On Loss Probabilities in Presence of Redundant Packets with Random Drop

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Abstract. We study the loss probabilities of messages in an M/M/1/K queueing system where in addition to losses due to buffer overflow there are random losses on the incoming and outgoing links. We focus on the influence of adding redundant packets to the messages. We obtain analytical results that allow us to investigate when does adding redundancy decrease the loss probabilities.

1 Introduction

Loss rate of packets is an important performance measure in telecommunication networks. Rapid progress in the development of fiber optics allows to achieve a bit error rate of 10^{-14}; information loss is then essentially due to congested nodes and buffer overflow. However, in wireless networks random losses also occur in the channels/links apart from congestion losses. Often, when messages are divided into several packets, the loss of a packet results in the loss of the whole message. In order to reduce the losses, one may add redundant packets so that lost packets can often be reconstructed. Indeed, there exist erasure recovery codes that, by adding k redundant packets to a message, enable to reconstruct up to k losses, see e.g.[4],[6], [8]. Note, however, that by adding redundant packets, the workload increases and thus the loss probability of a packet may increase [1]. Alternatively, if one wishes to have the workload unchanged, this means that the throughput of useful information transmitted by the source decreases. Thus there are two types of tradeoffs to be studied (according to whether we want to keep the total transmitted throughput the same, or only the throughput corresponding to useful transmitted information). In this paper we are concerned with studying the loss probabilities of messages in queueing systems where in addition to losses due to buffer overflow there are also random losses on the incoming and outgoing links to the bottleneck node. In particular, we study the tradeoffs mentioned in the previous paragraph.

The problem of analyzing loss probabilities due to congestion losses in the presence of redundant packets has been addressed in several papers in the past [1,6,4,3,8]. In [6], the authors have used an approximation based on an assumption of independence between consecutive losses, and shown that redundancy results in decrease of loss rate by a factor of 10 to 100. Exact numerical methods based on recursions [4] led to an opposite conclusion, i.e. that redundancy

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causes increase in loss probabilities. Explicit expressions for the losses have then been developed in [3,8] and references therein which allowed to obtain regions of parameters in which Forward Error Correction (FEC) is useful and others where it is not. In particular, in [3] information theoretical type of channel capacity has been obtained for channels with congestion losses (and general service and inter-arrival times). All these references studied models of where losses is only due to congestion. Such models are useful in fiber-optic networks, when the main source of losses in the network is indeed overflow of a bottleneck buffer. There are however other situations in which a non-negligible amount of losses may also occur at noisy links.

The goal of this paper is to determine the role of redundant packets in networks in which losses may be due to both phenomena: link *random losses* and losses due to *congestion losses*. We obtain expressions that permit us to study two scenarios for adding FEC. In the first, the global transmission rate is unchanged; when adding FEC we reduce the rate of useful information. We then analyze how does the received rate of useful information depend on the FEC. In the second scenario we keep the rate of useful information unchanged; adding FEC then increases the congestion and hence the losses, but allows one to recover some losses.

The paper is structured as follows. Section 2 presents the model and motivation. Section 3 presents our main results derived using an algebraic approach involving multidimensional generating functions. Section 4 provides numerical examples and discusses the region where adding redundancy improves performance. In Sec. 5 we employ a combinatorial approach using Ballot theorems to obtain explicit expressions for loss probabilities employing techniques developed in [8]. Section 6 concludes the paper.

2 The Model and Its Motivation

We consider networks consisting of a buffer that is in-between two noisy links. The latter is a suitable model for satellite connections in which there is a noisy uplink and a noisy downlink connection with further losses that may be due to congestion inside the satellite. We assume throughout that a packet that is corrupted before it arrives to the bottleneck queue is discarded and does not occupy any buffer space. In the analysis below we shall model random losses in the incoming link (uplink) and congestion losses at the node. We consider an M/M/1 queue with a finite buffer of size K (including the packet in service). We assume that losses can be caused either by a buffer overflow or randomly with probability r. The arrival process from the source is assumed to be Poisson with rate λ and the service times of packets is exponentially distributed with rate μ. Hence, the effective arrival process to the system (buffer) can be assumed to be Poisson with rate $\lambda_e = (1-r)\lambda$. Define $\bar{r} = 1 - r$, $\rho = \lambda_e/\mu$, and $\rho_r = \rho/\bar{r}$. We present a recursive scheme for computing $P(j, n)$ which is the probability of j losses (including random losses in the incoming link and congestion losses at the node) among n consecutive packets.

Remark 1. The case when there are losses in both the incoming and outgoing links can be analysed once we have $P(j, n)$. For example, let the random loss probability in the outgoing link be u and let $\mathcal{P}_{j,n}$ be the probability of j losses among n consecutive packets of a message when there are random losses with probability r in the incoming link, congestion losses due to buffer overflow at the node and random losses with probability u in the outgoing link. Then $\mathcal{P}_{j,n} = \sum_{w=0}^{j} \binom{n-j+w}{w} u^w (1-u)^{n-j} P(j-w, n)$.

Thus knowing $P(j, n)$, which is the loss probability in the model we consider (i.e., random losses in the incoming link and congestion losses at the node) one can obtain the loss probabilities for the case when random losses can occur both in the incoming and the outgoing links.

