# The $(\sigma + 1)$-Edge-Connectivity Augmentation Problem without Creating Multiple Edges of a Graph

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**Abstract.** The unweighted *k-edge-connectivity augmentation problem* ($k$ECA for short) is defined by "Given a $\sigma$-edge-connected graph $G = (V, E)$, find an edge set $E'$ of minimum cardinality such that $G' = (V, E \cup E')$ is $(\sigma+\delta)$-edge-connected and $\sigma+\delta = k$", where $E'$ is called a *solution* to the problem. Let $k$ECA(S,SA) denote $k$ECA such that both $G$ and $G'$ are simple.
The subject of the present paper is $(\sigma + 1)$ECA(S,SA) (or $k$ECA(S,SA) with $k = \sigma + 1$). Let $\mathcal{M}$ be any maximum matching of a certain graph $R(G)$ whose vertex set $V_R$ consists of vertices representing all leaves of $G$. From $\mathcal{M}$ we obtain an edge set $E'_0$, with $|E'_0| = |\mathcal{M}|$, such that each edge connects vertices in distinct leaves of $G$. Let $\mathcal{L}_1$ be the set of leaves to be created by adding $E'_0$ to $G$, and $\mathcal{K}_1$ the set of remaining leaves of $G$.
The main result is to propose two $O(\sigma^2|V|\log(|V|/\sigma)+|E|+|V_R|^2)$ time algorithms for finding the following solutions: (1) an optimum solution if $G$ has at least $2\sigma + 6$ leaves or if $|\mathcal{L}_1| \le |\mathcal{K}_1|$ and $G$ has less than $2\sigma + 6$ leaves; (2) a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$ and $G$ has less than $2\sigma + 6$ leaves.

## 1 Introduction

The unweighted *k-edge-connectivity augmentation problem* ($k$ECA for short) is described as follows: "Given a $\sigma$-edge-connected graph $G = (V, E)$, find an edge set $E'$ of minimum cardinality such that $G' = (V, E \cup E')$ is $(\sigma + \delta)$-edge-connected and $\sigma + \delta = k$." We often denote $G'$ as $G + E'$, and $E'$ is called a *solution* to the problem. Let $k$ECA(*,**) denote $k$ECA with the following restriction (i) and (ii) on $G$ and $E'$, respectively: (i) * is set to $S$ if $G$ is required to be simple, and * is left to mean that $G$ may be a multiple graph; (ii) ** is set to MA if creation of new multiple edges in constructing $G'$ is allowed, and is set to SA otherwise. In $k$ECA(*,SA), if $G$ is simple then so is $G'$, or if $G$ has multiple edges then any multiple edge of $G'$ exists in $G$. As for $k$ECA,

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$k$ECA(*,MA) has mainly been discussed so far. See [3,5,7,8,12,13,20,21,22,23] for the results. It is natural for us to assume that $|V| \geq \sigma + 2$ in $(\sigma + 1)$ECA(S,SA): in $(\sigma + 1)$ECA(*,SA), we may have $|V| \leq \sigma + 1$.

As related results, $k$ECA(S,SA) for $G$ having no edges was first discussed in [6], where the problem that is more general than $k$ECA(S,SA) is considered. An $O(|V|+|E|)$ algorithm for 2ECA(S,SA) can be obtained by slightly modifying the one given in [3] for 2ECA(*,MA). As for 3ECA(*,SA), [23] proposed an $O(|V| + |E|)$ algorithm for 3ECA(*,MA), and showed that if $|V| \geq 4$ then this algorithm finds an optimum solution to 3ECA(*,SA). Concerning $(\sigma + 1)$ECA(S,SA) with $|V| \geq \sigma + 2$ for $\sigma \in \{3, 4\}$, [15] proposed an $O(|V| \log |V| + |E|)$ algorithm. Other related results have been reported in [14,16]. T. Jordán showed in [10] that $k$ECA(S,SA) is NP-hard in general, and [2] proposed an $O(|V|^4)$ algorithm for $k$ECA(S,SA) for any fixed $k$.

The subject of the present paper is $(\sigma + 1)$ECA(S,SA), that is, $k$ECA(S,SA) with $k = \sigma + 1$. Let $\mathcal{M}$ be any maximum matching of the *leaf-graph* $R(G)$ whose vertex set $V_R$ consists of vertices representing all leaves of $G$. (The definition of $R(G)$ is going to be given later). From $\mathcal{M}$ we obtain a certain edge set $E_0'$, with $|E_0'| = |\mathcal{M}|$, such that each edge connects vertices in distinct leaves of $G$. Let $\mathcal{L}_1$ be the set of leaves to be created by adding $E_0'$ to $G$, and $\mathcal{K}_1$ the set of remaining leaves of $G$.

The main result of the paper is to propose two $O(\sigma^2|V| \log(|V|/\sigma) + |E| + |V_R|^2)$ time algorithms for finding the following solutions for $(\sigma + 1)$ECA(S,SA):

(1) an optimum solution if $G$ has at least $2\sigma + 6$ leaves or if $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ and $G$ has less than $2\sigma + 6$ leaves;
(2) a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$ and $G$ has less than $2\sigma + 6$ leaves.

A central concept in solving $k$ECA is a *t-edge-connected component* of $G$: a maximal set of vertices such that $G$ has at least $t$ edge-disjoint paths between any pair of vertices in the set [22]. A $t$-edge-connected component whose degree (the number of edges connecting vertices in the set to those outside of it) is equal to the edge-connectivity of $G$ is called a *leaf*. Although $(\sigma + 1)$ECA(S,SA) can be solved almost similarly to general $k$ECA(*,MA), the only difference is that the augmenting step has to choose a pair of leaves, each containing a vertex such that they are not adjacent in $G$. (Such a pair of leaves is called a *nonadjacent pair*.) This requires addition of some other characteristics or processes in finding solutions by means of structural graphs: a structural graph is introduced in [11], and is used as a useful tool that reduces time complexity in finding a solution to $k$ECA(*,MA) in [7,13].

This paper adopts the operation, called *edge-interchange*, in finding a solution, where it was introduced in [20,21] in order to reduce time complexity of [22]. A set of two nonadjacent pairs of leaves is called a *D-combination* if they are disjoint. The augmenting step in solving $(\sigma + 1)$ECA(S,SA) repeats both choosing a nonadjacent pair of leaves and enlarging a $(\sigma + 1)$-edge-connected component by means of edge-interchange (or an analogous operation). Hence obtaining an optimum solution requires finding a maximum set of nonadjacent pairs of leaves such that any two members in the set form a D-combination and, therefore, this is reduced to finding a maximum matching of the leaf-graph $R(G)$ of $G$. The point of $(\sigma+1)$ECA(S,SA) is that a solution $E'$ is closely related to a maximum matching $\mathcal{M}$ of $R(G)$.

The paper is organized as follows. Basic definitions and several basic results on $\sigma$-edge-connected componets and leaf-graphs are given in Section 2. In Section 3, results on maximum matchings of leaf-graphs are briefly mentioned. Edge-interchange operation is explained in Section 4. Section 5 discusses $(\sigma+1)$ECA(S,SA) when $G$ has less than $2\sigma+6$ leaves, and Section 6 considers $(\sigma+1)$ECA(S,SA) when $G$ has at least $2\sigma+6$ leaves.

