Reconstruction of Binary Matrices from Absorbed Projections

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Abstract. A generalization of the classical discrete tomography problem is considered: Reconstruct binary matrices from their absorbed row and column sums. We show that this reconstruction problem can be linked to a 3SAT problem if the absorption is characterized with the constant $\beta = ln(\frac{1+\sqrt{5}}{2})$.

Keywords: discrete tomography, reconstruction, absorption

1 Introduction

Let $A = (a_{ij})_{m\times n}$ be a binary matrix and let be $\beta \geq 1$. Then we can define the *absorbed row* and *column sums* of A $R_\beta(A)$ and $S_\beta(A)$, respectively, as

$$R_\beta(A) = R = (r_1, \ldots, r_m) \qquad \text{where} \qquad r_i = \sum_{j=1}^{n} a_{ij}\beta^{-j} \ , \quad i = 1, \ldots, m, \quad (1)$$

and

$$S_\beta(A) = S = (s_1, \ldots, s_n) \qquad \text{where} \qquad s_j = \sum_{i=1}^{m} a_{ij}\beta^{-i} \ , \quad j = 1, \ldots, n. \quad (2)$$

Then the *reconstruction problem of binary matrices with absorption* knowing the projections along horizontal and vertical lines can be posed as

Reconstruction $DA2D(\beta)$.

Instance: $\beta \geq 1$, m, n, $R \in \mathbb{N}^m$, *and* $S \in \mathbb{N}^n$

Task: *Construct a binary matrix* A *with size* $m \times n$ *such that*

$$R_\beta(A) \ = \ R \qquad \text{and} \qquad S_\beta(A) \ = \ S \ . \quad (3)$$

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If $\beta = 1$ then we have the classical reconstruction problem of binary matrices without absorption (as summaries see e.g. [1,2]). Other β values are suitable to describe the following model of the *emission discrete tomography.* Let us suppose that the discrete object represented by the binary matrix A is in an absorbing material having absorption coefficient μ. If we measure the horizontal and vertical projections of A, then we have the absorbed row and column sums, i.e., $R_\beta(A)$ and $S_\beta(A)$, where $\beta = e^\mu$. Some more explanation to the motivation of this problem see [3,4]

Select, for example, the mathematically interesting case $\beta = \beta_0$ where

$$\beta_0^{-1} \;=\; \beta_0^{-2} + \beta_0^{-3} \tag{4}$$

giving a solution

$$\beta_0 \;=\; \frac{1+\sqrt{5}}{2} \;. \tag{5}$$

In this paper we discuss the problem of reconstruction of binary matrices from their row and column sums in the case of absorption characterized with β_0. A necessary and sufficient condition of uniqueness in this class is published in [3,4].

In this paper we are going to connect this kind of reconstruction problem with 3SAT. The SAT and the different reconstruction problems have been connected already in [5,6].

2 β_0-Representation

Consider the row and column sums of the binary matrix A in the case of $\beta = \beta_0$:

$$r_i \;=\; \sum_{j=1}^{n} a_{ij}\beta_0^{-j} \;,\; i = 1, \ldots, m, \quad \text{and} \quad s_j \;=\; \sum_{i=1}^{m} a_{ij}\beta_0^{-i} \;,\; j = 1, \ldots, n. \tag{6}$$

Using the terminology of numeration systems we can say that the finite (binary) word $a_{i1}\cdots a_{in}$ is a (*finite*) *representation in base* β_0 (or a *finite* β_0-*representation*) of r_i for each $i = 1, \ldots, m$, and, similarly, $a_{1j}\cdots a_{nj}$ is a β_0-representation of s_j for each $j = 1, \ldots, n$. The equations (6) mean also that the row and column sums of A are nonnegative real numbers having a finite β_0-representation with n and m binary digits, respectively (including the eventually ending zeros).