3 Approach Using Generating Functions: Main Results

For the system with Poisson arrivals with rate λ_e and exponential transmission rate μ, in steady state, the probability of finding i packets in the system at an arbitrary epoch is given by $\Pi(i) = \rho^i / \sum_{l=0}^{K} \rho^l$. Define $Q_i(k)$ to be the probability that k packets out of i leave the system during an inter-arrival epoch. We have

$$Q_i(k) = \rho\alpha^{k+1}, \quad 0 \leq k \leq i-1, \qquad Q_i(i) = \alpha^i, \quad \text{where} \;\; \alpha := (1+\rho)^{-1}.$$

Denote by $P_i^a(j, n)$ the probability of j losses in a block of n consecutive packets, given that there are i packets in the system just before the arrival of the first packet in the block. Since the first packet in the block is arbitrary, we have

$$P(j, n) = \sum_{i=0}^{K} \Pi(i) P_i^a(j, n). \tag{1}$$

The recursive scheme for computing $P_i^a(j, n)$ is then for $i = 0, 1, ..., K-1$:

$$P_i^a(j, 1) = \begin{cases} \bar{r} & j = 0 \\ r & j = 1 \\ 0 & j \geq 2, \end{cases} , \text{ and } P_K^a(j, 1) = \begin{cases} 1 & j = 1 \\ 0 & j = 0, j \geq 2. \end{cases} \tag{2}$$

For $n \geq 2$ we have for $0 \leq i \leq K-1$

$$P_i^a(j, n) = \bar{r} \sum_{k=0}^{i+1} Q_{i+1}(k) P_{i+1-k}^a(j, n-1) + r \sum_{k=0}^{i} Q_i(k) P_{i-k}^a(j-1, n-1),$$

$$\text{and} \qquad P_K^a(j, n) = \sum_{k=0}^{K} Q_K(k) P_{K-k}^a(j-1, n-1).$$

Next, we state the main results, whose detailed proofs are given in the Appendix. Define $q(y, z) \triangleq \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} y^j z^{n-1} P(j, n)$. Let $x_1(y, z)$ and $x_2(y, z)$ be the solutions in x of $x^2 - (1 + \rho - r\rho yz)x + \bar{r}\rho z) = 0$:

$$x_1(y,z) = \frac{1+\rho - r\rho yz + \sqrt{(1+\rho - r\rho yz)^2 - 4\bar{r}\rho z}}{2}$$
$$x_2(y,z) = \frac{1+\rho - r\rho yz - \sqrt{(1+\rho - r\rho yz)^2 - 4\bar{r}\rho z}}{2}.$$

We shall often write simply x_1 and x_2 for $x_1(y,z)$ and $x_2(y,z)$. Define, for all $k \geq 1$, $\delta_k = x_1^k - x_2^k$, $\phi_k = (\bar{r} + ry)z\delta_{k-1} - \delta_k$. Let $R_K = (\sum_{l=0}^{K} \rho^l)^{-1}$.

Proposition 1. *The probability generating function (PGF) q is given by*

$$q(y,z) = \frac{R_K}{1-(\bar{r}+r\rho y)z}\left[(\bar{r}+ry)R_{K-1}^{-1} + y\rho^K \right.$$
$$\left. + z\rho(\alpha\rho)^K(\bar{r}(y-\alpha) - \alpha\rho y)A(y,z) + rzy(\alpha\rho)^K B(y,z)\right], \qquad (3)$$

where $A(y,z)$ and $B(y,z)$ solve

$$\begin{pmatrix} z\rho\alpha(\alpha x_1)^{K+1}(y(\bar{r}-\alpha x_1) - \bar{r}\alpha) & z\alpha^2(\bar{r}(x_1-\rho) + rx_1 y(\alpha x_1)^K) \\ z\rho\alpha(\alpha x_2)^{K+1}(y(\bar{r}-\alpha x_2) - \bar{r}\alpha) & z\alpha^2(\bar{r}(x_2-\rho) + rx_2 y(\alpha x_2)^K) \end{pmatrix} \begin{pmatrix} A(y,z) \\ B(y,z) \end{pmatrix}$$
$$= (-1)\begin{pmatrix} (1-\alpha x_1)\alpha x_1^{K+1} y + (1-\alpha x_1)\alpha x_1(ry+\bar{r})\left(\frac{1-x_1^K}{1-x_1}\right) \\ (1-\alpha x_2)\alpha x_2^{K+1} y + (1-\alpha x_2)\alpha x_2(ry+\bar{r})\left(\frac{1-x_2^K}{1-x_2}\right) \end{pmatrix}. \qquad (4)$$

For $y = 0$, Prop. 1 simplifies to: $q(0,z) = \bar{r}\left[R_{K+1}^{-1} - z\rho^K A(0,z)\right](R_K 1 - \bar{r}z)^{-1}$.

Having obtained the PGF, the explicit expressions for the required probabilities can be obtained by inverting $q(y,z)$. We next focus on $P_\rho(> j, n)$, the probability of losing more than j packets out of n. We investigate the cases of $j = 0, 1$, in order to be able to decide whether adding a redundant packet to each message results in a decrease of the loss probability. The proofs can be found in [2]. To stress the dependence of the different quantities (such as the p.g.f. q) on the random loss parameters, we shall sometimes add r and λ explicitly to the notation as subscript (e.g. we shall write $q_r^\lambda(y,z)$).

Corollary 1. *(i)* $q_r^\lambda(0,z) = q_0^{\bar{r}\lambda}(0,\bar{r}z)\bar{r}$, *(ii)* $P_r^\lambda(0,n) = \bar{r}^n P_0^{\bar{r}\lambda}(0,n)$.

Corollary 2. *The probability of losing one packet out of n consecutive packets, i.e., $P(1,n)$ is given by*

$$P(1,n) = [z^{n-1}]\left.\frac{\partial q(y,z)}{\partial y}\right|_{y=0} = [z^{n-1}]F_1(z) + [z^{n-1}]F_2(z)$$

$$\textit{with } F_1(z) = \frac{R_K}{1-\bar{r}z}\bar{r}\left[R_{K-1}^{-1} - z(\alpha\rho)^{K+1}A(0,z)\right]\left(-1 + \frac{zr\rho}{1-\bar{r}z}\right)$$
$$F_2(y) = \frac{R_K}{1-\bar{r}z}\left[R_{K-1}^{-1} + \rho^K - z(\alpha\rho)^{K+1}\bar{r}\dot{A}(0,z) + rz(\alpha\rho)^K B(0,z)\right]$$

where $A(0,z)$ and $B(0,z)$ are values at $y = 0$ of $A(y,z)$ and $B(y,z)$ defined in Proposition 1 and $\dot{A}(0,z)$ is the derivative of $A(y,z)$ with respect to y, evaluated at $y = 0$.

