Straight-Line Drawings on Restricted Integer Grids in Two and Three Dimensions
(Extended Abstract)*

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Abstract. This paper investigates the following question: Given an integer grid ϕ, where ϕ is a proper subset of the integer plane or a proper subset of the integer 3d space, which graphs admit straight-line crossing-free drawings with vertices located at the grid points of ϕ? We characterize the trees that can be drawn on a two dimensional $c \cdot n \times k$ grid, where k and c are given integer constants, and on a two dimensional grid consisting of k parallel horizontal lines of infinite length. Motivated by the results on the plane we investigate restrictions of the integer grid in 3 dimensions and show that every outerplanar graph with n vertices can be drawn crossing-free with straight lines in linear volume on a grid called a prism. This prism consists of $3n$ integer grid points and is universal – it supports all outerplanar graphs of n vertices. This is the first algorithm that computes crossing-free straight line 3d drawings in linear volume for a non-trivial family of planar graphs. We also show that there exist planar graphs that cannot be drawn on the prism and that the extension to a $n \times 2 \times 2$ integer grid, called a box, does not admit the entire class of planar graphs.

1 Introduction

This paper deals with crossing-free straight-line drawings of planar graphs in 2 and 3 dimensions. Given a graph G, we constrain the vertices in a drawing of G to be located at integer grid points and aim at computing drawings whose area/volume is small. A rich body of literature has been published on such straight-line drawings in 2d. Typically, these papers focus on lower bounds on

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the area required by drawings of specific classes of graphs and on the design of algorithms that possibly match these lower bounds. A very limited list of mile-stone papers in this field includes the works by de Fraysseix, Pach, and Pollack [7,8] and by Schnyder [20] who independently showed that every n-vertex triangulated planar graph has a crossing-free straight-line drawing such that the vertices are at grid points, the size of the grid is $O(n) \times O(n)$, and that this is worst case optimal; the work by Kant [16,17], Chrobak and Kant [3], Schnyder and Trotter [21], Felsner [12] and Chrobak, Goodrich, and Tamassia [4] who studied convex grid drawings of triconnected planar graphs in an integer grid of quadratic area; and the many papers proving that linear or almost-linear area bounds can be achieved for classes of trees, including the result by Garg, Goodrich and Tamassia [13] and the result by Chan [2]. Summarizing tables and more references can be found in the book by Di Battista, Eades, Tamassia, and Tollis [9].

While the problem of computing small-sized crossing-free straight-line drawings in the plane has a long tradition, the 3d counterpart has received less attention. Chrobak, Goodrich, and Tamassia [4] gave an algorithm for constructing 3d convex drawings of triconnected planar graphs with $O(n)$ volume and non-integer coordinates. Cohen, Eades, Lin and Ruskey [6] showed that every graph admits a straight-line crossing-free 3d drawing on an integer grid of $O(n^3)$ volume, and proved that this is asymptotically optimum. Calamoneri and Sterbini [1] showed that all 2-, 3-, and 4-colourable graphs can be drawn in a 3d grid of $O(n^2)$ volume with $O(n)$ aspect ratio and proved a lower bound of $\Omega(n^{1.5})$ on the volume of such graphs. For r-colourable graphs, Pach, Thiele and Tóth [18] showed a bound of $\theta(n^2)$ on the volume. Garg, Tamassia, and Vocca [14] showed that all 4-colorable graphs (and hence all planar graphs) can be drawn in $O(n^{1.5})$ volume and with $O(1)$ aspect ratio but using a grid model where the coordinates of the vertices may not be integral.

In this paper we study the problem of computing drawings of graphs on integer 2d or 3d grids that have small area/volume. The area/volume of a drawing Γ is measured as the number of grid points contained in or on a *bounding box* of Γ, *i.e.* the smallest axis-aligned box enclosing Γ. Note that along each side of the bounding box the number of grid points is one more than the actual length of the side.

