Minimum Weight Drawings of Maximal Triangulations*
(Extended Abstract)

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Abstract. This paper studies the drawability problem for minimum weight triangulations, i.e. whether a triangulation can be drawn so that the resulting drawing is the minimum weight triangulations of the set of its vertices. We present a new approach to this problem that is based on an application of a well known matching theorem for geometric triangulations. By exploiting this approach we characterize new classes of minimum weight drawable triangulations in terms of their skeletons. The skeleton of a minimum weight triangulation is the subgraph induced by all vertices that do not belong to the external face. We show that all maximal triangulations whose skeleton is acyclic are minimum weight drawable, we present a recursive method for constructing infinitely many minimum weight drawable triangulations, and we prove that all maximal triangulations whose skeleton is a maximal outerplanar graph are minimum weight drawable.

1 Introduction

The study of the combinatorial properties of fundamental geometric graphs such as minimum spanning trees, Delaunay triangulations, proximity graphs, rectangle of influence graphs, maximum weight triangulations, and Voronoi trees is motivated not only by the theoretical appeal of the questions that this study raises, but also by the importance that such geometric structures have in different application areas including computer graphics, computer aided manufacturing, communication networks, and computational biology. Geometric graphs are straight-line drawings that satisfy some additional geometric constraints (for example pairs of adjacent vertices are deemed to be "close" according to some definition of proximity, while not adjacent vertices are far from each other in the drawing). Thus, the study of the combinatorial properties of a given type of geometric graph can be naturally turned into the following graph drawing question:

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What are those graphs admitting the given type of drawing?. This question has attracted increasing interest in both the graph drawing and the computational geometry communities and several papers have been published on the topic in recent years, including [2,4,7,8,9,10,14,15,16,17]. See also [6] for a survey.

The present paper is devoted to minimum weight triangulations. Despite the relevance of minimum weight triangulations in areas like numerical analysis and computational geometry, the problem of computing these triangulations efficiently has not yet been solved and their basic combinatorial properties are still not well-understood. We provide new insight on the combinatorial properties of minimum weight triangulations by addressing the *minimum weight drawability problem*, i.e. the problem of determining whether a triangulation T admits a straight line drawing Γ that is a minimum weight triangulation of the set of its vertices; we call Γ a *minimum weight drawing* of T and we say that T is *minimum weight drawable*.

The minimum weight drawability problem was first studied in [11,12] where it was proved that all maximal outerplanar graphs are minimum weight drawable and a linear time drawing algorithm was presented. As a side effect of the combinatorial characterization in [11,12], a linear time algorithm for computing the minimum weight triangulation of a set of points that are the vertices of a regular polygon was shown. These results have motivated the investigation of minimum weight drawable triangulations such that not all the vertices belong to the outer face, which is the subject of [13] and of a recent paper by Wang, Chin, and Yang [20].

In [13] families of minimum weight drawable triangulations are characterized in terms of their *skeleton*, i.e. the subgraph induced by their interior vertices. Classes of drawable triangulations whose skeleton is either acyclic or maximal are described and it is proved that the skeleton of a minimum weight triangulation can be any forest. In the same paper, the relationship between minimum weight drawability and Delaunay drawability is investigated. A triangulation is *Delaunay drawable* if it admits a drawing that is the Delaunay triangulation of the set of its vertices; characterizations of Delaunay drawable triangulations can be found in the works by Dillencourt [8,7,9]. In [13] an infinite family of minimum weight drawable, but non-Delaunay-drawable, triangulations are constructed, each of which as an acyclic skeleton.

Wang, Chin, and Yang [20] focus on the minimum drawability of triangulations with acyclic skeletons and show examples of triangulations of this type that do not admit a minimum weight drawing, thus solving one of the open problems in [13]. Wang, Chin, and Yang also provide a partial characterization of minimum weight drawable triangulations having acyclic skeletons, by showing that all triangulations whose skeleton is a regular star graph admit a minimum weight drawing.

In this paper we look at the minimum weight drawability problem from a new perspective. Namely, we present a new technique for proving that a straight-line drawing is a minimum weight triangulation of the set of its vertices. The technique compares distances between adjacent vertices against distances between any possible pairs of vertices in the drawing and is based on an application of a matching theorem by Aichholzer et al. [1] which establishes a correspondence between the edges of any two triangulations computed on the same point set. By using this technique we can prove the correctness of new drawing algorithms for new classes of minimum weight drawable triangulations. The main results that we establish in this paper can be listed as follows.