4 Numerical Examples

In this section we compare the loss probabilities of a whole group of n consecutive packets, which we call a block, with and without j additional redundant packets. The group of packets that include the original block plus the additional redundant packets (if these are added) is called a frame. If at least n packets out of these consecutive $n + j$ packets reach the destination then no loss of frame occurs. In this section we restrict ourselves to the case of $j = 0$, i.e., no redundancy and $j = 1$, one redundant packet per n packets. Without loss of generality, we may scale the time so that the service rate is unity: $\mu = 1$. In the numerical examples we are looking only at the random losses in the incoming link with probability r and congestion losses. We take $K = 25$. When we numerically compared $P_\rho(> 0, n)$ with $P_\rho(> 1, n + 1)$ we always obtained $P_\rho(> 1, n + 1) < P_\rho(> 0, n)$, which should be of no surprise: this observation means that if redundancy is added in such a way that *the total load on the system remains unchanged* then indeed redundancy improves performance in terms of loss probabilities. However, the assumption that the total load remains the same means that the throughput of the *useful* information decreases (in real time applications this would mean that a higher compression rate should be used before transmission). This type of comparison (keeping the total load unchanged) has not been performed previously in [6,4,3,8] even for the case of congestion losses only. E.g., if we add k redundant packets to n (which gives frames of $n+k$) and if the load is unchanged, then this means that the throughput of useful information carried by a frame has decreased by a factor of $n/(n + k)$. Yet we have less losses of packets. Thus the question that needs to be addressed is whether we gain in *goodput* in this case. Let us define the goodput as the throughput arriving well to the destination. Then this is given by

$$\text{(input rate of blocks)} \times n/(n + k) \times P_\rho(\leq k, n + k).$$

So a meaningful thing to compare is $P_\rho(0, n)$ with $\frac{n}{n+1} P_\rho(\leq 1, n + 1)$ for fixed λ. In Fig. 1, we plot the relative gain, i.e.,

$$\frac{\frac{n}{n+1} P(\leq 1, n + 1) - P(0, n)}{P(0, n)}. \tag{5}$$

From Fig. (1) we observe that the benefits of adding FEC grows as the amount of random losses increases, and also as n increases. Also for very low r (very close to 0) and very low n (as compared to buffer size) we loose by adding FEC. Fig. (2) plots the same curve for $\lambda = 0.99$. We observe that curves for $\lambda = 0.3$ and $\lambda = 0.99$ are identical $r \geq 0.1$ and larger n and for r close to 0 the difference is very small. *Remark:* Consider a scenario in which there are only random losses (with probability r) and no congestion losses. Then we have: $P_\rho(0, n) = (1 - r)^n, \quad P_\rho(1, n) = nr(1 - r)^{n-1}$. If we want to study the effect of adding FEC on recovering from different type of losses we can compare the relative gain defined in (5) for the cases when $r = 0$ (congestion losses but no random losses) to the case when there are no congestion losses but only random

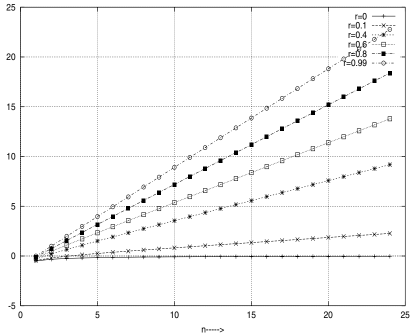

Fig. 1. $\dfrac{\frac{n}{n+1}P(\leq 1, n+1) - P(0,n)}{P(0,n)}$ as a function of n for varying r with $\lambda = 0.3$

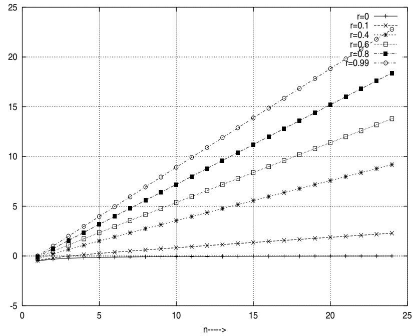

Fig. 2. $\dfrac{\frac{n}{n+1}P(\leq 1, n+1) - P(0,n)}{P(0,n)}$ as a function of n for varying r with $\lambda = 0.99$

losses with loss probabilities $P_\rho(0,n)$ and $P_\rho(1,n)$. We plot this comparison in Fig. (3) and observe that FEC is more helpful in recovering from random losses than congestion losses.

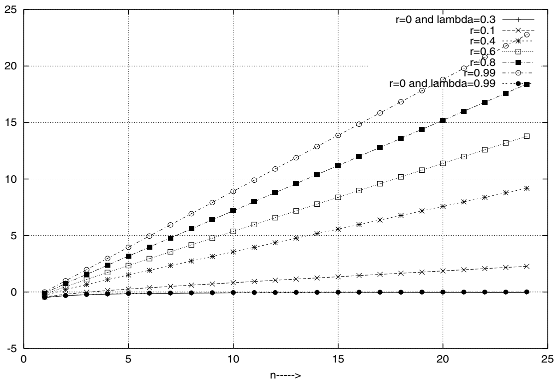

Fig. 3. Gain $\frac{\frac{n}{n+1}P(\leq 1,n+1)-P(0,n)}{P(0,n)}$ as a function of n for r varying from 0.1 to 0.99 for the scenario when there are no congestion losses. Also shown is the gain when there are no random losses ($r = 0$) and only congestion losses with $\lambda = 0.3$ and $\lambda = 0.99$. Observe that the curves for $r = 0$ and $\lambda = 0.3$ and $\lambda = 0.99$ have negligible differences.