We approach the drawing problem with the following point of view: Instead of "squeezing" a drawing onto a small portion of a grid of unbounded dimensions, we assume that a grid of *specified* dimensions is given and we consider which graphs have drawings that fit that restricted grid. For example, it is well-known that there are families of graphs that require $\Omega(n^2)$ area to be drawn in the plane, the canonical example being a sequence of $n/3$ nested triangles (see [8, 5,20]). Such graphs can be drawn on the surface of a 3 dimensional triangular prism of linear volume and using integer coordinates. Thus a natural question is whether there exist specific restrictions of the 3d integer grid of linear volume that can support straight-line crossing-free drawings of meaningful families of graphs. For planar graphs the best known results for 3 dimensional crossing-free straight-line drawings on an integer grid are by Calamoneri and Sterbini [1] who show $O(n^2)$ volume for general planar graphs and by Eades, Lin and Ruskey [6] who show $O(n \log n)$ volume for trees. By following the above described approach we have been able to design the first algorithms that draw significant families of planar graphs in an integer 3d grid requiring only $O(n)$ volume.

The main contributions of this paper are combinatorial characterizations and negative results on the drawability of graphs on 2d and 3d restricted integer grids and new drawing algorithms for some classes of graphs. An overview of the results is as follows:

- We characterize those trees that can be drawn on an integer restricted 2d grid consisting of k consecutive infinite horizontal grid lines (for a given positive integer k) and where edges can connect either collinear vertices or vertices that are one unit apart in their y-coordinates; we also present a linear time recognition and drawing algorithm for this class of trees.
- We study those trees that can be drawn on an integer restricted 2d grid of dimensions $c \cdot n \times k$, where c and k are two given integer positive constants and n is the number of vertices of the tree. In this case we relax one of our drawing constraints to allow adjacent vertices to be more than one unit apart in their y-coordinates, and show that this family of drawable trees coincides with those studied within the drawing convention of the previous item. A consequence of our characterization is that for any given k and c there always exist some trees that are not drawable on the $c \cdot n \times k$ grid.
- Motivated by the results on restricted integer 2d grids we explore the capability of restricted integer 3d grids for supporting linear volume drawings of graphs. In particular, we focus on two types of 3d integer grids to be defined subsequently, both having linear volume, called the *prism* and the *box*. We show that all outerplanar graphs can be drawn in linear volume on a prism. Note that this is the first result on 3d straight-line drawings of a significant class of planar graphs that achieves linear volume with integer coordinates.
- We further explore the class of graphs that can be drawn on a prism by asking whether the prism is a *universal* integer 3d grid for all planar graphs. We answer this question in the negative by exhibiting examples of planar graphs that cannot be drawn on a prism. We also investigate the relationship between prism-drawable and Hamiltonian graphs.
- We extend our study to box-drawability and present a characterization of the box-drawable graphs. While the box would appear to be a much more powerful grid than the prism, we prove that not all planar graphs are box-drawable.

2 Preliminaries

We assume familiarity with basic graph drawing, and computational geometry terminology; see for example [19,9]. Since in the remainder of the paper we shall study crossing-free straight-line drawings of planar graphs, from now on we shall simply talk about "graphs" to mean "planar graphs" and about "drawings" to mean "crossing-free straight line drawings". We use the terms "vertex" and "edge" for both the graph and its drawing.

We will draw graphs such that vertices are located at integer grid points. The dimensions of a grid are specified as the number of different grid points along each side of a bounding box of the grid. In 2 dimensions, a $p \times q$ grid consists of p grid points along the x-axis and of q grid points along the y-axis. In 3 dimensions, a $p \times q \times r$ grid consists of p grid points along the x-axis, q grid points along the y-axis, and r grid points along the z-axis; p, q and r are referred to as the x-, y-, and *z-dimension* of the grid, respectively.

We shall deal with the following grids and drawings.

- A 2d 1-*track* (or simply a *track*) is a $\infty \times 1$ grid; a 1-*track* drawing of a graph G is a drawing of G where the vertices are at distinct grid points of the track.
- A 2d *strip* is a $\infty \times 2$ grid; note that a strip consists of two tracks. A *strip drawing* of a graph G is a drawing of G with the vertices located at distinct grid points of the strip and the edges either connect vertices on the same track or connect vertices on different tracks.
- Let k be a given positive integer value. A 2d *k-track* grid is a $\infty \times k$ grid consisting of k consecutive parallel tracks. A *k-track drawing* of a graph G is a drawing of G where the vertices are at distinct grid points of the k-track and edges are only permitted between vertices that are either on the same track or that are one unit apart in their y-coordinates.
- Let k and c be two given positive integer values. In a *$c \cdot n \times k$-grid drawing* of a graph G the vertices are located at distinct grid points and the edges can connect any pair of vertices on that grid.
- We will also study two different types of $n \times 2 \times 2$ grids. A *box* is a $n \times 2 \times 2$ grid where each side of the bounding box is also a grid line. Therefore, a box has four tracks which lie on two parallel planes and are one grid unit apart from each other. A *prism* is a $n \times 2 \times 2$ grid obtained by removing a track from a box. Figure 1 shows an example of a box of size $4 \times 2 \times 2$ and an example of a prism.

