y = 0$ and can have arbitrary value. Furthermore, the diffusion equation implies $L_{xt} \stackrel{\text{def}}{=} L_{xxx} + L_{xyy}$. Then the scale space evolution of the critical curves follow from

$$\begin{cases} \partial_x L = \frac{1}{2}L_{xxx}x^2 + L_{xt}t + \frac{1}{2}L_{xyy}y^2 \\ \partial_y L = L_{xyy}xy + L_{yy}y + L_y \\ \det(H) = L_{xxx}x(L_{xyy}x + L_{yy}) - L_{xyy}^2 y^2. \end{cases}$$

Firstly we consider the case $L_y = 0$. Then Eq. (2) describes a Fold catastrophe (either annihilation or creation) at the origin, where the critical curve is positioned in the (x,t)-plane. A creation necessarily requires the constraint $L_{xxx}L_{xt} < 0$ at the catastrophe point. This constraint is sufficient.

Theorem 1. *At a catastrophe point in two spatial dimensions, if the third order derivatives of the general local form as given by Eq. (2) with $L_y = 0$, are uncorrelated, the number of creations has an a priori likelihood of $1/4$ relative to the total number of catastrophes. In n dimensions it is $\frac{1}{\pi}\arccos\frac{1}{\sqrt{n}}$.*

Proof. The requirement $L_{xxx}L_{xt} < 0$ can be rewritten to $L_{xxx}(L_{xxx} + L_{xyy}) < 0$. In the (L_{xxx}, L_{xyy})-space this constraint is satisfied by all point sets in the area spanned by the lines through the origin with direction vectors $(1,0)$ and $(1,-1)$, which is a quarter of the plane. For $n-D$ this extends to the area $L_{xxx}(L_{xxx}+L_{xy_1y_1}+\ldots+L_{xy_{n-1}y_{n-1}}) < 0$ in $(L_{xxx}, L_{xy_iy_i})$-space, with $\dim(\mathbf{y}) = n-1$. This representing two intersecting planes with normal vectors $(1,0,\ldots,0)$ and $(1,-1,\ldots,-1)$. They make an angle of ϕ radians, given by

$$\cos\phi = \frac{(1,0,\ldots,0)\cdot(1,-1,\ldots,-1)}{\mid(1,0,\ldots,0)\mid\cdot\mid(1,-1,\ldots,-1)\mid} = \frac{1}{\sqrt{n}}$$

Then the fraction of the space follows by taking twice this angle and dividing by the complete angle of 2π, *i.e.* $\frac{1}{\pi}\arccos\frac{1}{\sqrt{n}}$.

Note that if $n = 1$, the fraction of the space where creations can occur is zero, for $n = 2$ it is a quarter. The also interesting case $n = 3$ yields a fraction that is slightly more than a quarter, whereas for $n \to \infty$ the fraction converges to a half, see Figure 3a. That is: the higher the dimensions, the easier critical points can be created. The reason that in

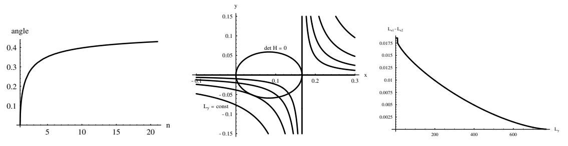

Fig. 3. a) The fraction of the space of the third order derivatives in which creations can occur as a function of the dimension according Theorem 1. b) Intersections of the curves $\det(H) = 0$ and $\partial_y L = 0$ with different values for L_y. For the value given by Theorem 2 the curves touch. c) Difference in intensity between the creation and the annihilation event for L_y increasing from 0 to its critical value.

practice in two dimensional images the number of creations observed is (much) smaller than a quarter, is caused by the role of the perturbation parameters. It is possible to give a tight bound to the perturbation of Equation (2) in terms of L_y:

Theorem 2. *A creation and subsequent annihilation event occur in Equation (2) if and only if*

$$| L_y | \leq \frac{3}{16} L_{yy}^2 \sqrt{\frac{-3L_{xxx}}{L_{xyy}^3}} \tag{3}$$