Let B_k denote the set of nonnegative real numbers having a β_0-representation with k binary digits ($k > 1$), formally,

$$B_k \;=\; \left\{ \sum_{i=1}^{k} a_i\beta_0^{-i} \mid a_i \in \{0,1\} \right\} \;. \tag{7}$$

Then

$$r_i \in B_n,\ i = 1, \ldots, m, \qquad \text{and} \qquad s_j \in B_m,\ j = 1, \ldots, n, \tag{8}$$

are necessary conditions for the existence of a matrix A with

$$R_{\beta_0}(A) = (r_1, \ldots, r_m) \qquad \text{and} \qquad S_{\beta_0}(A) = (s_1, \ldots, s_n)\ . \tag{9}$$

2.1 Switching in β_0-Representations

The β_0-representation is generally nonunique, because there are binary words with the same length representing the same number. For example, on the base of (4) it is easy to check the following equality between the 3-digit-length β_0-representations

$$100 \ = \ 011\ . \tag{10}$$

As direct consequences of (10), it is easy to see that

$$\begin{array}{rcl} 100 & = & 011 \\ 10x_3 00 & = & 01x_3 11 \\ 10x_3 0x_5 00 & = & 01x_3 1x_5 11 \\ 10x_3 0x_5 0x_7 00 & = & 01x_3 1x_5 1x_7 11 \\ & \cdots & \end{array} \tag{11}$$

where $x_3, x_5, x_7, \cdots$ denote the positions where both β_0-representations have the same (but otherwise arbitrary) binary digit. (That is, such kind of transformation $1(0x)^{k-1}00 \rightarrow 0(1x)^{k-1}11$ ($k \geq 1$) between the subwords of the β_0-representations can be performed without changing the represented value and without changing the values in the positions indicated by x's.) The transformations described by (10) and (11) are called *switchings.*

It is proved that any finite β_0-representation of a number can be get from its any other β_0-representation by switchings.

Lemma 1. *[3] Let $a_1 \cdots a_k$ and $b_1 \cdots b_k$ be different, k-digit-length β_0-representations of the same number. Then $b_1 \cdots b_k$ can be get from $a_1 \cdots a_k$ by a finite number of switchings.*

Consequence. If $a_1 \cdots a_k$ and $b_1 \cdots b_k$ are different, k-digit-length β_0-representations of the same number, then there are positions $i, i+1, i+2$ ($1 \leq i \leq k-2$) such that there is a switching between $a_1 \cdots a_k$ and $b_1 \cdots b_k$ on these positions.

2.2 β_0-Expansion

The k-digit-length β_0*-expansion* is a particular k-digit-length β_0-representation that can be computed by the "greedy algorithm": Let $r \in B_k$, then its β_0-expansion $a_1 \cdots a_k$ is determined as

$$\begin{array}{l} r_0 := r\ , \\ a_i := \lfloor \beta_0 \cdot r_{i-1} \rfloor, \quad r_i \ := \ \{\beta_0 \cdot r_{i-1}\}, \qquad i = 1, \ldots, k\ , \end{array} \tag{12}$$

where $\lfloor . \rfloor$ and $\{.\}$ denote the integer and fractional, respectively, part of the argument. It is clear that the k-digit-length β_0-expansion of any number $r \in B_k$ is uniquely determined (it is not the case with the k-digit-length β_0-representations as we saw it in the previous subsection).

The finite β_0-expansion is characterized by the following property .

Proposition 1. *[3] Let $a_1, \ldots, a_k \in \{0,1\}$ $(k \geq 1)$. The word $a_1 \cdots a_k$ is the β_0-expansion of a number $r \in B_k$ if and only if it has the form*

$$a_1 \cdots a_k \ = \ TUV \ , \text{ where } T = 0 \cdots 0, \quad T = 1 \cdots 1, \quad or \quad T = \lambda \ , \qquad (13)$$

(λ denotes the empty symbol),

$$U \ = \ U_1 \cdots U_u, \quad u \geq 0, \qquad such\ that \qquad U_i \ = \ 10 \cdots 0, \quad i = 1, \ldots, u, \qquad (14)$$

and each U_i contains at least one 0,

$$V \ = \ 1 \qquad or \qquad V \ = \ \lambda \qquad (15)$$

and at least one of T, U, and V is not the empty symbol λ.