All proofs are omitted becase of space limitation.

## 2 Preliminaries

### 2.1 Basic Definitions

Technical terms not specified here can be identified in [1,4,9,19]. An *undirected graph* $G=(V(G),E(G))$ consists of a finite and nonempty set of vertices $V(G)$ and a finite set of undirected edges $E(G)$, where $V(G)$ and $E(G)$ are often denoted as $V$ and $E$, respectively. An edge $e$ incident upon two vertices $u,v$ in $G$ is denoted by $e=(u,v)$ unless any confusion arises. We denote $V(e)=\{u,v\}$, or generally $V(K)=\{u,v\in V|(u,v)\in K\}$ for a subset $K\subseteq E$. For disjoint sets $X$, $X'\subset V$, we denote $(X,X';G)=\{(u,v)\in E|u\in X \text{ and } v\in X'\}$, where it is often written as $(X,X')$ if $G$ is clear from the context. We denote $d_G(X)=|(X,\overline{X};G)|$. This is called the *degree* of $X$ (in $G$). We set $d_G(S)=0$ if $S=\emptyset$. If $X=\{v\}$ then $d_G(\{v\})$ is denoted simply as $d_G(v)$ and is the total number of edges $(v,v')$, $v'\neq v$, incident upon $v$. We often denote $d_G(S)$ as $d(S)$ if $G$ is clear from the context. A path between vertices $u$ and $v$ is often called a $(u,v)$*-path* and denoted by $P_G(u,v)$, and is often written as $P(u,v)$ if $G$ is clear from the context. For two vertices $u$, $v$ of $G$, let $\lambda(u,v;G)$, or simply $\lambda(u,v)$, denote the maximum number of pairwise edge-disjoint paths between $u$ and $v$.

For a set $X\subseteq V$, let $G[X]$ denote the subgraph having $X$ as its vertex set and $\{(u,v)\in E|u,v\in X\}$ as its edge set. $G[X]$ is called the *subgraph* of $G$ *induced* by $X$ (or the *induced subgraph* of $G$ by $X$). *Deletion* of $X\subseteq V$ from $G$ is to construct $G[V-X]$, which is often denoted as $G-X$. If $X=\{v\}$ then we often denote $G-v$ for simplicity. Deletion of $Q\subseteq E$ from $G$ defines a spanning subgraph of $G$, denoted by $G-Q$, having $E-Q$ as its edge set. If $Q=\{e\}$ then we denote $G-e$. For a set $E'$ of edges such that $E'\cap E=\emptyset$, let $G+E'$ denote the graph $(V,E\cup E')$. If $E'=\{e\}$ then we denote $G+e$.

Let $K\subseteq E$ be any minimal set such that $G-K$ has more components than $G$. $K$ is called a *separator* of $G$, or in particular a $(X,Y)$-separator if any vertex of $X$ and any one of $Y$ are disconnected in $G-K$. If $X=\{u\}$ or $Y=\{v\}$ then it is denoted as a $(u,Y)$-separator or a $(X,v)$-separator, respectively. A *minimum* $(X,Y)$*-separator* $K$ of $G$ is a $(X,Y)$-separator of minimum cardinality. Such

$K$ is often called an $(X, Y)$-cut or an $|K|$-cut. It is known that a $(u, v)$-cut $K$ has $|K| = \lambda(u, v; G)$. A *minimum separator* $K$ of $G$ is a separator of minimum cardinality among all separators of $G$, and $|K|$ is called the *edge-connectivity* (denoted by $\sigma$) of $G$; particularly we call such $K \subseteq E$ a *minimum cut* (of $G$). $G$ is said to be *$k$-edge-connected* if $\lambda(G) \geq k$. A *$k$-edge-connected component* ($k$-component, for short) of $G$ is a subset $S \subseteq V$ satisfying the following (a) and (b): (a) $\lambda(u, v; G) \geq k$ for any pair $u, v \in S$; (b) $S$ is a maximal set that satisfies (a). Let $\Gamma_G(k)$ denote the set of all $k$-components of $G$. In a graph $G$ with $\lambda(G) = \sigma$, a $(\sigma + 1)$-component $S$ with $d_G(S) = \sigma$ is called a *leaf* $(\sigma + 1)$-*component* of $G$ (or a leaf of $G$, for short). It is known that $\lambda(G) \geq k$ if and only if $V$ is a $k$-component. Note that distinct $k$-components are disjoint sets. Each 1-component is often called a *component*.

Note that we assume that $|V| \geq \sigma + 2$ in $(\sigma + 1)$ECA(S,SA), the subject of the paper.

A *cactus* is an undirected connected graph in which any pair of cycles share at most one vertex. A *structural graph* $F(G)$ of $G$ with $\lambda(G) = \sigma$ is a representation of all minimum cuts of $G$ and is introduced in [11]. We use the term "nodes of $F(G)$" to distinguish them from vertices of $G$. $F(G)$ is an edge-weighted cactus of $O(|V|)$ nodes and edges such that each tree edge (an edge which is a bridge in $F(G)$) has weight $\lambda(G)$ and each cycle edge (an edge included in any cycle) has weight $\lambda(G)/2$. Let $F(G)$ be a structural graph of $G$. Particularly if $\sigma$ is odd then $F(G)$ is a weighted tree. (Examples of $G$ and $F(G)$ will be given in Figs. 1 and 2.) Each vertex in $G$ maps to exactly one node in $F(G)$, and $F(G)$ may have some other nodes, call *empty nodes*, to which no vertices of $G$ are mapped. Let $\epsilon(G) \subseteq V(F(G))$ denote the set of all empty nodes of $F(G)$. Note that any minimum cut of $G$ is represented as either a tree edge or a pair of two cycle edges in the same cycle of $F(G)$, and vice versa. Let $\rho$: $V \to V(F(G)) - \epsilon(G)$ denote this mapping. We use the following notations: $\rho(X) = \{\rho(v)|v \in X\}$ for $X \subseteq V$, and $\rho^{-1}(Y) = \{v \in V|\rho(v) \in Y\}$ for $Y \subseteq V(F(G))$. $\rho(\{v\})$ or $\rho^{-1}(\{v\})$ is written as $\rho(v)$ or $\rho^{-1}(v)$, respectively, for notational simplicity. For any cut $(X, V(F(G)) - X; F(G))$, if summation of weights of all edges contained in the cut is equal to $\sigma$ then $(\rho^{-1}(X), V - \rho^{-1}(X); G)$ is a $\sigma$-cut of $G$. Note that the cut of $F(G)$ consists of either one tree edge or a pair of two cycle edges in the same cycle of $F(G)$. Conversely, for any $\sigma$-cut $(X, V - X; G)$, $F(G)$ has at least one cut $(Y, V(F(G)) - Y; G)$ in which summation of weight of all edges contained in the cut is equal to $\sigma$, where $Y$ is a node set of $F(G)$ such that $\rho(X) = Y - \epsilon(G)$. Each $(\sigma + 1)$-component $S$ of $G$ is represented as a vertex $\rho(S) \in V(F(G)) - \epsilon(G)$ in $F(G)$, and, for any vertex $v \in V(F(G)) - \epsilon(G)$, $\rho^{-1}(v)$ is a $(\sigma + 1)$-component of $G$. For $v \in V(F(G))$, if summation of weights of all edges that are incident to $v$ in $F(G)$ equals to $\sigma$, then $v$ is called a *leaf node* (that is a degree-1 vertex in a tree or a degree-2 vertex in a cycle). Note that, for any leaf node $v$, $\rho^{-1}(v)$ is a leaf of $G$, conversely, for any leaf $L$ of $G$, $\rho(L)$ is a leaf node of $F(G)$. It is shown that $F(G)$ can be constructed in $O(|V||E|)$ time [11] or in $O(\sigma^2|V| \log(|V|/\sigma) + |E|)$ time [7].