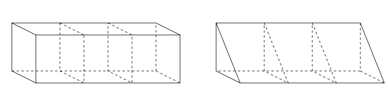

Fig. 1. A Box and a Prism

Note that k-track drawings differ from the so-called k-level drawings (see, e.g. [15]) as in a k-track drawing (consecutive) vertices on the same track are permitted to be joined by an edge and the given graph is undirected.

Let ϕ be one of the grids defined above. We say that a graph G is *ϕ drawable* if G admits a ϕ drawing Γ where each vertex is mapped to a distinct grid point of ϕ.

Property 1. A graph is 1-track drawable if and only if it is a simple path

While in a k-track drawing no edge can connect vertices that are on non-consecutive tracks, in a $c \cdot n \times k$-grid this is allowed. As the following property shows, this difference has immediate consequences on the families of k-track drawable and $c \cdot n \times k$-grid drawable graphs, and the graph K_4 provides an example.

Property 2. Let c, k be two positive integers. There exist graphs with n vertices that are $c \cdot n \times k$-grid drawable but are not k-track drawable.

3 Two-Dimensional Restricted Grids

In this section we characterize the family of k-track drawable trees and the family of $c \cdot n \times k$-grid drawable trees. In contrast to Property 2, we show that these two families of trees are actually the same. We also give linear time recognition and drawing algorithms for these trees. The approach is as follows. We first study strip-drawable trees, then we extend the result to the k-track grid, and finally we show that the result also holds on a $c \cdot n \times k$ grid.

3.1 Strip-Drawable Trees

By Property 1 we have that all paths are strip-drawable, since they are in fact 1-track drawable. A tree is defined as *2-strict* if it contains a vertex of degree greater than or equal to three. An immediate consequence of Property 1 is the following.

Property 3. A 2-strict tree is not 1-track drawable.

An edge is defined as a *core edge* if its removal results in two 2-strict components. For an edge $e = (u, v)$, we refer to the two subtrees resulting from its removal as T_u and T_v.

Lemma 1. *Core edges are connected.*

Proof. (sketch) Let $e_1 = (u, v)$ and $e_2 = (w, x)$ be 2 core edges and consider any edge on the tree between them. Each such edge receives one 2-strict component from T_u and one from T_x and thus must be core.

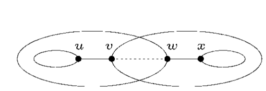

Fig. 2. Core edges are connected

Lemma 2. *A tree is strip drawable if and only if its core edges form a path.*

Proof. (sketch) (⇒) (by contradiction) By the previous lemma, if the core edges do not form a path, then there is a vertex v with at least 3 incident core edges $(v, a), (v, b), (v, c)$ – see Figure 3. If the subtrees T_a, T_b, T_c are drawable then by Property 3 their associated drawings $\Gamma_a, \Gamma_b, \Gamma_c$ each require 2 tracks. There is no location for v that permits a crossing-free connection to all 3 subdrawings.

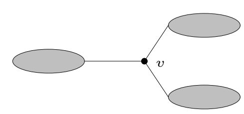

Fig. 3. T is not strip-drawable if the core edges are not a path

(⇐) If the core edges form a path of at least 2 vertices, then draw them consecutively on track t_1. Consider an arbitrary non-core edge $e = (u, v)$ with u on track t_1. Since e is non-core, T_v must *not* be 2-strict and is thus 1-track

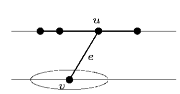

Fig. 4. Drawing a tree on a strip

drawable. Therefore v can be placed on track t_2 with the drawing of T_v also on the same track as in Figure 4.

There is one degenerate case to consider. If there are no core edges (*i.e.* a path of length 0), then either the tree has no vertex of degree 3 and is in fact 1-track drawable, or there exists at most one vertex v with neighbours $w_1, w_2, ...w_k$ and each T_{w_i} is *not* 2-strict. Each of the subtrees can thus be drawn on track t_2 and v on track t_1 as in Figure 5.