3 β_0-Representation and 3SAT Clauses

We are going to describe the β_0-representation by 3SAT expressions, that is, by Boolean expressions in conjunctive normal form with at most three literals in each clause. Let r be a real number having a k-digit long β_0-representation, $a_1 \cdots a_k$. Let $z_1, \ldots, z_k$ be Boolean variables and L be a Boolean function of $z_1, \ldots, z_k$, that is, $L = L(z_1, \ldots, z_k)$. We say that the Boolean values $a_1, \ldots, a_k$ satisfy L if $L(z_1 = a_1, \ldots, z_k = a_k)$ is true.

Now we are going to give the set of clauses, denoted by K, by which all k-digit length β_0-representations of any $r \in B_k$ can be described for any $k > 1$. Let $a_1 \cdots a_k$ the k-digit-length β_0-expansion of r. Then, by Proposition 1,

$$a_1 \cdots a_k = TUV \ ,$$

where T, U, and V are given by (13), and (14), respectively. Accordingly,

$$K = TT \cup UU \cup VV \ , \qquad (16)$$

where TT, UU, and VV denote the subsets of clauses describing the corresponding parts T, U, and V.

First, consider the non-constant part of the β_0-representations, $U = U_1 \cdots U_u$ $(u \geq 0)$. On the base of Lemma 1 we know that all β_0-representations of any $r \in B_k$ can be generated from the β_0-expansion of r by elementary switchings. Accordingly, the clauses UU have to describe the set of binary words generated from U_k by elementary switchings (see Fig. 1). The elementary switchings done in U can be classified into two classes according to the places of switchings:

(i) The switchings done in the positions of one U_i. (ii) The switchings done in the positions of U_i and U_{i+1}, i.e. the last 1 of U_i "overflows" into the first position of U_{i+1} as a consequence of switchings. There can be such a switching if the length of U_i is even and the length of U_{i+1} is not less than 3 (see the β_0-representations in Fig. 1 indicated by arrows).

There are two consequences of overflowing switchings: We have different clauses for U_i having even or odd length l_i and the sets of clauses of $U_i, i = 1, \cdots u$, are not completely independent.

The clauses of UU are given with the help of the Boolean variables $\gamma_j, \delta_j, \varphi_j, \psi_j$, and χ_j, $j = w_1, w_1 + 1, \cdots, w_u + l_u - 1$, i.e. for all the variables of UU. For each j exactly one of these variables has value 1 (see the clauses of $POSITIONS$ later). For this reason each binary word satisfying the clauses described by these variables can be represented in a 1-to-1 correspondence by a word of the alphabet $\{\gamma, \delta, \varphi, \psi, \chi\}$, indicating which variable has value 1 on that position. For example, $z_1 z_2 z_3 = \psi\gamma\delta$ means that $\gamma_1 = 0,\ \gamma_2 = 1,\ \gamma_3 = 0,\ \delta_1 = 0,\ \delta_2 = 0,\ \delta_3 = 1,\ \varphi_1 = 0,\ \varphi_2 = 0,\ \varphi_3 = 0,\ \psi_1 = 1,\ \psi_2 = 0,\ \psi_3 = 0,\ \chi_1 = 0,\ \chi_2 = 0,\ \chi_3 = 0$. The variables $\gamma_j, \delta_j, \varphi_j, \psi_j$, and χ_j describing the clauses of UU will be transformed to 0's and 1's as follows:

$$\varphi_j \Rightarrow a_j = 0,\ \delta_j \Rightarrow a_j = 1,\ \psi_j \Rightarrow a_j = 0,\ \gamma_j \Rightarrow a_j = 1,\ \chi_j \Rightarrow a_j = 1\ . \quad (17)$$

Continuing the previous example, then $\psi\gamma\delta = 011$.