Next we look at the case where the transmission of useful information is kept unchanged when adding redundancy. This implies that the total packet arrival rate increases due to adding redundancy. We assume that the rate at which frames arrive is the same for the two cases and is given by x. In case of no redundancy the rate at which packets arrive is $\lambda = \rho = nx$ and in case of redundancy $\lambda = \rho = (n+1)x$. A frame is lost in the latter case if more than one packet is lost out of $n+1$ consecutive packets. We are thus interested in the difference $D = P_{nx}(>0, n) - P_{(n+1)x}(>1, n+1)$. If $D > 0$ then the redundancy decreases the loss probability of messages. Observe that

$$\begin{aligned} D &= 1 - P_{nx}(0,n) - \left[1 - P_{(n+1)x}(0,n+1) - P_{(n+1)x}(1,n+1)\right] \\ &= P_{(n+1)x}(1,n+1) + P_{(n+1)x}(0,n+1) - P_{nx}(0,n). \end{aligned} \tag{6}$$

We next plot the relative gain $\frac{D}{P_{nx}(>0,n)}$ as a function of n for $x = 0.03$ (this means the load nx, varies from 0.03 (for $n = 1$) to 0.75 (for $n = 25$)) in Figure 4 and for $x = 0.4$ (load varying from 0.4 to 10) in Figure 5. The curves show that for fixed r, there exists a value of the frame size at which the gain obtained by adding FEC as defined in (6) is maximum. These figures can thus be used in order to optimize the size of blocks to which we add FEC.

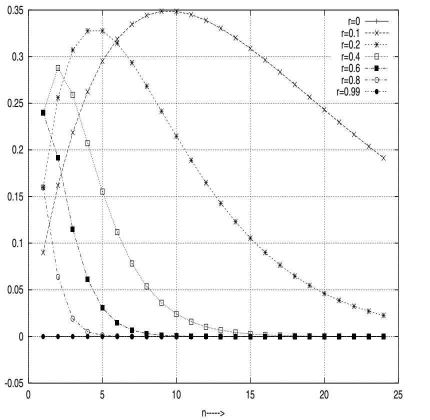

Fig. 4. $\frac{D}{P_{nx}(>0,n)}$ as a function of n for different r and $x = 0.03$. Observe that the load changes with n also.

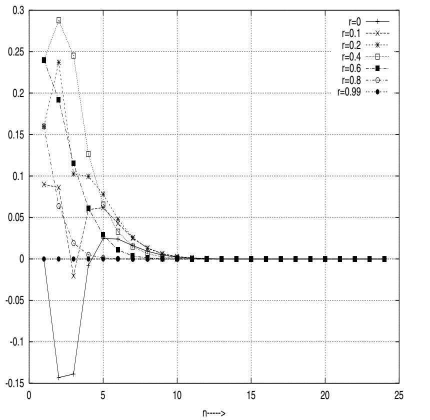

Fig. 5. $\frac{D}{P_{nx}(>0,n)}$ as a function of n for different r and $x = 0.4$

All the above curves establish that we benefit from adding redundancy when r is not very small, and this is a valid remark or observation at any load. However when the random loss probability is very low (close to 0) we may loose by adding redundancy.

5 Combinatorial Approach Using Ballot Theorems

We next employ combinatorial arguments together with the Ballot theorems [5] to alternatively obtain explicit expressions for all the probabilities of the previous section. In particular, we shall find the probability $P_i^a(j,n)$.

Consider the case when j [1] losses consist of $j_r(0 \leq j_r \leq j)$ random losses and $j_c(0 \leq j_c \leq j)$ congestion losses. The number of ways such an event can occur is $\binom{j}{j_c}$. We calculate the probability of one such outcome. The probability depends on the position of the lost packets in the frame. Let us denote by r_i the position of the ith random loss, $1 \leq i \leq j_r$ in the original frame. Also $i \leq r_i \leq n-(j_r-i)$. Thus $r_1 = 1$, when the first packet was lost by random loss and $r_{j_r} = n$, when the last packet was lost by random loss.

The following analysis is for the case of $j_r \geq 2, r_1 \neq 1, r_{j_r} \neq n$. We shall supplement the discussion with other cases $j_r \leq 1$ *and/or* $r_1 = 1$ *and/or* $r_{j_r} = n$ at appropriate places. Observe that the random losses can be *isolated* or they can occur in burst. In fact since our message length is finite (n), the probability that all the random losses occur in a burst is > 0 [2]. Also observe that only the packets of the original message which are not subject to random losses have the *potential* of getting lost at the queue due to buffer overflow (as these are the only packets that actually reach the queue). Thus we shall look at the packets of the original message between consecutive *random loss events*. A random loss event is formed consecutive random losses. Say that the packets coming to the queue between consecutive random loss events are forming an *interval*. Let T be the number of such intervals and k_i $(1 \leq i \leq T)$ be the number of consecutive random losses in the random loss event starting after the end of the ith interval and prior to the beginning of the $i+1$th interval. Thus the maximum value of T is $j_r + 1$ when all the random losses occur isolated and on the other extreme, the minimum value of T is 2 when all the random losses occur in a burst. Define $z(t) := \sum_{h=1}^{t} k_h$. We now distribute the j_c congestion losses in the T intervals of lengths $r_1 - 1, r_{1+k_1} - r_{k_1} - 1, r_{1+k_1+k_2} - r_{k_1+k_2} - 1, \ldots, r_{1+z(T)} - r_{z(T)} - 1$. Let n_y be the number of congestion losses in the yth such interval. Observe that (for $2 \leq y \leq j_r + 1$) we have $0 \leq n_y \leq \min(r_{1+z(y-1)} - r_{z(y-1)} - 1, j_c)$, and for $y = 1$, $0 \leq n_y \leq \min(r_1 - 1, j_c)$. Also, n_y satisfy $\sum_{y=1}^{T} n_y = j_c$. Now the number of ways in which n_y losses can occur in the yth interval is

$$\binom{r_{1+z(y-1)} - r_{z(y-1)} - 1}{n_y}$$

for $2 \leq y \leq T$ and is $\binom{r_1-1}{n_1}$ for $y = 1$.