Two edges $e_1$, $e_2$ are said to be *independent* if and only if $V(e_1) \cap V(e_2) = \emptyset$, and a set $Q \subseteq E$ is called an *independent set* or a *matching* of $G$ if and only if any pair of edges in $Q$ are independent. An independent set of maximum cardinality in $G$ is called a *maximum matching* of $G$.

**Proposition 1.** *[5] For distinct sets $X, Y \subset V$ of any graph $G = (V, E)$,*

$$d(X) + d(Y) = d(X - Y) + d(Y - X) + 2|(V - X \cup Y, X \cap Y)|, \tag{2.1}$$

$$d(X) + d(Y) = d(X \cap Y) + d(X \cup Y) + 2|(X - Y, Y - X)|. \tag{2.2}$$

Let $\lceil x \rceil$ ($\lfloor x \rfloor$, respectively) denote the minimum integer no smaller (the maximum one no greater) than $x$.

## 2.2 $\sigma$-Components and Leaf-Graphs

Let $\lambda(G) = \sigma > 0$. Let $X_1$, $X_2$ be distinct $(\sigma + 1)$-components of $G$. The pair $\{X_1, X_2\}$ are called an *adjacent pair* (denoted as $X_1 \chi X_2$) if any two vertices $w \in X_1$ and $w' \in X_2$ are adjacent in $G$, or called a *nonadjacent pair* (denoted as $X_1 \overline{\chi} X_2$) otherwise. Let

$$V_C = \{v | v \text{ represents an individual } (\sigma + 1)\text{-component of } G\}$$

and let $S(v) \in \Gamma_G(\sigma + 1)$ denote the one represented by $v \in V_C$. Let $C(G) = (V_C, E_C)$ be defined by $V_C$ and $E_C = \{(v, v') | v, v' \in V_C \text{ and } S(v) \overline{\chi} S(v')\}$, and it is called the *component graph* of G. Let $LF(G) = \{X \in \Gamma_G(\sigma+1) | X \text{ is a leaf of } G\}$ and $V_R = \{v | v \text{ represents an individual leaf of } G\} \subseteq V_C$. Let $Y(v)$ denote the leaf $(\sigma + 1)$-component represented by $v \in V_R$. Let $R(G) = (V_R, E_R)$ be the subgraph of $C(G)$ defined by $E_R = \{(v, v') \in E_C | v, v' \in V_R \text{ and } Y(v) \overline{\chi} Y(v')\}$, and it is called the *leaf-graph* of $G$.

*Property 1.* $R(G)$ is simple.

Let $Y_i$, $i = 1, 2, 3, 4$, be distinct leaves of $G$. A set of two nonadjacent pairs $\{Y_1, Y_2\}, \{Y_3, Y_4\}$ is called a *D-combination* if they are disjoint (that is, $\{Y_1, Y_2\} \cap \{Y_3, Y_4\} = \emptyset$). In general, for $2t$ distinct leaves $Y_i$, $i = 1, \ldots, 2t$, of $G$ with $t \geq 2$, a set of $t$ nonadjacent pairs $\{Y_1, Y_2\}, \ldots, \{Y_{2t-1}, Y_{2t}\}$ is called a *D-set* of $G$ if any two pairs of the set form a D-combination. Let $Y_1 \chi \{Y_2, Y_3\}$ denote that both $Y_1 \chi Y_2$ and $Y_1 \chi Y_3$ hold. A D-combination $\{\{Y_1, Y_2\}, \{Y_3, Y_4\}\}$ is called an *I-combination* (denoted as $\{Y_1, Y_2\} \angle \{Y_3, Y_4\}$) if either $Y_1 \chi \{Y_3, Y_4\}$ or $Y_2 \chi \{Y_3, Y_4\}$ holds. If neither $\{Y_1, Y_2\} \angle \{Y_3, Y_4\}$ nor $\{Y_3, Y_4\} \angle \{Y_1, Y_2\}$ holds then we denote $\{Y_1, Y_2\} \not\angle \{Y_3, Y_4\}$.

We first show some basic results on $R(G)$ and leaves of $G$.

**Proposition 2.** *Suppose that $G$ is simple. Then either $|Y| = 1$ or $|Y| \geq \sigma + 2$ for any $Y \in LF(G)$.*

Since each leaf $Y$ has $d_G(Y) = \sigma$, we obtain the next proposition by Proposition 2.

**Proposition 3.** *Suppose that $G$ is simple. If $\{Y_1, Y_2\} \subseteq LF(G)$ is an adjacent pair then $|Y_1| = |Y_2| = 1$.*

**Proposition 4.** $d_{R(G)}(v) \geq \max\{|V_R| - (\sigma + 1), 0\}$ *for any* $v \in V_R$.

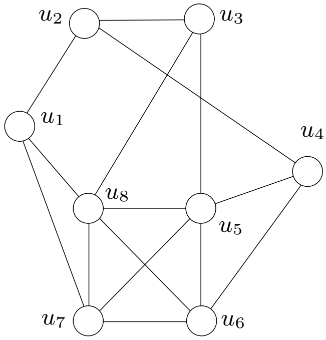


**Fig. 1.** A simple graph $G$ with $\lambda(G) = 3$ and $|LF(G)| = 4$.

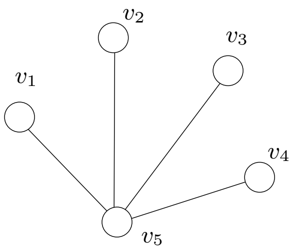


**Fig. 2.** A structural graph $F(G)$ of $G$ in Fig. 1, where all edge-weights are 3 and none of them are written. In this case leaves $Y_i$ in $LF(G)$ of the graph $G$ shown in Fig. 1 are represented as nodes $v_i$ of $F(G)$ for $i = 1, \ldots, 5$: it may happen that $G$ has a node to which no corresponding leaf of $LF(G)$ exists.