U_i	U_{i+1}	corresponding representations	
100000	10000	$\delta\varphi\varphi\varphi\varphi\varphi$ | $\delta\varphi\varphi\varphi\varphi$	
011000	10000	$\psi\gamma\delta\varphi\varphi\varphi$ | $\delta\varphi\varphi\varphi\varphi$	
010110	10000	$\psi\gamma\psi\gamma\delta\varphi$ | $\delta\varphi\varphi\varphi\varphi$	
100000	01100	$\delta\varphi\varphi\varphi\varphi\varphi$ | $\psi\gamma\delta\varphi\varphi$	
011000	01100	$\psi\gamma\delta\varphi\varphi\varphi$ | $\psi\gamma\delta\varphi\varphi$	
010110	01100	$\psi\gamma\psi\gamma\delta\varphi$ | $\psi\gamma\delta\varphi\varphi$	
010101	11100	$\psi\gamma\psi\gamma\psi\gamma$ | $\chi\gamma\delta\varphi\varphi$	$\longleftarrow$
100000	01011	$\delta\varphi\varphi\varphi\varphi\varphi$ | $\psi\gamma\psi\gamma\delta$	
011000	01011	$\psi\gamma\delta\varphi\varphi\varphi$ | $\psi\gamma\psi\gamma\delta$	
010110	01011	$\psi\gamma\psi\gamma\delta\varphi$ | $\psi\gamma\psi\gamma\delta$	
010101	11011	$\psi\gamma\psi\gamma\psi\gamma$ | $\chi\gamma\psi\gamma\delta$	$\longleftarrow$

Fig. 1. All β_0-representations of $U_i U_{i+1}$ generated by elementary switchings and the corresponding representations with the variables $\gamma, \delta, \varphi, \psi$, and χ (when $l_i = 6$ and $l_{i+1} = 5$). The positions of U_i and U_{i+1} are separated by vertical lines. The "overflowing" 1's are indicated by χ in the rows with arrows.

Let $B(U_i)$ denote the set of (binary) sequences of U_i. Clearly,

$$B(U_i) = \{01\}^b 1 \{0\}^c, \quad (18)$$

where b and c are nonnegative integers such that $b+1+c=l_i$. Then the binary sequences of U_iU_{i+1}, $B(U_iU_{i+1})$, can be given as

$$B(U_iU_{i+1}) = \begin{cases} B(U_i)B(U_{i+1}), & \text{if } l_i \text{ is odd} \\ B(U_i)B(U_{i+1}) \cup \{01\}^{l_i/2}1B^{(0)}(U_{i+1}), & \text{if } l_i \text{ is even,} \end{cases} \tag{19}$$

where $B^{(0)}(U_{i+1})$ denotes the set of subsequences created from those sequences of $B(U_{i+1})$, where the first element is 0, by omitting just this first 0. For example, if $l_i = 6$ and $l_{i+1} = 5$ then $B(U_i) = \{100000, 011000, 010110\}$, $B(U_{i+1}) = \{10000, 01100, 01011\}$, and $B^{(0)}(U_{i+1}) = \{1100, 1011\}$.

We can describe these sequences with the letters $\gamma, \delta, \varphi, \psi$, and χ as follows. Corresponding to (18) and (19)

$$B(U_i) = \{\psi\gamma\}^a\delta\{\varphi\}^b, \tag{20}$$

$$B(U_iU_{i+1}) = \begin{cases} B(U_i)B(U_{i+1}), & \text{if } l_i \text{ is odd} \\ B(U_i)B(U_{i+1}) \cup \{\psi\gamma\}^{l_i/2}\chi B^{(\psi)}(U_{i+1}), & \text{if } l_i \text{ is even.} \end{cases} \tag{21}$$

According to (17) ψ, φ denote 0, γ, δ, and χ denote 1. B and $B^{(\psi)}$ are defined in these sequences analogously to (18) and (19). Examples of generated in this way and the corresponding β_0-representations are in Fig. 1.

The following sets of clauses will define a subword U_i.

$$DELTA = \bigwedge_{j=w_i}^{w_i+l_i-2} (\delta_j \Rightarrow \varphi_{j+1}) \wedge \bigwedge_{j=w_i+1}^{w_i+l_i-1} (\delta_j \Rightarrow \gamma_{j-1}) \wedge \bigwedge_{j=1}^{[\frac{l_i}{2}]} \overline{\delta_{w_i+2j-1}} \wedge (\varphi_{w_i+1} \Rightarrow \delta_{w_i}) \ .$$

The position of δ is crucial, because knowing this position all the elements succeeding δ can be computed as it is described in the first part of this rule and all elements preceding δ can be computed as it is described in the second part. δ cannot be on an even position in the subword U_i. The last part of $DELTA$ expresses that if there is a φ in the second position then there is a δ in the first one.