We shall calculate the probability of one such event. We shall look at three types of intervals: A-starts with the first arrival after a random loss and ends with the last arrival before a random loss event; B-starts with the arrival of the first packet of the message (if $r_1 \neq 1$) and ends with the last arrival before the

[1] Observe that here we are looking at the case when the random losses (if any) occur before the frame enters the buffer. The complementary case of random losses occurring after the frame leaves the node can be handled as discussed in Remark 1. And then one can obtain the loss probabilities for the case when random losses can occur both in the outgoing and in the incoming link.

[2] Although bursty loss occurrence is more a characteristic of congestion losses.

first random loss event; C-starts with the first arrival after the last random loss event and ends with the arrival of the last packet of the message.

In a sample path with $j_r \geq 2, r_1 \neq 1, r_{j_r} \neq n$, and with A_i an interval of type A, the order of occurrence of the intervals is $B \rightarrow A_1 \rightarrow A_2 \ldots \rightarrow A_{T-2} \rightarrow C$. For $j_r \geq 2, r_1 = 1, r_{j_r} \neq n$, the order is $A_1 \rightarrow A_2 \ldots \rightarrow A_{T-1} \rightarrow C$ and no interval of type B. For $j_r \geq 2, r_1 \neq 1, r_{j_r} = n$, the order is $B \rightarrow A_1 \ldots A_{T-1}$ and no interval of type C. Similarly, for $j_r \geq 2, r_1 = 1, r_{j_r} = n$, there will be no interval of type either B or of type C. For $j_r = 1$, there can either be intervals $B \rightarrow C$ or C or B and no interval of type A can occur.

Let the queue length at the beginning of the yth interval be α and at the end of the interval be β. We thus need to calculate the probability of a path that starts with α packets in the buffer, ends with β packets in the buffer, has n_y losses in it by congestion and consists of $a_y = (r_{1+z(y-1)} - r_{z(y-1)} - 1)$ arrival events. We employ the arguments as in [8] to evaluate this probability. However here in our analysis we also need to know the queue length at the arrival of the last packet of an interval. We shall denote this probability by $P_{(\alpha,\beta)}(n_y, a_y)$. Let f_j denote the jth lost packet. We shall decompose an interval into three types of events as follows: (i) $\mathcal{V}_\alpha(f_1)$-the first packet to be lost is f_1 given that upon the arrival of the first packet of the interval there are α packets in the buffer; (ii)$\mathcal{S}(f_l, f_{l+1})$-packet f_{l+1} is lost given that packet f_l was lost; (iii)$\mathcal{U}(f_{n_y}, \beta)$-packet f_{n_y} is the last to be lost and the queue length at the arrival of the last packet of the interval is β.

Observe that an interval consists of the succession of events $\mathcal{V}_\alpha(f_1), \mathcal{S}(f_1, f_2)$, $\mathcal{S}(f_2, f_3), \ldots, \mathcal{S}(f_{n_y-1}, f_{n_y}), \mathcal{U}(f_{n_y}, \beta)$. Let $v_\alpha(f_1)$, $s(f_l, f_{l+1})$ and $u(f_{n_y}, \beta)$ be the probabilities of the event $\mathcal{V}_\alpha(f_1), \mathcal{S}(f_l, f_{l+1})$ and $\mathcal{U}(f_{n_y}, \beta)$, respectively. Thus $P_{(\alpha,\beta)}(n_y, a_y)$ is given by

$$\sum_{f_1=1}^{a_y-n_y+1} \sum_{f_2=f_1+1}^{a_y-n_y+2} \cdots \sum_{f_{n_y}=f_{n_y-1}+1}^{a_y} v_\alpha(f_1)s(f_1, f_2)\ldots s(f_{n_y-1}, n_y)u(f_{n_y}, \beta).$$

The computation of the probabilities $v_\alpha(f_1)$ and $s(f_l, f_{l+1})$ is similar to that in [8]. For their computation, as well as of $u(f_{n_y}, \beta)$ see [2].

Proposition 2. *The probabilities $v_\alpha(f_1), s(f_l, f_{l+1})$ and $u(f_{n_y}, \beta)$ are given as*

$$v_\alpha(f_1) = \begin{cases} 0 & f_1 \leq K - \alpha \\ \frac{\rho}{\rho+1}.\phi_{2f_1-K+\alpha-3}(\alpha+1, K) & o.w. \end{cases} \qquad \alpha \neq K, \tag{7}$$

$$v_K(f_1) = \begin{cases} 1 & f_1 = 1 \\ 0 & o.w. \end{cases}, \qquad s(f_l, f_{l+1}) = \frac{\rho}{\rho+1}.\phi_{2(f_{l+1}-f_l-1)}(K, K) \tag{8}$$

$$u(f_{n_y}, \beta) = \begin{cases} \phi_{2(a_y-f_{n_y})+K-\beta}(K, \beta) & f_{n_y} < a_y \\ 1 & f_{n_y} = a_y \text{ and } \beta = K \\ 0 & f_{n_y} = a_y \text{ and } \beta \neq K \end{cases} \tag{9}$$

where $\phi_\eta(\alpha,\beta)$ *is defined as the probability of a path that starts with* α *packets in the buffer, ends with* β *packets in the buffer and consists of* η *events (arrivals and departures) and is defined as* $\phi_\eta(\alpha,\beta) = \epsilon_\eta(\alpha,\beta) + \sum_{r=1}^{\mathcal{H}} W_\alpha Y^{r-1} Z^{\mathcal{T}}$, *where, for* $\alpha \geq 1, \beta \geq 1$ *where* $\epsilon_\eta(\alpha,\beta)$ *is given by*