### 2.3 Examples

Let $G = (V, E)$ with $|V| \geq \sigma + 2$ and $\lambda(G) = \sigma$ be any given simple graph. Let $OPT(M)$ or $OPT(S)$ denote the cardinality of an optimum solution to $(\sigma+1)$ECA(*,MA) or to $(\sigma+1)$ECA(S,SA) for $G$, respectively. For $\sigma = 3$, we give an example such that $OPT(S) = OPT(M)+1$. For the graph $G$ with $|LF(G)| = 4$ shown Fig. 1, $R(G)$ is given in Fig. 3. The set of edges $\{(u_1, u_3), (u_2, u_4)\}$ is an optimum solution to 4ECA(*,MA), while $\{(u_1, u_3), (u_2, u_8), (u_3, u_7)\}$ is an optimum solution to 4ECA(S,SA) and, therefore, $OPT(S) = 3 = OPT(M) + 1$.

## 3 Maximum Matchings of Leaf-Graphs

One of requirements in finding a solution to $(\sigma+1)$ECA(S,SA) or $(\sigma+1)$ECA(*, SA) with $\sigma \geq 1$ is to obtain a largest D-set. Hence, in this section, the cardinality of a maximum D-set is investigated by considering a maximum matching $\mathcal{M}$ of $R(G)$.

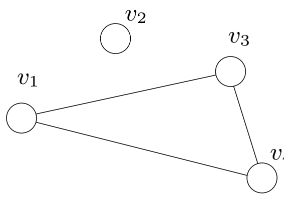


**Fig. 3.** The leaf-graph $R(G)$ of $G$ in Fig. 1.

Let $\mathcal{M}$ denote any fixed maximum matching of $R(G)$ in the following discussion unless otherwise stated, where we assume that $\lambda(G) = \sigma \geq 1$.

**Proposition 5.** *$|\mathcal{M}|$ satisfies one of the following (1)–(3).*

*(1) If $|V_R| \geq 2\sigma + 1$ or if $\sigma$ is even and $|V_R| = 2\sigma$ then $|\mathcal{M}| = \lfloor |V_R|/2 \rfloor$.*
*(2) If $\sigma$ is odd and $|V_R| = 2\sigma$ then*

$$\lfloor |V_R|/2| \rfloor - 1 \leq |\mathcal{M}| \leq \lfloor |V_R|/2 \rfloor.$$

*(3) If $|V_R| \leq 2\sigma - 1$ then*

$$\max\{0, \min\{|V_R| - \sigma, \lfloor |V_R|/2 \rfloor\}\} \leq |\mathcal{M}| \leq \lfloor |V_R|/2 \rfloor.$$

**Corollary 1.** *Suppose that $|V_R| = 2\sigma$ and $\sigma = 2m + 1$. If $|\mathcal{M}| = \lfloor |V_R|/2 \rfloor - 1$ then $G = (V, E)$ is a complete bipartite graph with $V = X \cup Y$, $X \cap Y = \emptyset$, $|X| = |Y| = \sigma$ and $E = \{(x, y) | x \in X, y \in Y\}$.*

The relationship among $G$, $C(G)$ and $R(G)$ shows the following proposition concerning $|V_R|$, $|\mathcal{M}|$ and $|E'|$ of any optimum solution $E'$ to $(\sigma+1)$ECA(S,SA).

**Proposition 6.** *Let $E'$ be any solution to $G$ in $(\sigma + 1)$ECA(S,SA) and $\mathcal{M}$ be a maximum matching of $R(G)$. Then*

$$|V_R| - |\mathcal{M}| \leq |E'|. \tag{3.1}$$

## 4 Augmentation by Edge-Interchange

We explain an operation called edge-interchange which was originally introduced in [20,21] for an efficient augmentation. It is also used in [14,15,16,17,18]. Let $LF(G) = \{Y_1, \ldots, Y_q\}$ ($q = |LF(G)|$) denote the class of all leaves of $G$ and choose $y_i \in Y_i$ as the representative of $Y_i$. Let

$$Y(G) = \{y_i | Y_i \in LF(G)\}, \;\; q \geq 2, \text{ and } r = \lceil q/2 \rceil.$$

We can easily prove the next proposition.

**Proposition 7.** *If there is a set $E'$ of edges, each connecting vertices of $G$, such that $E' \cap E = \emptyset$ and $V(E') = Y(G) \subseteq S$ for some $(\sigma+1)$-component $S$ of $G+E'$, then $S = V$.*

Let $Y$ stand for $Y(G)$ in the rest of the section.

### 4.1 Attachments

We have $d_G(Y_i) = \sigma$ and $\lambda(y_i, y_j; G) = \sigma$ for any $y_i, y_j \in Y$ $(i \neq j)$. An edge set $F$ is called an *attachment* (for $G$) if and only if the following (1) through (4) hold:

(1) $V(F) \subseteq Y$,
(2) $F \cap E(G) = \emptyset$,
(3) $V(e) \neq V(e')$ $(\forall e, e' \in F, e \neq e')$, and
(4) if $q$ $(= |LF(G)|)$ is odd then $F$ has at most one pair $f, f'$ such that $|V(f) \cap V(f')| = 1$; or if $q$ is even then $F$ has no such pair.

Let $F$ be any attachment for $G$. For each $e = (u, v) \in F$, $G + F$ has a new $(\sigma + 1)$-component, denoted by $\mathcal{A}(e, G + F)$, containing $V(e)$.

We are going to show that we can find a minimum attachment $Z(\sigma + 1) = \{e_1, \ldots, e_r\}$ $(r = \lceil q/2 \rceil)$ such that $\lambda(G + Z(\sigma + 1)) = \sigma + 1$. Although there are two cases: $r = 1$ and $r \geq 2$, we discuss only the latter case in the following. (Note that if $r = 1$ then we immediately obtain the desired attachment $F$.)

### 4.2 Finding a Minimum Attachment

Suppose that there are an attachment $F$ for $G$ and vertices $y_{ij} \in Y - V(F)$, $1 \leq i, j \leq 2$, where $y_{11}$, $y_{12}$, $y_{21}$ are distinct, and if $y_{22}$ is equal to one of the other three then we assume that $y_{22} = y_{21}$ (see Fig. 4). We use the following

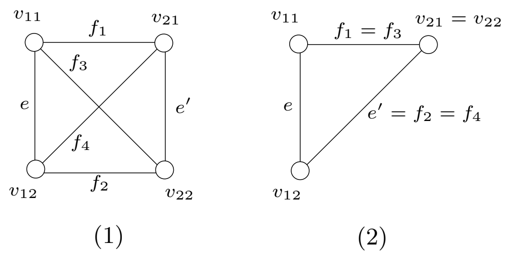


**Fig. 4.** The edges $e, e'$ and $f_i$, $1 \leq i \leq 4$: (1) $y_{21} \neq y_{22}$; (2) $y_{21} = y_{22}$.

notations:

$$L = G + F, \quad e = (y_{11}, y_{12}), \quad e' = \begin{cases} (y_{21}, y_{22}) & \text{if } y_{21} \neq y_{22} \\ (y_{12}, y_{21}) & \text{if } y_{21} = y_{22}, \end{cases}$$

$$\mathcal{A}(e) = \mathcal{A}(e, L + \{e, e'\}), \quad \mathcal{A}(e') = \mathcal{A}(e', L + \{e, e'\}),$$

$$f_1 = (y_{11}, y_{21}), \quad f_2 = (y_{12}, y_{22}), \quad f_3 = (y_{11}, y_{22}), \quad f_4 = (y_{12}, y_{21}),$$

where we set $f_1 = f_3$ and $e' = f_2 = f_4$ if $y_{21} = y_{22}$, and

$$\mathcal{A}(f_i) = \begin{cases} \mathcal{A}(f_i, L + \{f_1, f_2\}) & \text{if } 1 \leq i \leq 2 \\ \mathcal{A}(f_i, L + \{f_3, f_4\}) & \text{if } 3 \leq i \leq 4. \end{cases}$$

Note that $e, e', f_i \notin E(L), 1 \leq i \leq 4$. We have the following two cases.