$$PHI = \bigwedge_{j=w_i+1}^{w_i+l_i-2} (\varphi_j \Rightarrow \varphi_{j+1}) \wedge \overline{\varphi_{w_i}} \ .$$

In other words, φ can be followed only by φ and φ cannot stand on the first position of the subword.

$$GAMMAPSI = \bigwedge_{j=w_i+2}^{w_i+l_i-1} (\gamma_j \Rightarrow \psi_{j-1}) \wedge \bigwedge_{j=w_i}^{w_i+l_i-2} (\psi_j \Rightarrow \gamma_{j+1}) \ .$$

The only predecessor of γ is ψ and the only successor of ψ is γ.

$$CHI = \bigwedge_{j=w_i+1}^{w_i+l_i-1} \overline{\chi_j} \wedge (\chi_{w_i} \Rightarrow \gamma_{w_i+1}) \; .$$

χ can stand only on the first position.

$$GAMMA = \bigwedge_{j=w_i}^{w_i+l_i-1} (\gamma_j \Rightarrow \overline{\varphi_{j+1}}) \; .$$

γ cannot be followed by φ.

$$POSITIONS = \bigwedge_{j=1}^{[\frac{l_i}{2}]} (\varphi_{w_i+2j} \vee \gamma_{w_i+2j}) \wedge (\delta_{w_i} \vee \psi_{w_i} \vee \chi_{w_i}) \wedge \bigwedge_{j=1}^{[\frac{l_i}{2}]} (\delta_{w_{i+2j-1}} \vee \psi_{w_{i+2j-1}} \vee \varphi_{w_{i+2j-1}}) \; .$$

On an even position in a subword can be φ or γ, on the first position in the subword can stand δ, ψ, or χ, and on odd positions in the subword can stand δ, ψ, or φ. These are the only clauses containing 3 variables.

$$EVEN = (\gamma_{w_i+l_i-1} \Rightarrow \chi_{w_i+l_i}) \; .$$

Actually $w_i + l_i = w_{i+1}$, the first element of the subword U_{i+1}. This means that a subword with even length can influence the next subword. In this case the first element is a χ followed by γ.

$$ODD = \overline{\chi_{w_i+l_i}} \wedge \overline{\psi_{w_i+l_i-1}} \; .$$

A subword with odd length cannot influence the next subword, this means that the first element of the next subword cannot be χ and the last element cannot be ψ.

$$DISJ = \bigwedge_{j=1}^{l_i} (A_j \Rightarrow \overline{B_j}), \text{ for symbols } A, B \in \{\varphi, \psi, \gamma, \delta, \chi\}, \text{ where } A \neq B \; .$$

The clauses mean that exactly one of the variables φ, ψ, γ, δ, and χ has the value 1, for each $j = 1, \cdots, l_i$.

The clauses for a subword U_i. Knowing the length of the subword U_i we can construct a corresponding 3SAT expression:

$$K_i = \begin{cases} DELTA \wedge PHI \wedge GAMMAPSI \wedge CHI \\ \wedge GAMMA \wedge POSITIONS \wedge ODD, & \text{if } l_i \text{ is odd} \\ \\ DELTA \wedge PHI \wedge GAMMAPSI \wedge CHI \\ \wedge GAMMA \wedge POSITIONS \wedge EVEN, & \text{if } l_i \text{ is even} \end{cases} \tag{22}$$

The clauses describing UU. Let $\Gamma = (\gamma_1, \cdots, \gamma_k)$, $\Delta = (\delta_1, \cdots, \delta_k)$, $\Phi = (\varphi_1, \cdots, \varphi_k)$, $\Psi = (\psi_1, \cdots, \psi_k)$, and $X = (\chi_1, \cdots, \chi_k)$ be the vectors of Boolean variables. Then $UU = UU(r; \Gamma, \Delta, \Phi, \psi, X)$ is defined as follows:

$$UU = \bigwedge_{i=1}^{u} K_i \ .$$

The clauses describing TT and VV. In these clauses the same variables are as used in UU. Since the subwords corresponding to T and V have constant values in each β_0-representation of the same $r \in B_k$, the clauses describing these parts are