Proof. The catastrophes satisfy $\partial_x L = \partial_y L = \det H = 0$. Since the solution of the system

$$\begin{aligned} \partial_y L &= L_y + y(L_{yy} + L_{xyy}x) &&= 0 \\ \det H &= L_{xxx}x(L_{yy} + L_{xyy}x) - L_{xyy}^2 y^2 &&= 0 \end{aligned} \tag{4}$$

only contains spatial coordinates, their intersections define the spatial coordinates of the catastrophes. The catastrophe points form the local extrema of the critical curve in $(x, y; t)$-space, *i.e.* at these points the tangent vector has no scale component. If the curves given by Eq. (4) touch, there is only a point of inflection in the critical curve, *i.e.* the critical curve in $(x, y; t)$-space has a (Fold) catastrophe point. At this point of inflection, the spatial tangent vectors of the curves defined by Eq. (4) are equal. Solving the system Eq. (4) with respect to y results in

$$y = -\frac{L_y}{L_{yy} + L_{xyy}x} = \pm\frac{1}{L_{xyy}}\sqrt{L_{xxx}x(L_{yy} + L_{xyy}x)}\,.$$

The equality of the tangent vectors at the point of inflection x_i, y_i yields

$$\frac{\partial}{\partial x}\left(-\frac{L_y}{L_{yy} + L_{xyy}x}\right)|_{x_i,y_i} = \frac{\partial}{\partial x}\left(\pm\frac{1}{L_{xyy}}\sqrt{L_{xxx}x(L_{yy} + L_{xyy}x)}\right)|_{x_i,y_i}$$

Solving both equalities results in

$$(x_i, y_i, L_y) = \left(-\frac{L_{yy}}{4L_{xyy}}, \pm\sqrt{\frac{-3L_{xxx}L_{yy}^2}{16L_{xyy}^3}}, \mp\frac{3L_{yy}^2}{16L_{xyy}}\sqrt{\frac{-3L_{xxx}}{L_{xyy}}}\right),$$

which gives the boundary values for L_y.

Note that Eq. (3) has only real solutions if $L_{xxx}L_{xyy} < 0$, *i.e.* at the D_4^- (morsified) catastrophe. As a consequence of Theorem 2, creations only occur if the perturbation is small enough. Again, this perturbation occurs in the coordinate system, obtained by the alignment of the catastrophe in the plane defined by $y = 0$.

Example 1. Taking $L_{xxx} = 6, L_{xyy} = -12, L_{yy} = 2$ yielding $L = x^3 - 6xy^2 - 6xt + y^2 + 2t + L_y$, we obtain the "generic creation example" as given in section 2 with perturbation. Then Theorem 2 gives $| L_y | \leq \frac{1}{32}\sqrt{6}$ as a – relatively small compared to the other derivative values– bound for the occurrence of a creation – annihilation couple. In Figure 3b the ellipse $\det(H) = 0$ is plotted, together with the curves $\partial_y L = 0$ for $L_y = 0$ (resulting in two straight lines at $y = 0$ and $x = \frac{1}{6}$, intersecting at $(x, y) = (\frac{1}{6}, 0)$), and $L_y = 2^{-i}\sqrt{6}, i = 4, \ldots, 7$. For $i > 5$, the perturbation is small enough and the intersection of $\partial_y L = 0$ and $\det H = 0$ contains two points. Thus a creation-annihilation is observed. If $i = 5$, L_y has its critical value and the curves touch. For larger values the curves do not intersect each other.

Obviously the perturbation L_y can be larger if L_{yy} increases. If so, the structure becomes more elongated. It is known by various examples of creations given in literature that elongated structures play an important role. In fact, the quintessential property is scale anisotropy. Another reason that creations are rarely found is that their lifetime is rather limited: with increasing t the created critical points annihilate. If the scale steps are taken too large, one simply misses the creation – annihilation couple. This may be regarded as a dual expression for the previous explanation. In the chosen coordinate system this can be calculated explicitly.