$$\sum_{\Upsilon}\left[\binom{\eta}{\frac{\eta+\alpha-\beta}{2} - \Upsilon(K+1)} - \binom{\eta}{\frac{\eta-\alpha-\beta}{2} - \Upsilon(K+1)}\right]\left(\frac{\rho}{1+\rho}\right)^{\frac{\eta-\alpha+\beta}{2}}\left(\frac{1}{1+\rho}\right)^{\frac{\eta+\alpha-\beta}{2}},$$

$$W_\alpha = \left(\epsilon_\alpha(\alpha,0), \epsilon_{\alpha+2}(\alpha,0), \ldots, \epsilon_{\alpha+2(\mathcal{H}-1)}(\alpha,0)\right)$$
$$Z = \left(\epsilon_{\eta-\alpha}(0,\beta), \epsilon_{\eta-\alpha-2}(0,\beta), \ldots, \epsilon_{\eta-\alpha-2(\mathcal{H}-1)}(0,\beta)\right)$$
$$Y = \begin{pmatrix} 0 & \epsilon_2(0,0) & \epsilon_4(0,0) & \ldots & \epsilon_{2(\mathcal{H}-1)}(0,0) \\ 0 & 0 & \epsilon_2(0,0) & \ldots & \epsilon_{2(\mathcal{H}-2)}(0,0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & \epsilon_2(0,0) \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix}, \qquad \mathcal{H} = 1 + \frac{\eta-\alpha-\beta}{2}$$

and $\epsilon_\eta(0,\beta) = \epsilon_{\eta-1}(1,\beta)$, $\beta \geq 1$, $\epsilon_\eta(\alpha,0) = \frac{1}{1+\rho}\epsilon_{\eta-1}(\alpha,1)$, $\alpha \geq 1$, $\epsilon_\eta(0,0) = \frac{1}{1+\rho}\epsilon_{\eta-2}(1,1)$, *where* $-\infty < \Upsilon < \infty$ *takes on values in the sum in the definition of* $\epsilon_\eta(\alpha,\beta)$ *in (10) so that the binomial coefficients are proper, for e.g. in the first sum in (10)* $\frac{\eta+\alpha-\beta}{2} > \Upsilon(K+1)$ *and* $\eta > \frac{\eta+\alpha-\beta}{2} - \Upsilon(K+1)$.

We also need the probability of the evolution of a path after the end of interval A_i and before the start of interval A_{i+1} and having $k_i (\geq 1)$ packets lost by random losses. Observe that the duration of this random loss event has the distribution of the sum of $k_i + 1$ independent $\exp(\lambda)$ distributed random variables, i.e., Erlang(k_i+1,λ). Let X_i be the number of service completions $\exp(\mu)$ in an interval with distribution $F * F * \ldots (k - times) = F^{*k}$ where $F \sim \exp(\lambda)$ and $*$ denotes the convolution operation. Then the probability that A_i ends with β_1 packets (including the last arrival in the interval A_i) in the buffer and A_{i+1} starts with β_2 packets (not including the first arrival in the interval A_{i+1}) in the buffer and has k_i random losses can be written as

$$P(X_i = \beta_1 - \beta_2, k_i) = \begin{cases} \int_0^\infty \frac{e^{-\mu s}(\mu s)^{(\beta_1-\beta_2)}}{(\beta_1-\beta_2)!} dF^{*(k_i+1)}(s) & \text{if } 0 < \beta_2 \leq \beta_1 \\ \sum_{m=\beta_1}^\infty \int_0^\infty \frac{e^{-\mu s}(\mu s)^m}{m!} dF^{*(k_i+1)}(s) & \text{if } \beta_2 = 0 \\ 0 & \beta_2 > \beta_1. \end{cases}$$

Remark 2. Indeed, the end of service times are a Poisson process with intensity μ. *The PGF of the number of such points during a fix interval* T *is* $G(z) = \exp(-\mu(1-z)T)$. *If* T *is a random interval then it is* $G(z) = E[\exp(-\mu(1-z)T] = T^*(\mu(1-z)$ *where* $T^*(s)$ *is the Laplace Stieltjes transform of* T. *If* T *were exponential* (λ) *then this would give*

$$G(z) = \frac{\lambda}{\lambda + \mu(1-z)} = \frac{1}{z}\frac{\theta z}{1-(1-p)z} \text{ where } \theta = \frac{\lambda}{\lambda+\mu} = \frac{\rho}{1-\rho}.$$

We see that $G(z)$ is the PGF of $Y = X - 1$ where X has a geometric distribution with parameter θ, so $P(Y = n) = (1-\theta)^n\theta$. The number of points in an Erlang($k_i + 1, \lambda$) RV, say X_i, has thus the distribution of the convolution of $k_i + 1$ copies of Y, which gives:

$$P(X_i = n) = \sum_{y_1+\ldots+y_n=k_i+1} \frac{(k_i+1)!}{y_1!y_2!\ldots y_n!}\theta^n(1-\theta)^{k_i+1}$$

This can now be used to for the expressions in (10).

We will now consider a path that starts with i packets in the buffer, in which out of n packets in a frame, j_r packets are lost by random losses j_c packets are lost by congestion losses, $j_c + j_r = j$ and has T *intervals*. Let r_i be the position of the ith random loss. Let $P_p^i(j_c, j_r, T, n)$ be the probability of such a path [3]. Then for $r_1 \neq 1$ and $r_{j_r} \neq n$,