$$TT = \begin{cases} \gamma_1 = \cdots = \gamma_{l_t} = 0, \delta_1 = \cdots = \delta_{l_t} = 0, \\ \varphi_1 = \cdots = \varphi_{l_t} = 0, \psi_1 = \cdots = \psi_{l_t} = 0, & \text{if } T = 0 \cdots 0; \\ \chi_1 = \cdots = \chi_{l_t} = 1, \\ \\ \gamma_1 = \cdots = \gamma_{l_t} = 0, \delta_1 = \cdots = \delta_{l_t} = 1, \\ \varphi_1 = \cdots = \varphi_{l_t} = 0, \psi_1 = \cdots = \psi_{l_t} = 0, & \text{if } T = 1 \cdots 1; \\ \chi_1 = \cdots = \chi_{l_t} = 0, \\ \\ \phi, & \text{if } T = \lambda \ , \end{cases} \tag{23}$$

and

$$VV = \begin{cases} \gamma_k = 0, \delta_k = 1, \varphi_k = 0, \psi_k = 0, \chi_k = 0, & \text{if } V = 1; \\ \phi, & \text{if} V = \lambda \ , \end{cases} \tag{24}$$

The clauses describing K. As we saw TT, UU, VV, and so K are defined with the help of r, Γ, Δ, Φ, Ψ, and X, i.e.,

$$K = K(r; \Gamma, \Delta, \Phi, \Psi, X) \ .$$

K is given by (16) explicitly.

Theorem 1. *Let $r \in B_k$ and $a_1, \ldots, a_k$ be a binary word. $a_1, \ldots, a_k$ is a β_0-representations of r if and only if there are vectors $\Gamma, \Delta, \Phi, \Psi$, and X of Boolean values such that $a_1, \ldots, a_k$ is transformed by these vectors by (17) and $K(r; \Gamma, \Delta, \Phi, \Psi, X)$ is true.*

Proof. Let $a_1 \cdots a_k$ be a k-digit-length β_0-representation of r. The corresponding word of $\gamma, \delta, \varphi, \psi$, and χ is uniquely determined on the base of the forms (20) and (21). It is easy to check that all clauses of K (i.e. TT, VV, $DELTA$, $\cdots$, $DISJ$) are satisfied by any word given by (20) and (21).

In order to prove the other direction, consider an arbitrary word W satisfying the clauses of K. W has the uniquely determined structure TUV, where T same as (13), V same as (15) and U is a word of $\gamma, \psi, \varphi, \delta$, and, χ. We have to show that U is a sequence of subsequences U_i, each of them satisfying (20) and (21). Knowing r we can determine the lengths l_i and positions of all U_i, $i = 1, \cdots u$.

Now we identify the subsequence U_i with length l_i starting from the end of U.

1. l_i is odd. According to $POSITIONS$, in the l_ith position can be δ, ψ, or φ.
   a. In the l_i position there is a δ. Now we have to prove that before δ there are only pairs of $\psi\gamma$. From $DELTA$ it follows that in the position $l_i - 1$ there is a γ. Let γ the position $2j$, before δ. From $GAMMAPSI$ it follows that in the position $2j - 1$ there is a ψ. From $POSITIONS$ it follows that in the position $2j - 2$ there can be φ or γ. If in the position $2j - 2$ is a φ, then according to PHI in the position $2j - 1$ should be φ which is a contradiction (from $DISJ$) . This means, that in the position $2j - 2$ is a γ, and let $j = j - 1$. This step has to be repeated till $j > 1$. If $j = 1$, i.e. in the second position is γ, then from $POSITIONS$ we have that in the first position can be δ, ψ, or χ. If in the first position is δ then from $DELTA$ follows that in the second position should be φ which is a contradiction. Conform to the equations (20) and (21), in the first position can be ψ or χ, in this last case there is an overflow.
   b. In the l_i position there is a ψ This in contradiction with ODD.
   c. In the l_i position there is a φ. From $POSITIONS$ it follows that in the previous position can be φ or γ. If it is γ, then from $GAMMA$ it follows that in the l_ith position cannot be φ which is a contradiction. This means, that in the position $l_i - 1$ is φ. If $l_i - 2 = 1$ then in this position is δ (from $DELTA$). If $l_i - 2 > 1$ then from $POSITIONS$ it follows that in the position $l_i - 2$ can be δ, ψ, or φ. If in the position $i - 2$ is δ then similar to Case a. we can prove that U_i satisfies (20) and (21). If in the position $l_i - 2$ is φ then similar to Case c. we can prove that U_i satisfies (20) and (21). If in the position $l_i - 2$ is ψ then from $GAMMAPSI$ follows that in the position $l_i - 1$ is γ and this is in contradiction with $DISJ$.
2. l_i is even. According to $POSITIONS$ in the position l_i can be φ or γ. If it is φ then using a similar deduction as in Case C. we can prove that U_i satisfies (20) and (21). If in the position l_i is γ, then conform $EVEN$ in the