Case I: $\mathcal{A}(e) \cap \mathcal{A}(e') = \emptyset$; Case II: $\mathcal{A}(e) \cap \mathcal{A}(e') \neq \emptyset$ (that is, $\mathcal{A}(e) = \mathcal{A}(e')$).

For Case I, we are going to show that there are two edges $f, f'$, with $V(f) \cup V(f') = V(e) \cup V(e')$, such that

$$\mathcal{A}(e) \cup \mathcal{A}(e') \subseteq \mathcal{A}(f, L + \{f, f'\}) = \mathcal{A}(f', L + \{f, f'\}).$$

That is, we can add two edges so that one $(\sigma+1)$-component containing $\mathcal{A}(e) \cup \mathcal{A}(e')$ may be obtained. Finding and adding such a pair of edges $f, f'$ is called *edge-interchange* (with respect to $V(e_1) \cup V(e_2)$).

Suppose that $\mathcal{A}(e) \cap \mathcal{A}(e') = \emptyset$. Note that $y_{21} \neq y_{22}$ in this case. Let $K$ be any fixed $(\mathcal{A}(e), \mathcal{A}(e'))$-cut of $L + \{e, e'\}$, and let $B_i$, $1 \leq i \leq 2$, denote the two sets of vertices in $L + \{e, e'\}$ such that $B_1 \cup B_2 = V$, $B_2 = V - B_1$, $K = (B_1, B_2; L + \{e, e'\})$, $\mathcal{A}(e) \subseteq B_1$ and $\mathcal{A}(e') \subseteq B_2$. $|K| = \sigma = \lambda(y_1, y_2; L'')$ for any $y_i \in B_i$, $1 \leq i \leq 2$, where $L''$ denotes $L$, $L + e$, $L + e'$ or $L + \{e, e'\}$. $K$ is a $(y_1, y_2)$-cut of $L$. Suppose that $f$ and $f'$ satisfy either (i) or (ii):

(i) $f = f_1$, $f' = f_2$, or (ii) $f = f_3$, $f' = f_4$,

where $\{f, f'\} \cap E(L) = \emptyset$ .

The next proposition shows a property of edge-interchange.

**Proposition 8.** *If $\mathcal{A}(e) \cap \mathcal{A}(e') = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) = \emptyset$ then $\mathcal{A}(f_3) \cap \mathcal{A}(f_4) \neq \emptyset$, that is, $\mathcal{A}(f_3) = \mathcal{A}(f_4)$.*

Let $\{f, f'\}$ denote the following pair of edges:

$$\{e, e'\} \text{ if } \mathcal{A}(e) = \mathcal{A}(e') \text{ (the case with } V(e) \cap V(e') = \emptyset \text{ is included);}$$

$$\{f_1, f_2\} \text{ if } \mathcal{A}(e) \cap \mathcal{A}(e') = \emptyset \text{ and } \mathcal{A}(f_1) = \mathcal{A}(f_2);$$

$$\{f_3, f_4\} \text{ if } \mathcal{A}(e) \cap \mathcal{A}(e') = \mathcal{A}(f_1) \cap \mathcal{A}(f_2) = \emptyset.$$

Clearly, $\{f, f'\} \cap E(L) = \emptyset$. Such a pair $f, f'$ are called an *augmenting pair* (with respect to $\{y_{11}, y_{12}, y_{21}, y_{22}\}$) of $L$.

**Corollary 2.** *Let $L' = L + \{f, f'\}$ for any augmenting pair $f, f'$. Then $L' - f'$ has no $\sigma$-cut separating $V(f')$ from $V(f)$. That is, if $L' - f'$ has a $\sigma$-cut $K$ separating a vertex of $V(f')$ from $V(f)$ then $K$ separates the two vertices of $V(f')$.*

From Corollary 2, other important properties (Proposition 9–11) of edge-interchange are obtained.

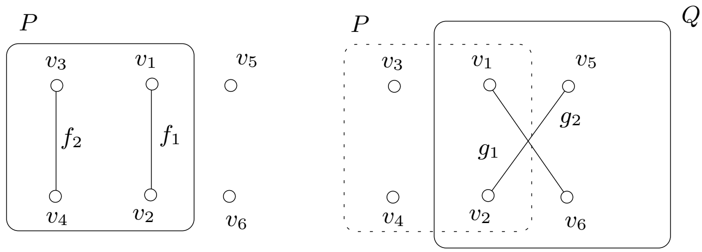


**Fig. 5.** The two $(\sigma+1)$-components $\mathcal{A}(f_1, G+\{f_1, f_2\})$ and $\mathcal{A}(g_1, G+\{g_1, g_2\})$ produced by two augmenting pairs $\{f_1, f_2\}$ and $\{g_1, g_2\}$, respectively.

**Proposition 9.** *Suppose that* $G$ *has six leaves* $Y_i \in LF(G)$ $(1 \le i \le 6)$*, and choose* $y_i \in Y_i$ *as a representative of each* $Y_i$*. Suppose that* $\{f_1, f_2\}$ *is an augmenting pair with respect to* $\{y_i | 1 \le i \le 4\}$ *of* $G$*. If* $\mathcal{A}(f_1, G + \{f_1, f_2\})$ *is a leaf then, for each* $i \in \{1, 2\}$*, there is an augmenting pair* $\{g_1, g_2\}$ *with respect to* $V(f_i) \cup \{y_5, y_6\}$ *of* $G$ *such that* $\mathcal{A}(g_1, G + \{g_1, g_2\})$ *is not a leaf (see Fig. 5).*

By Proposition 9, we obtain the following procedure that is a modified version of the procedure given in [15]. It finds a sequence of edges $e_1, \ldots, e_r$ $(r = \lceil |LF(G)|/2 \rceil \ge 1)$ by repeating edge-interchange operation, where handling the case with $|LF(G)| = 2$ is included. Note that edges with which we are concerned are those connecting vertices belonging to distinct leaves. If an edge $g$ connects a vertex in a leaf $Y_i$ and another vertex in a leaf $Y_j$ $(i \ne j)$ then, for simplicity, we say that $g$ *connects* $Y_i$ *and* $Y_j$.