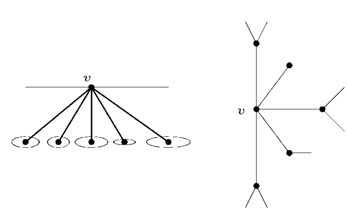

Fig. 5. A degenerate core path

Lemma 3. *Let T be a tree with n vertices. There exists an $O(n)$-time algorithm that recognizes whether T is strip-drawable and, if so, computes a strip drawing of T.*

Proof. (sketch) Note that a tree is 2-strict iff it has more than 2 leaves; thus counting leaves is the crucial operation. First the core edges must be established and then the path condition on the core edges checked.

With each edge $e = (u, v)$ we associate 2 counters: l_u will be the number of leaves in T_u, and l_v will be the number of leaves in T_v. Let l be the number of leaves in the entire tree T. Then clearly $l_u + l_v = l$. By the previous observation, e is a core edge iff both l_u and $l_v > 2$.

Choose an arbitrary non-leaf vertex r as a root. Each vertex v reports the number of leaves in the subtree below it to its parent u – thus establishing l_u for the edge (u, v) and hence l_v. If v has no children then it is a leaf and reports 1. A simple recursive function can be used to implement this counting step in linear time.

Finally, checking that the core edges form a path is also easily accomplished in linear time as is the production of a strip drawing.

Theorem 1. *A tree T with n vertices is strip drawable if and only if its core edges form a path. Furthermore, there exists an $O(n)$-time algorithm that determines whether T is strip drawable and, if so computes a strip drawing of T.*

3.2 k-Track and $c \cdot n \times k$-Grid Drawable Trees

The results of Theorem 1 can be extended to k tracks by generalizing some of the concepts of the previous section. A tree is *k-strict* if it contains a vertex adjacent to at least three subtrees that are $(k-1)$-strict. An edge is a *k-core edge* if its removal results in two k-strict components. The proofs of some of the following lemmas are similar to the case when $k = 2$ and are omitted in this extended abstract.

Property 4. A k-strict tree is not $(k-1)$-track drawable.

Lemma 4. *k-core edges are connected.*

Lemma 5. *A tree is k-track drawable if and only if the k-core edges form a path.*

The following lemma shows that the families of k-track drawable trees and $c \cdot n \times k$-grid drawable trees coincide.

Lemma 6. *Let c, k be two positive integer. A tree T is $c \cdot n \times k$-grid drawable if and only if it is k-track drawable.*

Proof. (sketch) Since a k-track drawing is a restricted form of a $c \cdot n \times k$ grid drawing where c is any given positive integer constant, it suffices to show that if T has a $c \cdot n \times k$ grid drawing, then the k-core edges form a path which can be shown by contradiction – the proof follows the form of Lemma 2.

Theorem 2. *Let T be a tree with n vertices and let c and k be two positive integer constants. The following three statements are equivalent:*

1. *T is $c \cdot n \times k$-grid drawable.*
2. *T is k-track drawable.*
3. *the k-core edges of T form a path.*

Furthermore, there exists an $O(n)$-time algorithm that determines whether T satisfies the above conditions and computes a k-track and a $c \cdot n \times k$-grid drawing of T.

One consequence of the previous theorem is the existence of non-drawable trees – ternary trees for example provide the critical strict components.

Corollary 1. *The complete ternary tree of height k is not drawable on a $c \cdot n \times (k-1)$ grid, for any positive integer c.*

4 Three-Dimensional Drawings of Outerplanar Graphs

In this section we show that all outerplanar graphs are prism-drawable by providing a linear time algorithm that computes this drawing. This is the first known three-dimensional straight-line drawing algorithm for the class of outerplanar graphs that achieves $O(n)$ volume on an integer grid.

A high level description of our drawing algorithm, called `Algorithm Prism Draw`, is as follows. Let G be an outerplanar graph with a specified *outerplanar embedding*, i.e. a circular ordering of the edges incident around each vertex such that all vertices of G belong to the external face. `Algorithm Prism Draw` computes a prism drawing of G by executing two main steps. Firstly a 2d drawing of G is computed on a grid that consists of $O(n)$ horizontal tracks and such that adjacent vertices are at grid points whose y-coordinates differ by at most one by visiting G in a breadth-first fashion. Secondly, the drawing is "wrapped" on the faces of a prism by folding it along the tracks.