- We characterize those maximal triangulations with acyclic skeleton that admit a minimum weight drawing. In [13] only a partial characterization was presented.
- We devise a method for recursively constructing minimum weight drawable triangulations whose skeleton is a maximal triangulation. As an application of this method, we show minimum weight drawable triangulations with maximal skeletons that are *not* Delaunay drawable. The other previously known members of this family of triangulations all had acyclic skeletons [13].
- We show that all maximal triangulations whose skeleton is a maximal outerplanar graph are minimum weight drawable. This extends the result of [11] where the minimum weight drawability of maximal outerplanar graphs is proved.

2 Preliminaries

We assume familiarity with basic computational geometry, graph drawing and graph theory concepts. For further details see [3,5,18].

Theorem 1. **[1]** *Let P be a finite set of points in the plane and consider two triangulations T and T' of P. There exists a perfect matching between T and T' with the property that matched edges either cross or are identical.*

The *skeleton* $S(T)$ of a triangulation T is the graph induced by the set of its internal vertices. For example, the skeleton of a maximal outerplanar graph is the empty graph, the skeleton of a wheel graph consists of just one vertex, namely, the center of the wheel.

We will often be concerned with graphs G each of whose edges has an associated weight, namely the length of the corresponding segment in some straight-line drawing of G. We denote the weight of an edge e by $w(e)$. The weight of a set E of edges refers to the sum of the weights of the edges in E and is denoted by $w(E)$, as is the weight of a graph G or a drawing Γ.

3 Feasible and Forced Edges

Let P be a finite set of points in the plane. We denote by $Seg(P)$ the set of all segments having both endpoints in P. A set $E \subseteq Seg(P)$ is *feasible* if E is contained in some minimum weight triangulation of P; E is *forced* if it is contained in every minimum weight triangulation of P. Edges of the convex hull of P are clearly forced, as is any segment which is not crossed by any other segment connecting two points of P.

Our algorithms compute drawings where some edges are forced and the other edges are feasible. In order to show the correctness of our constructions, we rely on the following lemma, which gives a sufficient condition under which a set of edges is feasible. The lemma is an application of Theorem 1. In the lemma, for any $E \subseteq Seg(P)$, we denote by $I(E) \subseteq Seg(P)$ those edges which intersect at least one edge of E.

Lemma 1. *Let $E \subseteq Seg(P)$ be such that*

1. *E is planar.*
2. *Every edge of E is* light*, that is, for every $e \in E$, every edge crossing e is at least as long as e.*
3. *For all $s\prime \in I(E)$ and all $s \in I(I(E)) - E - I(E)$ such that s crosses $s\prime$, $\mid s\prime \mid \geq \mid s \mid$.*

Then E is feasible.

Sketch of Proof. Let T be a triangulation of P such that $E \not\subseteq T$. We will show how to modify T to obtain a triangulation containing E and having weight at most $w(T)$. Let $E' = E - T$, and let $G = T - I(E')$. Note that G is planar and that no edge of G intersects any edge of the (planar) set E', and so $G \cup E'$ is planar. Since any planar set of edges can be extended to a triangulation by the addition of zero or more edges, we can extend $G \cup E'$ to a triangulation T'. Hence, by Theorem 1, there exists a matching between T' and T in which every edge of T' is matched either with itself or with an edge of T that crosses it. Since no edge of E' is in T, E' is matched to a subset M of T so that every edge in E' crosses the edge in M that it matches. Therefore $M \subseteq T \cap I(E')$. Then $w(M) \geq w(E')$, since each edge of E' is light. So, $w(T) = w(G) + w(M) + w((T \cap I(E')) - M) \geq w(G) + w(E') + w((T \cap I(E')) - M) = w(G \cup E') + w((T \cap I(E')) - M)$.

Now, we extend $G \cup E'$ to a triangulation by adding $\mid (T \cap I(E')) - M \mid$ edges. Notice that each of the edges we add must be in $I(I(E)) - E - I(E)$, since $E \subseteq G \cup E'$, and edges in $I(E)$ would cross edges in $G \cup E'$. Thus, by Condition 3 of the lemma, each of the added edges can be at most as long as any edge in $I(E)$ that it crosses. Because $T \cap I(E')) - M \subset I(E)$, we have that the weight of the added edges is at most $w(T \cap I(E')) - M)$. Therefore, by the inequality above we conclude that the new triangulation has weight at most $w(T)$.

In the next sections we present several applications of Lemma 1 to the minimum weight drawability problem.