next position is χ and conform CHI in the position $l_i + 2$ is γ, which means that U_i satisfies (20) and (21).

4 The Reconstruction Algorithm

In order to solve the reconstruction problem $DA2D(\beta_0)$ we express the β_0-representations of the absorbed row and column sums with 3SAT clauses. Boolean variables $\Gamma^{(h)} = (\gamma_{ij}^{(h)})_{m\times n}$, $\Delta^{(h)} = (\delta_{ij}^{(h)})_{m\times n}$, $\Phi^{(h)} = (\varphi_{ij}^{(h)})_{m\times n}$, $\Psi^{(h)} = (\psi_{ij}^{(h)})_{m\times n}$, and $X^{(h)} = (\chi_{ij}^{(h)})_{m\times n}$ are for describing relations of column sums (h stands for horizontal), and $\Gamma^{(v)} = (\gamma_{ij}^{(v)})_{m\times n}$, $\Delta^{(v)} = (\delta_{ij}^{(v)})_{m\times n}$, $\Phi^{(v)} = (\varphi_{ij}^{(v)})_{m\times n}$, $\Psi^{(v)} = (\psi_{ij}^{(v)})_{m\times n}$, and $X^{(v)} = (\chi_{ij}^{(v)})_{m\times n}$ for describing relations of column sums (v stands for vertical), Let, furthermore, $\Gamma_{i\cdot}^{(h)} = (\gamma_{i1}^{(h)}), \cdots, \gamma_{in}^{(h)})$ be the ith row of $\Gamma^{(h)}$, $i = 1, \cdots, m$ and $\Gamma_{\cdot j}^{(v)} = (\gamma_{1j}^{(v)}), \cdots, \gamma_{mj}^{(V)})^T$ be the jth column of $\Gamma^{(v)}$, $j = 1, \cdots, n$. $\Delta_{i\cdot}^{(h)}$, $\Phi_{i\cdot}^{(h)}$, $\Psi_{i\cdot}^{(h)}$, $X_{i\cdot}^{(h)}$, $\Delta_{\cdot j}^{(v)}$, $\Phi_{\cdot j}^{(v)}$, $\Psi_{\cdot j}^{(v)}$, and $X_{\cdot j}^{(v)}$, be defined similarly.

The clauses describing the rows and columns. Now we can describe a whole row of the discrete set to be reconstructed by the following subset of clauses:

$$K^{(h)}(r_i; \Gamma_{i\cdot}^{(h)}, \Delta_{i\cdot}^{(h)}, \Phi_{i\cdot}^{(h)}, \Psi_{i\cdot}^{(h)}, X_{i\cdot}^{(h)}) = TT \wedge UU \wedge VV,\ i = 1, \cdots, m\ ,$$

where TT, UU, and VV are defined in the previous section. All clauses describing the absorbed row sums are given by

$$\begin{aligned} L^{(h)} &= L^{(h)}(R, \Gamma^{(h)}, \Delta^{(h)}, \Phi^{(h)}, \Psi^{(h)}, X^{(h)}) \\ &= \bigwedge_{i=1}^{m} K^{(h)}(r_i; \Gamma_{i\cdot}^{(h)}, \Delta_{i\cdot}^{(h)}, \Phi_{i\cdot}^{(h)}, \Psi_{i\cdot}^{(h)}, X_{i\cdot}^{(h)})\ . \end{aligned} \tag{25}$$