Theorem 3. *The maximum lifetime of a creation given by Equation (2) is*

$$t_{lifetime} = \frac{-L_{xxx}L_{yy}^2}{2L_{xyy}^2(L_{xxx} + L_{xyy})}.$$

The difference in intensity of the critical point that is created and subsequently annihilated is

$$\frac{L_{xxx}(2L_{xxx} - L_{xyy})L_{yy}^3}{6L_{xyy}^3(L_{xxx} + L_{xyy})}.$$

Proof. Observe that the lifetime is bounded by the two intersections of $\partial_y L = 0$ and $\det(H) = 0$, see Figure 3b. As $| L_y |$ increases from zero, the two points move towards each other over the arch $\det(H) = 0$ until they reach the value given by theorem 2 with lifetime equal to zero. The largest arch length is obtained for $L_y = 0$. Then the spatial coordinates are found by $\partial_y L(x, y; t) = y(L_{xyy}x + L_{yy}) = 0$ and $\det H = L_{xxx}x(L_{xyy}x + L_{yy}) - L_{xyy}^2 y^2 = 0$ *i.e.* $(x, y) = (0, 0)$ and $(x, y) = (-\frac{L_{yy}}{L_{xyy}}, 0)$ The location in scale space is given by $\partial_x L(x,y;t) = \frac{1}{2}L_{xxx}x^2 - \frac{1}{2}L_{xyy}y^2 + L_{xt}t = 0$. Consequently, the first catastrophe takes place at the origin - since also $t = 0$ - with zero intensity. The second one is located at

$$(x,y;t) = (-\frac{L_{yy}}{L_{xyy}}, 0; \frac{-L_{xxx}L_{yy}^2}{2L_{xyy}^2(L_{xxx}+L_{xyy})})$$

with intensity

$$L_{cat} = \frac{L_{xxx}(2L_{xxx} - L_{xyy})L_{yy}^3}{6L_{xyy}^3(L_{xxx}+L_{xyy})}.$$

Then the latter is also the maximum difference in intensity.

Example 2. To show the effect of the movement along the arch $\det(H) = 0$, see Figure 3c. Without loss of generality we took again $L_{xxx} = 6, L_{xyy} = -12, L_{yy} = 2$. Firstly, the two solutions to $\nabla L = 0 \wedge \det(H) = 0$ were calculated as function of L_y. Secondly, the difference of the intensity of the solutions was calculated for 766 subsequent values of $L_y, L_y \in [0, \ldots, \frac{1}{32}\sqrt{6}]$. It is clearly visible that the intensity decreases monotonously with an increase of L_y. For this example we find that the lifetime is $\frac{1}{72}$, the difference in intensity $\frac{1}{18}$.

From the proof of Theorem 3 it is again apparent that L_{yy} plays an important role in enabling a (long)lasting creation. To observe this in more detail, note that the curve $\det H = 0$ is an ellipse (see also Figure 3b). Replacing x by $x - \frac{L_{yy}}{2L_{xyy}}$, it is centred at the origin. Setting $L_{xyy} = \frac{1}{b}$ and $L_{xxx}L_{xyy} = -\frac{1}{a^2}$, we find

$$\det H = 0 \Leftrightarrow x^2 + \frac{a^2}{b^2}y^2 = L_{yy}^2\frac{b^2}{4} .$$

Assuming that we have a creation, $a^2 > 0$. The ellipse is enlarged with an increase of L_{yy}^2. Obviously, at the annihilations of the Hyperbolic Umbilic catastrophe $a^2 < 0$, so $\det H = 0$ then describes a hyperbola.

$D_5^{\pm}$ Parabolic Umbilic catastrophes. In the previous section we saw that the geometry significantly changed by taking either the term $-xy^2$, or the term $+xy^2$. Let us therefore, ignoring the perturbation terms λ_1, λ_2, and λ_3, define the scale space Parabolic Umbilic catastrophe germ by

$$L(x,y;t) = \frac{1}{4!}x^4 + \frac{1}{2!}x^2t + \frac{1}{2!}t^2 + \delta(\frac{1}{2}xy^2 + xt) \tag{5}$$

where $\delta = \pm 1$ and t takes the role of λ_4. Its critical curves and catastrophes follow from

$$\begin{cases} L_x = \frac{1}{6}x^3 + xt + \delta(t + \frac{1}{2}y^2) \\ L_y = \delta xy \\ \det(H) = \delta x(\frac{1}{2}x^2 + t) - y^2 \end{cases}$$