$$P_p^i(j_c, j_r, T, n)$$
$$= r^{j_r}(1-r)^{n-j_r} \sum_{\beta_g=0,0\le g\le T-1}^{\beta_g=K} \sum_{\alpha_h=1,0\le h\le T-1}^{\alpha_h=K} \sum_{r_1=2}^{n-j_r} \sum_{k_1=1}^{j_r} \sum_{k_2=1}^{j_r-k_1} \cdots \sum_{k_{T-2}=1}^{j_r-\sum_{h=1}^{T-3} k_h}$$
$$\sum_{a_2=1}^{n-j_r-a_1} \sum_{a_3=1}^{n-j_r-\sum_{i=1}^{2} a_i} \cdots \sum_{a_{T-1}=1}^{n-j_r-\sum_{i=1}^{T-2} a_i} \sum_{n_1=0}^{\min(r_1-2,j_c)} \sum_{n_2=0}^{\min(a_2,j_c-n_1)} \cdots \sum_{n_{T-1}=0}^{\min(a_{T-1},j_c-\sum_{h=1}^{T-2} n_h)}$$
$$P_{(i,\beta_0)}(n_1, a_1)P(X_1 = \beta_0 - \alpha_1, k_1)P_{(\alpha_1,\beta_1)}(n_2, a_2)$$
$$P(X_2 = \beta_1 - \alpha_2, k_2)\ldots P_{(\alpha_{T-2},\beta_{T-2})}(n_{T-1}, a_{T-1})P(X_{T-1} = \beta_{T-2} - \alpha_{T-1}, k_{T-1})$$
$$P_{(\alpha_{T-1},\beta_{T-1})}(n_T, a_T).$$

where $\sum_{k=1}^{i} f_k = 0$ for $i \le 0$ and $a_1 = r_1 - 1$, $a_T = n - j_r - \sum_{i=1}^{T-1} a_i$, $k_{T-1} = j_r - \sum_{h=1}^{T-2} k_h$, $n_T = j_c - \sum_{h=1}^{T-1} n_h$. One can similarly obtain expressions for the other cases ($j_r \le 2$) and/or $r_1 \neq 1$ and/or $r_{j_r} \neq n$ etc. Having obtained the expressions we have

$$P_p^i(j_c, j_r, n) = \sum_T P_p^i(j_c, j_r, T, n) \text{ and } P_p^i(j, n) = \binom{j}{j_c} P_p^i(j_c, j_r, n).$$

And finally, $P_p(j, n) = \sum_{i=0}^{K} \Pi(i)P_p^i(j, n)$. The probability $P_p(j, n)$ here is the same as the probability $P(j, n)$ in Sec. 3.

6 Conclusion

We have studied the steady state loss probabilities of messages in an $M/M/1/K$ queue where there are both random losses and congestion losses using an algebraic approach involving generating functions and a second approach based on ballot theorems. The explicit expressions we obtained allowed us to investigate numerically when it is profitable to add FEC, and what should the optimal block size be when we add a single redundant packet per block (e.g. using a XOR operation).

[3] We use the subscript p to distinguish the notation from Sec. 3

Appendix: Proof of Proposition 1

Define $\pi_{j,n}(x) \triangleq \sum_{i=0}^{K} x^i P_i^a(j,n),\ n \geq 1,\ j \geq 0$. (3) implies for $n \geq 2$, that

$$\pi_{j,n}(x) = \bar{r} \sum_{i=0}^{K-1} x^i \sum_{k=0}^{i+1} Q_{i+1}(k) P_{i+1-k}^a(j, n-1)$$
$$+ r \sum_{i=0}^{K-1} x^i \sum_{k=0}^{i} Q_i(k) P_{i-k}^a(j-1, n-1) + x^K \sum_{k=0}^{K} Q_K(k) P_{K-k}^a(j-1, n-1).$$

We substitute (3) in the last equation, introduce $\pi_{j,n}(x)$ and also use the facts that $\pi_{j,n}(0) = P_0^a(j,n)$ and $1 - \rho\alpha = \alpha$. We then obtain for $n \geq 2$, $j \geq 1$, after some algebra [2]

$$\begin{aligned}
\pi_{j,n}(x) &= \frac{\bar{r}\rho\alpha^2}{1-\alpha x}\left(\frac{1}{\alpha x}\pi_{j,n-1}(x) - (\alpha x)^K \pi_{j,n-1}(\alpha^{-1})\right) \\
&- \frac{\bar{r}\rho\alpha^2}{1-\alpha x}\left(\frac{1}{\alpha x} - (\alpha x)^K\right)\pi_{j,n-1}(0) + \bar{r}\alpha\frac{1-(\alpha x)^K}{1-\alpha x}\pi_{j,n-1}(0) \\
&+ r\frac{\rho\alpha}{1-\alpha x}\left(\pi_{j-1,n-1}(x) - (\alpha x)^K \pi_{j-1,n-1}(\alpha^{-1})\right) \\
&+ r\alpha\frac{1-(\alpha x)^K}{1-\alpha x}\pi_{j-1,n-1}(0) + \alpha\rho(\alpha x)^K \pi_{j-1,n-1}(\alpha^{-1}) + \alpha(\alpha x)^K \pi_{j-1,n-1}(0).
\end{aligned} \tag{10}$$

Define, with some abuse of notation, the generating function of $P_i^a(j,n)$ $\pi(x,y,z) \triangleq \sum_{j=0}^{\infty}\sum_{n=1}^{\infty} y^j z^{n-1} \pi_{j,n}(x)$. When we fix y and $|z| < 1$, the above generating function is polynomial in x, and therefore an analytic function. In order to use (10), which holds only for $n \geq 2$ and $j \geq 1$, we note that $\sum_{j=1}^{\infty}\sum_{n=2}^{\infty} y^j z^{n-1}\pi_{j,n}(x) = \pi(x,y,z) - \pi(x,0,z) - \pi(x,y,0) + \pi(x,0,0)$. We obtain after some algebra [2]

$$\begin{aligned}
&\pi(x,y,z) - \pi(x,0,z) \\
&= yx^K + r\frac{1-x^K}{1-x}y + \bar{r}\frac{\rho\alpha^2 z}{(1-\alpha x)\alpha x}\left[\pi(x,y,z) - \pi(x,0,z)\right] + \frac{r\rho\alpha y z}{1-\alpha x}\pi(x,y,z) \\
&+ \rho\alpha(\alpha x)^K\left(y - \frac{(\bar{r}\alpha + r y)}{1-\alpha x}\right) z\left[\pi(\alpha^{-1},y,z) + \pi(0,y,z)/\rho\right] \\
&+ \frac{\bar{r}\alpha^2(x-\rho)}{(1-\alpha x)\alpha x} z\left[\pi(0,y,z) - \pi(0,0,z)\right] \\
&+ \frac{\bar{r}\rho\alpha^2(\alpha x)^K}{1-\alpha x} z\left[\pi(\alpha^{-1},0,z) + \pi(0,0,z)/\rho\right] + \frac{r\alpha y z}{1-\alpha x}(\alpha x)^K \pi(0,y,z).
\end{aligned} \tag{11}$$