Similarly, the columns can be described by

$$K^{(v)}(s_j; \Gamma_{\cdot j}^{(v)}, \Delta_{\cdot j}^{(v)}, \Phi_{\cdot j}^{(v)}, \Psi_{\cdot j}^{(v)}, X_{\cdot j}^{(v)}) = TT \wedge UU \wedge VV,\ j = 1, \cdots, n\ ,$$

and

$$\begin{aligned} L^{(v)} &= L^{(v)}(R, \Gamma^{(v)}, \Delta^{(v)}, \Phi^{(v)}, \Psi^{(v)}, X^{(v)}) \\ &= \bigwedge_{i=1}^{m} K^{(v)}(s_j; \Gamma_{\cdot j}^{(v)}, \Delta_{\cdot j}^{(v)}, \Phi_{\cdot j}^{(v)}, \Psi_{\cdot j}^{(v)}, X_{\cdot j}^{(v)})\ . \end{aligned} \tag{26}$$

The clauses describing the binary matrix. The last step is to define the connections between the Boolean matrices

$$\begin{aligned} CONN \;=\;& (\bigwedge_{i1,j1} \varphi^{(h)}_{i1,j1} \Rightarrow \overline{\gamma^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \varphi^{(h)}_{i1,j1} \Rightarrow \overline{\delta^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \varphi^{(h)}_{i1,j1} \Rightarrow \overline{\chi^{(v)}_{i1,j1}}) \wedge \\ & \wedge (\bigwedge_{i1,j1} \psi^{(h)}_{i1,j1} \Rightarrow \overline{\gamma^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \psi^{(h)}_{i1,j1} \Rightarrow \overline{\delta^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \psi^{(h)}_{i1,j1} \Rightarrow \overline{\chi^{(v)}_{i1,j1}}) \wedge \\ & \wedge (\bigwedge_{i1,j1} \gamma^{(h)}_{i1,j1} \Rightarrow \overline{\varphi^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \gamma^{(h)}_{i1,j1} \Rightarrow \overline{\psi^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \delta^{(h)}_{i1,j1} \Rightarrow \overline{\varphi^{(v)}_{i1,j1}}) \wedge \\ & \wedge (\bigwedge_{i1,j1} \delta^{(h)}_{i1,j1} \Rightarrow \overline{\psi^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \chi^{(h)}_{i1,j1} \Rightarrow \overline{\varphi^{(v)}_{i1,j1}}) \wedge (\bigwedge_{i1,j1} \chi^{(h)}_{i1,j1} \Rightarrow \overline{\psi^{(v)}_{i1,j1}}) \;. \end{aligned}$$

The 3SAT expression describing the whole discrete set is:

$$L^{(h)} \wedge L^{(v)} \wedge CONN \;. \tag{27}$$

That is, in order to solve the reconstruction problem $DA2D(\beta_0)$ we have to do the following steps:

1. Determine the β_0-expansions of r_i, $i = 1, \cdots, m$, and $j = 1, \cdots n$.
2. On the base of β_0-expansions give the 3SAT expression (27).
3. Solve the 3SAT problem using an efficient SAT solver (e.g. CSAT, see [7]).
4. If there is a solution of the 3SAT problem, give the binary matrix solution on the base of (17).

5 Discussion

A method is given to solve the reconstruction problem $DA2D(\beta_0)$, i.e., to reconstruct a binary matrix from it absorbed row and column sums, when the absorption can be represented by the special value β_0. It is shown that the problem $DA2D(\beta_0)$ can be transformed to a 3SAT expression such that if there is a solution of the 3SAT expression then it gives also a solution of the reconstruction problem (see Section 4).

It is a natural question that how this method can be extended to other values of β. We believe that this idea is specific and cannot be generalised directly to all possible values of β. However, it is relative easy to show that very similar results are true for β's having the property

$$\beta^{-1} = \beta^{-2} + \beta^{-3} + \cdots + \beta^{-l} \;,$$

where $l \geq 3$. Then the switchings can be described by similar relations as in (11), β-representations can be given similarly as in Section 3, and so the reconstruction problem can be reduced to a 3SAT problem in such cases.

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