So the catastrophe points are located at the origin (a double point) and at $(x, y; t) = (-\frac{3}{2}\delta, 0; -\frac{9}{8}\delta^2)$. The latter is a simple annihilation (a fold catastrophe), the former is a cusp catastrophe (three critical point change into one) for both values of δ. Adding small perturbations by choosing the parameters λ_1, λ_2, and λ_3, the morsified Cusp catastrophe remains. The critical curves at the Cusp breaks up into two curves, one with a Fold catastrophe, one without a catastrophe.

$D_6^\pm$ Second Umbilic Catastrophes. Ignoring the perturbation terms $\lambda_1, \ldots, \lambda_4$ for the moment, the scale space expression of the $D_6^\pm$-catastrophes are given by

$$L(x, y; t) = \frac{1}{5!}x^5 + \frac{1}{3!}x^3 t + \frac{1}{2!}x t^2 + \delta(\frac{1}{2}x y^2 + x t) \,, \tag{6}$$

where t takes the role of λ_5 and $\delta = \pm 1$. Its critical curves and catastrophes follow from

$$\begin{cases} L_x = \frac{1}{4!}x^4 + \frac{1}{2}x^2 t + \frac{1}{2}t^2 + \delta(t + \frac{1}{2}y^2) \\ L_y = \delta x y \\ \det(H) = \frac{1}{6}\delta x^2(x^2 + 6t) - y^2 \end{cases}$$

Setting $y = 0$, several catastrophes occur: At $(x, y; t) = (\pm\sqrt{-6\delta}, 0; \delta)$ two Fold annihilations if $\delta = -1$, at the origin a creation and at $(x, y; t) = (0, 0; -2\delta)$ again an annihilation, see Figure 4a for $\delta = 1$ and Figure 4b for $\delta = -1$. It is clear that

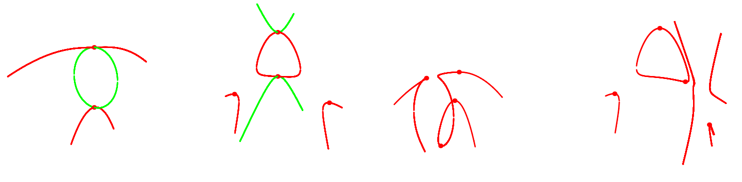

Fig. 4. Critical paths of the $D_6^\pm$-catastrophe. a) Unperturbed, $\delta = 1$. b) Unperturbed, $\delta = -1$. c) Perturbed, $\delta = 1$. d) Perturbed, $\delta = -1$.

the morsification by t of the D_6^+ yields a D_4^- scatter followed (while increasing scale) by a D_4^+ double annihilation at the origin. The D_6^- shows a D_4^- scatter at the origin, followed by again a D_4^- scatter at some higher scale. Both images show that a part of the critical curve forms a loop: The created critical points annihilate with each other. So if the perturbations are small (or if the measurement contains some uncertainty), one might not be able to distinguish between the involved Fold catastrophes. However, the scale space representation causes a separation into two non-generic catastrophes already mentioned. Further morsification gives more insight in the way *critical curves* can behave. By taking $\lambda_1, \ldots, \lambda_4 \neq 0$, the generic critical curves shown in Figure 4c-d are obtained. The morsification of the D_6^+ shows two critical curves behaving in an aesthetic way, combining the morsifications of the $D_4^\pm$ catastrophes, *i.e.* containing Fold annihilations and creations. Both created critical points on the right critical curve in Figure 4c annihilate at some larger scale. The morsification of the D_6^-, on the other hand, still shows the loop close to the origin. Consequently, in contrast to the elliptic Umbilic catastrophe, now *both created branches annihilate with each other*: the critical curve in the centre of Figure 4d is a closed loop in scale space.