`Algorithm Prism Draw`

input: An outerplanar graph G with a given outerplanar embedding.
output: A prism drawing of G.

`Step 1. The 2d Drawing Phase:` A 2d grid drawing Γ of G where vertices are assigned to distinct tracks is computed as follows.

- Add a dummy vertex d on the external face and an edge connecting d to an arbitrary vertex v.

– mark d; i:=0; currx:=0
– draw v on track t_0 by setting X(v) := currx; Y(v) :=i
– currx := currx +1
– mark v
– while there are unmarked vertices of G do
  • visit the vertices on track t_i from left to right and for each encountered vertex u do
    ∗ let w be a marked neighbour of u in G
    ∗ visit the neighbours of u in counterclockwise order starting from w, and for each encountered vertex r such that r is unmarked do
      · draw r on track t_{i+1} by setting X(r) := currx; Y(r) := i+1
      · currx := currx +1
      · mark r
  • i := i+1

`Step 2: The 3d Wrapping Phase:` A prism drawing Γ' is obtained by folding Γ along its tracks as follows.
– for each vertex v of Γ define its coordinates $X'(v)$, $Y'(v)$ and $Z'(v)$ in Γ' by setting:
  • $X'(v) := X(v)$
  • if $Y(v) = 0, 1 \ mod\ 3$ then $Y'(v) := 0$, else $Y'(v) := 1$
  • if $Y(v) = 0, 2 \ mod\ 3$ then $Z'(v) := 0$, else $Z'(v) := 1$

`End of Algorithm Prism Draw`

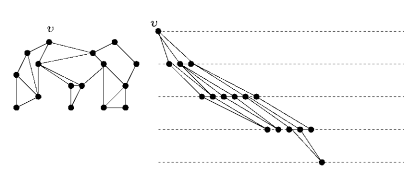

Fig. 6. An outerplanar graph drawn by `Step 1` of `Algorithm Prism Draw`.

Figure 6 shows an example of the output of Step 1 of the algorithm. The correctness of `Algorithm Prism Draw` is established via the following observations:

– No two vertices of Γ are assigned the same X-coordinate.
– Every vertex assigned to track t_{i+1} has a neighbour on track t_i, for $i \geq 0$.
– No pair of edges between vertices on adjacent tracks intersect.

Theorem 3. *Every outerplanar graph G with n vertices admits a crossing-free straight line grid drawing in 3 dimensions in optimal $O(n)$ volume. Furthermore, there exists an algorithm that computes such a drawing of G in $O(n)$ time and with the vertices of G drawn on the grid points of a prism.*

5 Prism-Drawable Graphs

Since by Theorem 3 all outerplanar graphs can be drawn on a prism, it is natural to investigate the class of graphs that are prism-drawable. For example, note that the family of planar graphs consisting of a sequence of nested triangles and that are known to require $\Omega(n^2)$ area in the plane, can be drawn on the prism (and thus have $O(n)$ volume). It is also clear that since any drawing on the prism can be augmented by edges to form a convex polytope, by the theorem of Steinitz only planar graphs are prism-drawable. In this section, we give a characterization for the prism-drawable graphs and show that not all planar graphs are in this class. Figure 7 shows a graph, and its prism drawing.

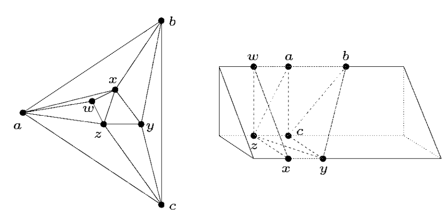

Fig. 7. A prism-drawable graph G and its drawing

5.1 Characterization of Prism-Drawable Graphs

An essential prerequisite of our characterization of prism-drawable graphs, is the study of the strip-drawable graphs since a prism effectively consists of three strips. We define a *spine* in a graph as a sequence of adjacent vertices with no chord. The characterization of strip-drawable graphs proposed in this section notes that in a strip drawing, there must exist 2 potential spines and that edges incident to vertices on both spines must not cross. This formulation will be generalized in subsequent sections to larger grid sets.