We note that in order to establish the proof of Proposition 1, it follows from (1) that it suffices to obtain $\pi(x,y,z)$ at $x = \rho$, since $q(y,z) = R_K \pi(\rho,y,z)$. From (11), we have

$$\left[\pi(\rho,y,z) - \pi(\rho,0,z)\right]\left(1 - (\bar{r} + r\rho y)z\right) = y\rho^K + r\frac{1-\rho^K}{1-\rho}y$$

$$+z\left(y-\bar{r}-\frac{ry}{\alpha}\right)(\rho\alpha)^{K+1}\left[\pi(\alpha^{-1},y,z)+\pi(0,y,z)/\rho\right]$$
$$+z\bar{r}(\rho\alpha)^{K+1}\left[\pi(\alpha^{-1},0,z)+\pi(0,0,z)/\rho\right]+r\rho yz\left[\pi(\rho,0,z)+\frac{(\alpha\rho)^K}{\rho}\pi(0,y,z)\right].$$

To compute the function $\pi(\rho,y,z)$ it suffices to compute the functions in the square brackets as well as $\pi(\rho,0,z)$. To do that, we first compute $\pi_{0,n}$ by proceeding in the same manner as in (10). Since $P^a_K(0,n)=0$ we have for $n\geq 2$,

$$\pi_{0,n}(x)=\bar{r}\tfrac{\rho\alpha^2}{1-\alpha x}\tfrac{1}{\alpha x}\pi_{0,n-1}(x)-\bar{r}\tfrac{\rho\alpha^2}{1-\alpha x}(\alpha x)^K\pi_{0,n-1}(\alpha^{-1})$$
$$+\bar{r}\alpha\tfrac{1-(\alpha x)^K}{1-\alpha x}\pi_{0,n-1}(0)-\bar{r}\tfrac{\rho\alpha^2}{1-\alpha x}\left(\tfrac{1}{\alpha x}-(\alpha x)^K\right)\pi_{0,n-1}(0).$$

Taking the generating function of both sides and substituting $\pi(x,0,0)=\bar{r}\frac{1-x^K}{1-x}$, we get

$$(1-\alpha x)\alpha x\pi(x,0,z)=\bar{r}\frac{1-x^K}{1-x}(1-\alpha x)\alpha x+\bar{r}\rho\alpha^2 z\pi(x,0,z)$$
$$-\bar{r}\rho\alpha^2(\alpha x)^{K+1}z\left[\pi(\alpha^{-1},0,z)+\pi(0,0,z)/\rho\right]+\bar{r}\alpha^2(x-\rho)z\pi(0,0,z). \quad (12)$$

From (11), we have

$$\left((1-\alpha x)\alpha x-\rho\alpha^2\bar{r}z\right)\left[\pi(x,y,z)-\pi(x,0,z)\right]$$
$$=(1-\alpha x)\alpha yx^{K+1}+(1-\alpha x)\alpha xr\frac{1-x^K}{1-x}y$$
$$+z\rho\alpha(\alpha x)^{K+1}\left[(y(1-\alpha x)-(\bar{r}\alpha+ry)\right]\times\left[\pi(\alpha^{-1},y,z)+\pi(0,y,z)/\rho\right]$$
$$+\bar{r}\rho\alpha^2(\alpha x)^{K+1}z\left[\pi(\alpha^{-1},0,z)+\pi(0,0,z)/\rho\right]+\alpha^2 r\rho xyz\pi(x,y,z)$$
$$+\alpha^2\bar{r}(x-\rho)z\left[\pi(0,y,z)-\pi(0,0,z)\right]+\alpha^2 rxyz(\alpha x)^K\pi(0,y,z). \quad (13)$$

Substituting (12) in (13) yields

$$\left((1-\alpha x)\alpha x-\rho\alpha^2\left(\bar{r}z+rxyz\right)\right)\pi(x,y,z)$$
$$=(1-\alpha x)\alpha yx^{K+1}+(1-\alpha x)\alpha x\left(ry+\bar{r}\right)\frac{1-x^K}{1-x}$$
$$+z\rho\alpha(\alpha x)^{K+1}\left(y\left(\bar{r}-\alpha x\right)-\bar{r}\alpha\right)\times\left[\pi(\alpha^{-1},y,z)+\pi(0,y,z)/\rho\right]$$
$$+z\alpha^2\left(\bar{r}(x-\rho)+rxy(\alpha x)^K\right)\pi(0,y,z). \quad (14)$$

For each $i=1,2$, when $x=x_i(y,z)$, the term that multiplies $\pi(x,y,z)$ in the left hand side of equation (14) vanishes. Since $\pi(x,y,z)$ is polynomial in x and therefore analytic in x, the left hand side of (14) vanishes at $x=x_i(y,z)$. Thus by substituting x_i for x into (14), we obtain two equations (4) with two unknowns: $A(y,z)=\left[\pi(\alpha^{-1},y,z)+\pi(0,y,z)/\rho\right]$ and $B(y,z)=\pi(0,y,z)$. Equation (3) of the proposition, finally, follows from (14) with $x=\rho$ and since $q(y,z)=R_K\pi(\rho,y,z)$.

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