Morsification summary. All non-Fold catastrophes morsify to Fold catastrophes and Morse critical points. The morsification gives insight in the structure around the catastrophe point regarding the critical curves. The morsification of the Umbilic catastrophes (the D_k) show that the trajectories in scale space of the created critical points fall into several classes. The morsified D_4^+-catastrophes describes two Fold annihilations. The morsified D_4^- catastrophe describes the creation of a pair of critical points and the annihilation of one of them with another critical point. So while tracing a critical branch of a critical curve both an annihilation and a creation event are traversed. The morsified D_6^+ catastrophe describes the creation of a pair of critical points and the annihilation of both of them with two other critical points. So while tracing a critical branch of a critical curve successively an annihilation, a creation and again an annihilation event are traversed. The morsified D_6^--catastrophe describes an *isolated* closed critical curve, appearing *ex nihilo* with two critical branches that disappear at some larger scale. So the morsified D_4^- (and its extension, the D_6^+) and D_6^--catastrophes describe essentially different creation events.

An important result lays on the area of tracing critical points. If one traces *only* critical points starting from the *initial* image, one will find the "D_4^-" creations, since they emerge as the starting point of a part of a critical curve that annihilates with one of the initial critical points. However, one will miss the "D_6^-" loops that occur somewhere in scale space, since they have no relation whatsoever to the critical points in the initial image. So fine-to-coarse tracing of critical points will not always yield the right result. Note that the full morsification of the non-generic catastrophes always yields the generic Fold annihilations and creations and Morse critical points.

4 Applications

In this section we give some examples to illustrate the theory presented in the previous sections. We will focus on the critical curves emerging from a creation event. Firstly, we will show the actual presence of a sequence of creation and annihilation events, modelled by the D_4^--catastrophe, on the (artificial) MR image of Figure 5a. This image is taken from the web site http://www.bic.mni.mcgill.ca/brainweb. Secondly, an example of creation *ex nihilo*, the D_6^--catastrophe, is shown by means of the classic "bridge"-image of Figure 6a and the MR image. The *practical* usefulness of critical curves (as they provide the ability for an uncommitted hierarchical structure and segmentation), as well as their non-generic modelling, is described by the authors in several papers, *e.g.* [10]. Results on an MR, a CT and a noise image with respect to the area where creations are possible have been presented elsewhere [9].

The artificial MR image of Figure 5a was used as initial image for the scale space image. For visualisation purposes, we restricted to the scale range $8.37 - 33.1$. The image at scale 8.37 (with only the large structures remaining) is shown in Figure 5b. This image contains 7 extrema.

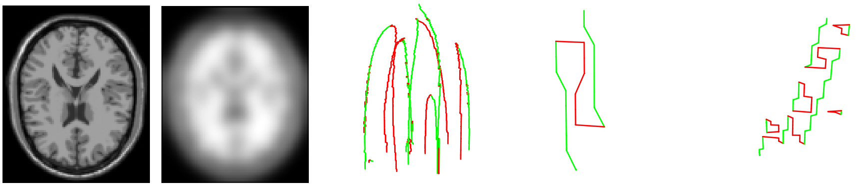

Fig. 5. a: 181 x 217 artificial MR image. b) Image on scale 8.37 c) Critical paths of the MR image in scale range $8.37 - 33.1$. d) Close-up of one of the critical paths of the MR image, showing a subsequent annihilation – creation event. e) Close-up, showing subsequent annihilation – creation events and loop events.

The scale space image in this scale range contains 161 logarithmically sampled scales. At all scales the spatial critical points were calculated and connected, forming the critical paths. Figure 5c shows these critical paths in the $(x, y; t)$-space. The bright curves represent the extrema, the dark ones the saddles. At the (approximate) catastrophe locations the curves are connected. Globally, the image shows annihilating pairs of critical points. Locally, however, the presence of extra branches of critical curves is visible. A close-up of one of the critical paths is shown in Figure 5d. It clearly shows a critical curve containing two subsequent Fold annihilation – creation events. The critical curve evidently shows the the appearance of an annihilation-creation-pair described by the D_3^- morsification. Note that the creation events would have been missed if the sampling was taking coarser, yielding one critical curve without protuberances in scale direction. Sampling without connecting critical paths yields the observation of temporarily created extrema (and saddles).