Theorem 4. *A graph G is strip-drawable iff it is possible to augment G with edges to produce a graph G' which contains two pairs of adjacent vertices r, b and r', b' and there exists a spine from r to r', a spine from b to b' with all vertices of G on the two spines and if there exists an edge (r_i, b_j) then there are no edges of the form (r_k, b_l) with ($k < i$ and $l > j$) or ($k > i$ and $l < j$).*

Proof. (sketch) Refer to Figure 8. Given a strip drawing, it is clear that edges can be added along the 2 tracks to form the 2 spines, and to add an edge between the leftmost pair on the two tracks and between the rightmost pair on the 2 tracks. Since no pair of edges intersect, the non-crossing conditions are maintained.

Given an augmented graph with the required properties, a valid strip drawing is obtained by drawing each spine on a separate track, and the edge conditions

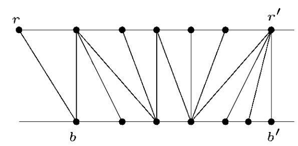

Fig. 8. Strip characterization

ensure there are no crossings. Vertices on each spine can be placed at consecutive integral X-coordinates thus ensuring that the strip is at most of length n.

The characterization of prism drawable graphs generalizes Theorem 4 to 3 dimensions, namely it must be possible to augment a given graph to obtain 3 spines with 2 lids (3 cycles) and between each pair of spines the non-crossing condition on edges must hold. We omit the proof.

Theorem 5. *A graph G is prism-drawable iff it is possible to augment G with edges to produce a graph G' which contains two three cycles r, b, g and r', b', g' and there exists a spine from r to r', a spine from b to b', a spine from g to g' with all vertices of G on the three spines and for each pair of spines $x \rightarrow x'$ and $y \rightarrow y'$ ($x, y = r, b, g,\ x \neq y$) , if (x_i, y_j) is an edge, then there are no edges of the form (x_k, y_l) with ($k < i$ and $l > j$) or ($k > i$ and $l < j$).*

5.2 Prism-Drawability and Planarity

Define a graph G as *strictly prism-drawable* if it is prism-drawable and all prism drawings of G have at least one edge on each facet of the prism. Note that K_4 is strictly prism-drawable. Our goal is to show the existence of series-parallel graphs that are not prism-drawable, and the graph P of Figure 9 is a more useful strictly prism-drawable graph for this purpose.

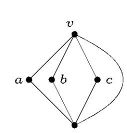

Fig. 9. A strictly prism-drawable graph P

Lemma 7. *The graph P in Figure 9 is strictly prism-drawable.*

Lemma 8. *Let G be a 1-connected graph that has a cut vertex whose removal separates the graph into h strictly prism-drawable components ($h \geq 3$); then G is not prism-drawable.*

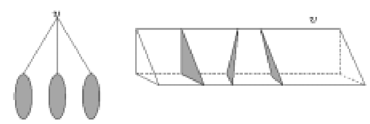

Fig. 10. A non-prism-drawable graph

Proof. (sketch) Figure 10 outlines the necessary construction. Consider a prism drawing Γ_i of G_i. Γ_i has a 3-cycle which defines a plane that intersects all three facets of the prism, because G_i is strictly prism-drawable. Thus, each Γ_i slices the prism into $h+1$ slices. Now there is no location for v that permits it to be connected crossing-free to all Γ_i without crossing at least one 3-cycle.

It is not critical that v be a cut vertex in the previous proof – any vertex connected to 3 or more strictly prism-drawable subgraphs will suffice and thus the previous lemma generalizes; however we omit the stronger result in this extended abstract.

Theorem 6. *There exist series-parallel graphs that are not prism-drawable.*

Proof. (sketch) The graph in Figure 11 has the required properties.

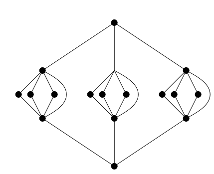

Fig. 11. A series-parallel graph that is not prism-drawable

Theorem 7. *Not all maximal planar graphs are prism-drawable. Also, the family of maximal planar prism drawable graphs is a proper subset of the family of Hamiltonian planar graphs.*

Proof. (sketch) It is not hard to show that all prism drawings of maximal planar graphs are Hamiltonian. Since there exist maximal planar graphs which are not Hamiltonian it immediately follows that not all maximal planar graphs are prism-drawable.

It is also possible to use the spine characterization more directly to produce planar graphs that are not prism-drawable – even Hamiltonian.