**Procedure** *FIND_EDGES;*

**begin**

1. $G_1 \leftarrow G$; $\pi \leftarrow LF(G)$; $i \leftarrow 1$; $E'_1 \leftarrow \emptyset$;
2. **while** $\pi \ne \emptyset$ **do**

    **begin**

3. **if** $|\pi| = 2$ **then**
4. $f_i \leftarrow$ an edge connecting the two leaves of $\pi$; $E''_i \leftarrow \{f_i\}$;
5. **else if** $|\pi| \le 5$ **then**
6. Find an augmenting pair $E''_i = \{f_i, f'_i\}$ by Proposition 8;
7. **else** /* $|\pi| \ge 6$ */
8. Find an augmenting pair $E''_i = \{f_i, f'_i\}$ by Proposition 9;
9. $E'_{i+1} \leftarrow E'_i \cup E''_i$; $G_{i+1} \leftarrow G_i + E''_i$; $\pi \leftarrow \pi - \{Y(v) | v \in V(E''_i)\}$; $i \leftarrow i + 1$;

    **end**

**end**;

**Proposition 10.** $G_{i+1}$ *has a leaf containing* $\mathcal{A}(f_i, G_{i+1})$ *if and only if* $|LF(G_i)|$ $= 5$ *just after the execution of Step 9 in FIND_EDGES.*

Note that executing Step 6 or Step 8 once can be done in $O(|V_R|)$ by using a structural graph $F(G)$, and we can construct $F(G)$ in $O(\sigma^2|V|\log(|V|/\sigma)+|E|)$ time (see [7]). The details are omitted here.

The next proposition holds for the edge set $E'$ produced by *FIND_EDGES*.

**Proposition 11.** *Let $Z(\sigma+1)=\{e_1,\ldots,e_r\}$ ($r=\lfloor|LF(G)/2\rfloor$) be given by FIND_EDGES. Then $Z(\sigma+1)$ is a minimum attachment such that $\lambda(G')=\sigma+1$, where $G'=G+Z(\sigma+1)$. Furthermore the procedure runs in $O(\sigma^2|V|\log(|V|/\sigma)+|E|+|V_R|^2)$ time.*

## 5 $(\sigma+1)$ECA(S,SA) for $G$ Having Less Than $2\sigma+6$ Leaves

We denote $LF(G)=\{Y_i|1\leq i\leq q\}$ ($q=|LF(G)|$), $Y(G)=\{y_1,\ldots,y_q\}$ and $V_R=\{v_1,\ldots,v_q\}$, where each $y_i$ is represented as $v_i$ in $R(G)$. First we consider the case where $G$ has two or three leaves.

**Proposition 12.** *If $q=2$ then the following (1) or (2) holds.*

*(1) If $Y_1\overline{\chi}Y_2$ then $|\mathcal{M}|=1$, there are two vertices $y_i\in Y_i$, $i=1,2$, such that $E'=\{(y_1,y_2)\}$ is a solution, and $OPT(S)=OPT(M)=1$.*
*(2) If $Y_1\chi Y_2$ then $|\mathcal{M}|=0$, there are three vertices $y_i\in Y_i$ ($i=1,2$), $x\in V-(Y_1\cup Y_2)$ such that $E'=\{(y_1,x),(y_2,x)\}$ is a solution, and $OPT(S)=2=OPT(M)+1$.*

**Proposition 13.** *If $q=3$ and there exist two leaves $Y_1$, $Y_2$ with $Y_1\overline{\chi}Y_2$ then $|\mathcal{M}|=1$, there are distinct edges $e_1,e_2$ such that $E'=\{e_1,e_2\}$ is a solution, and $OPT(S)=OPT(M)=2$.*

Next we consider the remaining case where $3\leq q<2\sigma+6$. For each $e'=(x',y')\in\mathcal{M}$, we can choose two vertices $x\in Y(x')$, $y\in Y(y')$, and let $e=(x,y)$ be an edge, which is not included in $E$. We fix such an edge $e$ for each $e'\in\mathcal{M}$, and let

$$E'_0=\{e=(x,y)\mid(x',y')\in\mathcal{M}\}.$$

**Proposition 14.** $|E'_0|=|\mathcal{M}|$ *and* $E'_0\cap E=\emptyset$.

In the rest of this section, we consider the graph $G+E'_0$. First we define two sets $\mathcal{L}_1$ and $\mathcal{K}_1$ as follows.

Let $G_1=G+E'_0$ and let $\mathcal{L}_1$ be the set of new leaves of $G_1$ created by adding $E'_0$ to $G$. Clearly $|\mathcal{L}_1|\leq|\mathcal{M}|$. Let $\mathcal{K}_1=LF(G+E'_0)-\mathcal{L}_1$ ($\subseteq LF(G)$). Since $\mathcal{M}$ is a maximum matching of $R(G)$, Proposition 3 shows that each leaf in $\mathcal{K}_1$ consists of only one vertex and that the set of vertices $\mathcal{K}'_1=\{x\mid\{x\}\in\mathcal{K}_1\}$ induces a complete graph of $G$ and of $G+E'_0$.

We are going to propose an $O(\sigma^2|V|\log(|V|/\sigma)+|E|+|V_R|^2)$ time algorithm such that it finds an optimum solution if $|\mathcal{L}_1|\leq|\mathcal{K}_1|$ and such that a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1|>|\mathcal{K}_1|$. Note that we have $|\mathcal{L}_1|\leq|\mathcal{K}_1|$ if $|\mathcal{M}|\leq\lfloor|V_R|/3\rfloor$.

**Proposition 15.** *Let* $\{y'_1\}, \{y'_2\} \in \mathcal{K}_1$ $(y'_1 \neq y'_2)$ *and* $Y_1, Y_2 \in \mathcal{L}_1$ $(Y_1 \neq Y_2)$*. If* $\{(y_1, y'_1), (y_2, y'_2)\}$ *is not an augmenting pair with* $y_1 \in Y_1$ *and* $y_2 \in Y_2$ *then there are* $y_3 \in Y_1$ *and* $y_4 \in Y_2$ *such that* $\{(y_4, y'_1), (y_3, y'_2)\}$ *is an augmenting pair and* $(y_4, y'_1), (y_3, y'_2) \notin E$ *(See Fig. 6).*

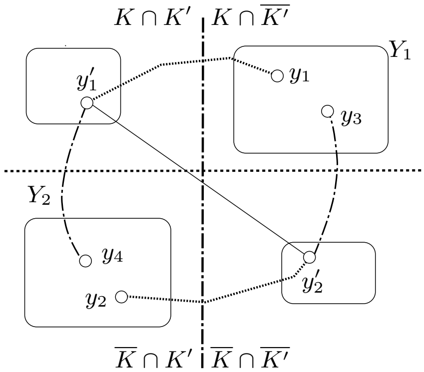


**Fig. 6.** A situation for Proposition 15

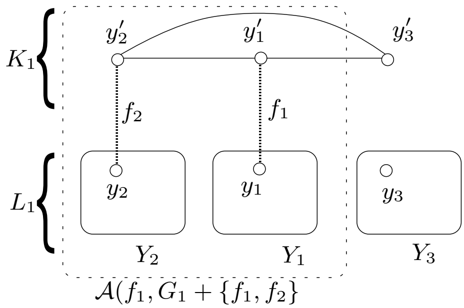


**Fig. 7.** $\mathcal{A}(f_1, G + \{f_1, f_2\})$ in the proof of Proposition 16

We obtain the next proposition by Propositions 9 and 15.