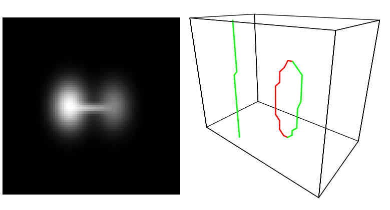

Fig. 6. a: Artificial bridge image. b) Critical paths of the bridge image.

Figure 6a shows the classical "bridge"-image: two mountains of different height (blobs with different intensity) connected by a small ramp and a deep valley between the mountains. This image was described by Lifshitz and Pizer [11] as possible initial image yielding a creation event in scale space. Firstly, there is only one maximum of the left blob. The right blob is not a maximum, since it is connected to the other blob by the ramp. Secondly, at some scale the ramp changes into a bridge with a deep dip in it due to the surrounding deep valleys: a maximum (right blob) – saddle (dip of the bridge) pair is created. Finally, at a large scale a saddle – extremum annihilation occurs. If the saddle annihilates with the left extremum, it can be modelled by the D_4^- catastrophe, as in the previous section. However, as shown by Figure 6b, it can also annihilate with the newly created extremum. This figure shows the critical paths of the scale space image of 6a. The left string represents the extremum of the brightest blob, the loop represents the created and annihilated maximum-saddle pair. The same behaviour is observed at the MR scale space image. Figure 5e shows a close-up of one of the critical curves. Besides several aforementioned subsequent Fold annihilation – creation events along the critical curve, here clearly also several "loop events" occur.

5 Discussion

In this paper we investigated the (deep) structure on various catastrophe events in Gaussian scale space. Although it is known that pairs of critical points are annihilated and created (the latter if the dimension of the image is 2 or higher), it is important to describe also the local structure of the image around the non-generic events. These events might be encountered in practical usage of scale spaces and the non-generic catastrophes can be used to model these occurrences. We therefore embedded catastrophes in scale space. Scale acts as one of the perturbation parameters. The morsification of the catastrophes yields generic Fold annihilations and creations of pairs of critical points.

The A_k series can be used to model (almost) simultaneous annihilations of pairs of critical points at a location (or indistinguishable region) in scale space. If k is even, it models the annihilation of k critical points, if k is odd, it models the collision of k critical points where $k - 1$ annihilate and one remains.

For creations the D_k series can be used. Creations occur in different types. Critical paths in scale space can have protuberances, a subsequent occurrence of an annihilation and a creation. In scale space images this is visible by the creation of an extremum-saddle pair, of which one critical point annihilates at some higher scale with an already present critical point, while the other remains unaffected. It is also possible that critical paths form loops: the created pair annihilates at some higher scale. The possibility for both types to occur in practice was shown in the artificial MR image. This phenomena is known from physics, where it is used to describe the creation and successive annihilations of "virtual" elementary particles (and even the universe). Furthermore we showed that the protuberances in the critical paths, expressed in canonical coordinates, occur only in case of a small local perturbation. In addition, creations are less likely to happen due to a special constraint on the combination of third order derivatives and local perturbation. We gave a dimension dependent expectation of this event and an upper bound for the perturbation in canonical coordinates.

The lifetime of a created pair is enlarged if the local structure is elongated. This was derived from the canonical formulation and visualised by the example of the bridge image in section 4. Since the number of possible catastrophes is infinite, there is an infinite number of possible non-generic constellations in which ("infinite") critical points are annihilated and created. We restricted ourselves to the situations in which at most 6 critical points annihilate and in which critical points are created, the latter divided into models representing protuberances and loops.

Finally, the calculations were based on the canonical coordinates. In general, it is not trivial to transform the local coordinate system to these nice formulated catastrophe germs. In that sense, the numerical values have no direct meaning. They do describe, however, the qualitative behaviour of the critical curves close to the location of the catastrophes and can therefore be used to model the type of behaviour encountered in practical usage of a scale space. We gave examples of the appearances of this behaviour in section 4 based on an artificial MR image.

The theory described in this paper extends the knowledge of the deep structure of Gaussian scale space, especially with respect to the behaviour of critical curves in the vicinity of creation events and the scale space lifetime of the created critical points. It emphasises the relevance of investigating the complete scale space image, instead of a series of images at different scales.

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