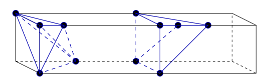

Fig. 12. K_5 and $K_{3,3}$ drawn on a box

6 Box-Drawable Graphs

The family of box-drawable graphs is clearly a superset of the class of prism-drawable graphs. Furthermore, there exist non-planar graphs that are box-drawable; for example K_5 and $K_{3,3}$ are box-drawable as shown in figure 12. Note however that K_6 is not box-drawable and we will show that there exist planar graphs (and even series-parallel graphs) that are not box-drawable. We refer to the 6 possible strips on which edges can be drawn as the *facets* of the box although two such facets appear inside the bounding box.

Theorem 8. *A graph G is box-drawable iff it is possible to augment G with edges to produce a graph G' which contains two four cycles r, b, g, o and r', b', g', o' and there exists a spine from r to r', a spine from b to b', a spine from g to g', and a spine from o to o' with all vertices of G on the four spines and for each pair of spines $x \to x'$ and $y \to y'$, if (x_i, y_j) is an edge, then there are no edges of the form (x_k, y_l) with ($k < i$ and $l > j$) or ($k > i$ and $l < j$).*

Proof. (sketch) As in the previous characterizations for the strip and prism, a box drawing can be augmented to complete the spines and the 6 facets must form valid strips. In this case there appears to be an extra condition necessary to ensure that the 2 pairs of strips on the 2 diagonally opposite spines do not intersect, however these 2 pairs of spines can be separated to ensure no diagonal crossings appear in the interior of the box.

Examples of graphs that are not box-drawable can be constructed using Theorem 8.

Theorem 9. *Not all planar graphs are box-drawable.*

7 Conclusions, Extensions, and Open Problems

In this paper we showed that all outerplanar graphs can be drawn in linear volume on a prism. We gave efficient characterizations of the trees that are drawable in 2 dimensions on an $n \times k$ grid. Classes of planar graphs that are not prism-drawable nor box-drawable were also provided. There remain several interesting problems and directions for further research.

1. Can all outerplanar graphs be drawn in $o(n^2)$ area on a 2d integer grid? Does there exist a 2d universal grid set of $o(n^2)$ area that supports all outerplanar graphs?

2. Characterize the graphs drawable on an $n \times k$ grid.
3. Can all planar graphs be drawn in $O(n)$ volume on a 3d integer grid? Does there exist a 3d universal grid set of $O(n)$ volume that supports all planar graphs?
4. **Aspect Ratio**: Our results about linear volume come at the expense of aspect ratio. Is it possible to achieve both linear volume and $o(n)$ aspect ratio for outerplanar graphs? We conjecture that it is in fact not possible in 2d to simultaneously attain $O(n)$ area and $O(1)$ aspect ratio for some classes of planar graphs.

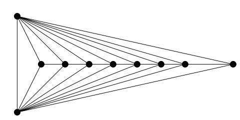

Fig. 13. A graph S_n with poor aspect ratio?

Conjecture 1. There is no fixed constant k for which the family of graphs S_n (in Figure 13) can be drawn in a 2d integer grid of size $k\sqrt{n} \times \sqrt{n}$.

Note that the graph S_n can be drawn on a $n \times 3$ grid (and hence in $O(n)$ area but with *linear* aspect ratio).

5. **Grid Drawings and Pathwidth**: The notion of the *pathwidth* of a graph has been well-studied in the graph theory literature – for definitions, see for example the book by Diestel [10]. A connection between pathwidth and layered graphs was established in [11]. It is not difficult to show that for trees there is a strong connection between pathwidth and grid drawings as summarized in the following propositions.

Proposition 1. *For a tree T, pathwidth(T)=*$\min_k$ *(T is drawable on a $n \times k$ grid).*

Proposition 2. *If G is a planar graph then pathwidth(G) $\leq$* $\min_k$ *(G is drawable on k-tracks).*

We have not been able to determine whether there are planar graphs with *pathwidth*(G) $<$ $\min_k$ (G is drawable on a k-strip). If not, a linear time algorithm for recognizing graphs of pathwidth $\leq k$ would provide us with a linear time algorithm to recognize k-track drawable graphs. We believe that 2-strip drawable graphs can be recognized in polynomial time, however the complexity of the recognition of k-strip drawable graphs for $k \geq 3$ remains open.

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