**Proposition 16.** *Assume that* $|\mathcal{L}_1| \geq 3$ *and* $|\mathcal{K}_1| \geq 3$*. Then there exists an augmenting pair* $\{f_1, f_2\}$ *such that* $f_1 = (y_1, y'_1) \notin E \cup E'_0$, $f_2 = (y_2, y'_2) \notin E \cup E'_0$, $\{\{y'_1\}, \{y'_2\}\} \subseteq \mathcal{K}_1$ $(y'_1 \neq y'_2)$, $\mathcal{L}_1$ *has two distinct sets* $Y_1, Y_2$ *with* $y_1 \in Y_1$, $y_2 \in Y_2$ *and* $\mathcal{A}(f_1, G + \{f_1, f_2\})$ *is not a leaf. Furthermore* $\mathcal{L}_1 \cup \mathcal{K}_1 - \{\{y'_1\}, \{y'_2\}\}, Y_1, Y_2\}$ *is the set of all leaves in* $G_1 + \{f_1, f_2\}$*. (See Fig. 7)*

Next we are going to discuss the case where $|\mathcal{L}_1| \leq 2$ or $|\mathcal{K}_1| \leq 2$.

**Proposition 17.** *Suppose that* $|\mathcal{L}_1| \leq 2$ *and* $|\mathcal{L}_1| \leq |\mathcal{K}_1|$*. Then there exists a set* $E'_2 = \{f_1, \ldots, f_{|\mathcal{K}_1|}\}$ *such that* $\lambda(G_1 + E'_2) \geq \sigma + 1$ *and* $E'_2 \cap (E \cup E'_0) = \emptyset$*.*

It remains to consider the cases ($|\mathcal{L}_1| \geq 3$ and $|\mathcal{K}_1| \leq 2$) and ($|\mathcal{L}_1| \leq 2$ and $|\mathcal{L}_1| > |\mathcal{K}_1|$), for which the next proposition holds.

**Proposition 18.** *Suppose that one of the following (1)–(3) holds: (1)* $|\mathcal{L}_1| \geq 3$ *and* $|\mathcal{K}_1| \leq 2$*; (2)* $|\mathcal{L}_1| = 2$ *and* $|\mathcal{K}_1| = 1$*; (3)* $|\mathcal{L}_1| = 2$ *and* $|\mathcal{K}_1| = 0$*. Let* $q_1 = |LF(G_1)|$ *and* $r_1 = \lceil \frac{q_1}{2} \rceil$*. Then there exists a set* $E''_2 = \{f_1, \ldots, f_{r_1}\}$ *such that* $\lambda(G_1 + E''_2) \geq \sigma + 1$ *and* $E''_2 \cap (E \cup E'_0) = \emptyset$*.*

The discussion from Propositions 16 through 18 is summarized in the following procedure *FIND_EDGES2*.

**Procedure** *FIND_EDGES2;*

**begin**

1. $G_0 \leftarrow G$; $\pi \leftarrow LF(G)$; $E'_0 \leftarrow \emptyset$; $\rho \leftarrow \emptyset$;
2. Find an edge set $E'_0$ as in Proposition 14; $G_1 \leftarrow G_0 + E'_0$;
   Determine $\mathcal{L}_1$ and $\mathcal{K}_1$; $i \leftarrow 1$;
3. **while** $\mathcal{K}_i \neq \emptyset$ **do**
   **begin**
4. **if** $|\mathcal{L}_i| \geq 3$ and $|\mathcal{K}_i| \geq 3$ **then**
   Find an augmenting pair $\{f, f'\}$ by Proposition 16, $E''_i \leftarrow \{f, f'\}$;
5. **else if** $|\mathcal{L}_i| \leq 2$ and $|\mathcal{L}_i| \leq |\mathcal{K}_i|$ **then**
   Find an edge set $E''_i$ by Proposition 17;
6. **else**
   Find an edge set $E''_i$ by Proposition 18;
7. Construct $\mathcal{K}_{i+1}$ and $\mathcal{L}_{i+1}$; $E'_i \leftarrow E'_{i-1} \cup E''_i$; $G_{i+1} \leftarrow G_i + E''_i$; $i \leftarrow i + 1$;
   **end**;
8. **if** $\lambda(G_i) = \sigma$ **then**/* the case with $|\mathcal{L}_i| \neq 0$ */
   Find an edge set $E''_i$ by Proposition 18; $E'_{i+1} \leftarrow E'_{i-1} \cup E''_i$;

**end**;

**Proposition 19.** *FIND_EDGES2 produces an optimum solution if* $|\mathcal{L}_1| \leq |\mathcal{K}_1|$.

**Proposition 20.** *FIND_EDGES2 gives a* $\frac{3}{2}$*-approximate solution if* $|\mathcal{L}_1| > |\mathcal{K}_1|$.

*Remark 1.* Let $\mathcal{M}$ be any maximum matching of $R(G)$. If $|\mathcal{M}| \leq \lfloor \frac{|LF(G)|}{3} \rfloor$ then $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ and we can find an optimum solution in polynomial time. If $\lfloor \frac{|LF(G)|}{3} \rfloor < |\mathcal{M}| \leq \lfloor \frac{|LF(G)|}{2} \rfloor$ then $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ or $|\mathcal{L}_1| > |\mathcal{K}_1|$. Since the proof of NP-completeness of $k$ECA(S,SA) in [10] is given for the case with $|\mathcal{M}| = \lfloor \frac{|LF(G)|}{2} \rfloor$, we consider approximate solutions if $|\mathcal{L}_1| > |\mathcal{K}_1|$.

**Theorem 1.** *Suppose that* $|LF(G)| \leq 2\sigma + 6$. *Then FIND_EDGES2 can find an optimum solution if* $|\mathcal{L}_1| \leq |\mathcal{K}_1|$, *or a* $\frac{3}{2}$*-approximate solution if* $|\mathcal{L}_1| > |\mathcal{K}_1|$, *in* $O(\sigma^2 |V| \log(|V|/\sigma) + |E|)$ *time.*

## 6 $(\sigma + 1)$ECA(S,SA) for $G$ Having at Least $2\sigma + 6$ Leaves

In this case, Proposition 5(3) shows that any maximum matching $\mathcal{M}$ of $R(G)$ has $|\mathcal{M}| = \lfloor \frac{|LF(G)|}{2} \rfloor$. First, some basic results on nonadjacent pairs and edge interchange operation are going to be given.

**Proposition 21.** *Suppose that there are a nonadjacent pair of leaves* $Y_1, Y_2 \in LF(G)$ *and two vertices* $y_i \in Y_i$, $i = 1, 2$, *with* $(y_1, y_2) \notin E$, *such that* $G' = G + \{(y_1, y_2)\}$ *has a leaf* $S$ *containing* $Y_1 \cup Y_2$. *Let* $\mathcal{L}' = \{Y \subseteq S | Y \in \Gamma_G(\sigma + 1)\}$, $X = Y_1 \cup Y_2$ *and* $Z = \bigcup_{Y \in LF(G) - \{Y_1, Y_2\}} Y$. *Then* $|(X, Z; G)| \leq \sigma - 1$ *if* $|\mathcal{L}'| \geq 3$.

The next proposition can be proved by using Propositon 21.

**Proposition 22.** *Suppose* $\sigma \geq 3$ *and let* $\mathcal{M}' = \{(v_{2i-1}, v_{2i}) | 1 \leq i \leq m\} \subseteq \mathcal{M}$ *for some* $m \leq |\mathcal{M}|$, *and put* $Y_j = Y(v_j)$ *for each* $v_j \in V_R$.

*(1) If* $|\mathcal{M}'| \geq 2$ *and there are distinct indices* $i, j$ *with* $1 \leq i, j \leq m$ *such that* $\{Y_{2i-1}, Y_{2i}\} \nparallel \{Y_{2j-1}, Y_{2j}\}$ *then (i) and (ii) hold.*

*(i) These leaves are partitioned into a D-combination* $\{\{L'_1, L'_2\}, \{L'_3, L'_4\}\}$ *having four vertices* $y_t \in L'_t$, $t = 1, 2, 3, 4$, *such that* $G + \{(y_1, y_2), (y_3, y_4)\}$ *has a* $(\sigma + 1)$*-component* $S$ *containing all* $L'_t$, $t = 1, 2, 3, 4$.

*(ii) The* $(\sigma+1)$*-component* $S'$ *of* $G + \{(y_1, y_2)\}$ *such that* $L'_1 \cup L'_2 \subseteq S'$ *is not a leaf.*

*(2) If* $|\mathcal{M}'| \geq \lceil \sigma/2 \rceil + 1$ *and no such pair of indices as in (1) exist then, for each* $(v_{2i-1}, v_{2i}) \in \mathcal{M}'$, *there are vertices* $y_{2i-1} \in Y_{2i-1}$ *and* $y_{2i} \in Y_{2i}$ *such that* $G' = G + \{(y_{2i-1}, y_{2i})\}$ *is a simple graph having a* $(\sigma + 1)$*-component* $X$ *which is not a leaf and which contains* $Y_{2i-1} \cup Y_{2i}$.

**Proposition 23.** *Suppose that there is a set* $\mathcal{M}' = \{(v_{2i-1}, v_{2i}) | 1 \leq i \leq m\} \subseteq \mathcal{M}$ *for some* $m$ *with* $\sigma + 2 \leq m \leq |\mathcal{M}|$, *and put* $Y_i = Y(v_i)$ *for each* $v_i \in V_R$. *Then there is an edge* $(v_{2h-1}, v_{2h}) \in \mathcal{M}'$ *with* $\{Y_1, Y_2\} \nparallel \{Y_{2h-1}, Y_{2h}\}$.

By combining Propositions 9, 22 and 23, we obtain the following proposition.

**Proposition 24.** *Suppose that there is a set* $\mathcal{M}' = \{f_i = (v_{2i-1}, v_{2i}) | 1 \leq i \leq m\} \subseteq \mathcal{M}$ *for some* $m$ *with* $\sigma + 3 \leq m \leq |\mathcal{M}|$, *and put* $Y_i = Y(v_i)$ *for each* $v_i \in V_R$. *Then there exists an augmenting pair* $\{e'_1, e'_2\}$ *with respect to* $Y_1, Y_2, Y_{2j-1}, Y_{2j}$ *such that* $G + \{e'_1, e'_2\}$ *is simple and has no leaf* $S$ *with* $Y_1 \cup Y_2 \cup Y_{2j-1} \cup Y_{2j} \subseteq S$, *where* $\{f_1, f_j\} \subseteq \mathcal{M}'$.

Based on Proposition 24, the next procedure *FIND_EDGES3* is obtained.

**Procedure** *FIND_EDGES3;*
  **begin**
$G_1 \leftarrow G$; $\pi \leftarrow LF(G)$; $i \leftarrow 1$; $E'_0 \leftarrow \emptyset$;
**while** $\pi \neq \emptyset$ **do**
  **begin**
  **if** $|\pi| \leq 3$ **then**
    Find an edge set $E''_i$ as $E'$in Proposition 12(1) or 13;
  **else**
    **begin** /* $|\pi| \geq 4$ */
    Find a matching $\mathcal{M}'' = \{(v_{2p-1}, v_{2p}) | 1 \leq p \leq m'\}$ of $R(G_i)$,
      where if $|\pi| \leq 2\sigma + 6$ then $m' \leftarrow \lfloor \pi/2 \rfloor$, otherwise $m' \leftarrow \sigma + 3$;
    **if** $|\pi| \leq 2\sigma + 6$ **then**
      **begin**
      Choose $E'_s \subseteq E'_i$ with $|E'_s| = \sigma + 3 - m'$ appropriately;
      $\mathcal{M}' \leftarrow \mathcal{M}'' \cup \{(v, w) \in E_R | (v', w') \in E'_s, v' \in Y(v), w' \in Y(w)\}$;

```
            /* M' is a matching on R(G) in the case.*/
          end;
        else
          M' ← M'';
8.      Find an augmenting pair E''_i as {e'_1, e'_2} in Proposition 24
          by choosing f_1 ∈ M'';          /* Note that |M'| = σ + 3. */
9.      if f_j ∈ M' − M'' for f_j of Proposition 24 then
          begin /* In the case with |π| ≤ 2σ + 6 */
          E'_i ← E'_i − {(y_{2j−1}, y_{2j})}, G_i ← G_i − {(y_{2j−1}, y_{2j})}, where
            y_{2j−1} ∈ Y_{2j−1} and y_{2j} ∈ Y_{2j};
        end;
10.   E'_{i+1} ← E'_i ∪ E''_i; G_{i+1} ← G_i + E''_i;
      π ← π − {Y(v)|v ∈ V(E''_i)}; i ← i + 1;
      end;
    end;
```

**Proposition 25.** *Any set final $E'_i$ obtained at the termination of FIND_EDGES3 is a minimum attachment such that $\lambda(G') = \sigma + 1$, where $G' = G + E'$.*

**Theorem 2.** *If $G$ has at least $2\sigma + 6$ leaves then the algorithm FIND_EDGES3 correctly finds a solution $E'$ to $(\sigma+1)$ECA(S,SA) for any given $G$ with $\lambda(G) = \sigma$ in $O(\sigma^2|V|\log(|V|/\sigma) + |E| + |V_R|^2)$ time.*

## 7 Concluding Remarks

The paper has proposed

(1) an $O(\sigma^2|V|\log(|V|/\sigma) + |E| + |V_R|^2)$ time algorithm for finding an optimum solution if $G$ has at least $2\sigma + 6$ leaves or if $|\mathcal{L}_1| \leq |\mathcal{K}_1|$ and $G$ has less than $2\sigma + 6$ leaves,
(2) an $O(\sigma^2|V|\log(|V|/\sigma) + |E|)$ time one for a $\frac{3}{2}$-approximate solution if $|\mathcal{L}_1| > |\mathcal{K}_1|$ and $G$ has less than $2\sigma + 6$ leaves.

We can improve the first algorithm to an $O(\sigma^2|V|\log(|V|/\sigma) + |E|)$ time one by devising how to check whether or not $\{f_1, f_2\}$ is an augmenting pair, and whether or not $\mathcal{A}(f_1, G + \{f_1, f_2\})$ is a leaf in Proposition 9.

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