4 Acyclic Skeletons

In [20] it is shown that not all triangulations with acyclic skeleton are minimum weight drawable. On the positive side, in [13] the following partial characterization of maximal triangulations with acyclic skeleton is proved.

Lemma 2. *Let T be a maximal triangulation and let $S(T)$ be the skeleton of T. If $S(T)$ is a path, then T is minimum weight drawable and a minimum weight* *drawing of T can be computed in linear time with the real RAM model of computation.*

In this section we complete the characterization of those maximal triangulations with acyclic skeleton that are minimum weight drawable. We start by characterizing acyclic skeletons of maximal triangulations.

Lemma 3. *Let T be a maximal triangulation and let $S(T)$ be the skeleton of T. If $S(T)$ is acyclic, then it is a tree with at most three leaves.*

Proof. Proof omitted in extended abstract.

Let T be a maximal triangulation whose skeleton is acyclic. By Lemma 3, two cases are possible: Either the skeleton of T is a tree with exactly one vertex of degree 3 or it is a path. If it is a path, the minimum weight drawability of T is guaranteed by Lemma 2. The next lemma studies the remaining case.

Lemma 4. *Every maximal triangulation whose skeleton is a tree with three leaves is minimum weight drawable.*

Proof. Proof omitted in extended abstract.

We can summarize the results of this section as follows.

Theorem 2. *Let T be a maximal triangulation with n vertices and let $S(T)$ be the skeleton of T. If $S(T)$ is acyclic, then T is minimum weight drawable, and a minimum weight drawing of T can be computed in $O(n)$ time in the real RAM model of computation.*

5 Maximal Skeletons

In this section we study the minimum weight drawability of triangulations whose skeleton is maximal. In Subsection 5.1 we show a recursive method to construct minimum weight drawable triangulations each having a skeleton that is a maximal triangulation. In subsection 5.2 we study the minimum weight drawability when the skeleton is a maximal outerplanar graph.

5.1 Skeletons That Are Maximal Triangulations

The next theorem allows us to describe a recursive method for drawing certain triangulations as minimum weight triangulations

Theorem 3. *Let T be a maximal triangulation and let $S(T)$ be its skeleton. If $S(T)$ is such that:*

1. *$S(T)$ is a maximal triangulation, and*
2. *$S(T)$ is minimum weight drawable,*

then T is minimum weight drawable.

Sketch of Proof. Let a_0, a_1, and a_2 be the three vertices of the outer face of T. Let v_0, v_1, and v_2 be the three vertices of the outer face of $S(T)$ and let Γ' be a minimum weight drawing of $S(T)$. We show how to construct a minimum weight drawing Γ of T from Γ' by adding vertices a_0, a_1, and a_2 and their incident edges. There are two cases to consider: Either (i) each a_i $(i = 0, 1, 2)$ is adjacent to two vertices of the outer face of $S(T)$, or (ii) there exists a vertex of the outer face of T adjacent to v_0, v_1, and v_2.

For Case (i), suppose a_i is adjacent to v_i and to v_{i+1} (in the rest of the proof we always assume $i = 0, 1, 2$ and all subscripts taken mod 3). Each a_i is represented as a point in the plane so that the following geometric constraints are satisfied (see Figure 1(a)):

Constraint 1: The coordinates of the vertices are chosen so that a_1, a_2, and a_3 form the convex hull of the new set of points.

Constraint 2: Vertex a_i is connected to v_i and to v_{i+1} by segments that do not intersect any edge of Γ'. The segment connecting a_i to v_{i+2} intersects Γ'.

Constraint 3: The distance between a_i and each vertex of Γ' is larger than length of the longest edge of the outer face of Γ'.

Let Γ be the resulting drawing and let P be the set of vertices of Γ. We prove that Γ is a minimum weight triangulation of P. All edges of the type $a_i a_{i+1}$ are forced because of *Constraint 1*. Similarly, the edges connecting a_0, a_1 and a_2 to v_0, v_1, and v_2 are forced because of *Constraint 2*.

Let E be the edges of the type $v_i v_{i+1}$. Clearly E is planar. $I(E)$ consists of segments connecting a vertex of Γ' to a vertex of the convex hull of P. Thus, by Constraint 3 every edge of E is light. Also notice that $I(I(E)) - E - I(E)$ consists of segments connecting pairs of vertices of Γ', which by *Constraint 3* are shorter than any element of $I(E)$. Therefore, by Lemma 1 we conclude that E is feasible.

Now, since E is feasible and is a (convex) triangle, all edges of the type $a_i a_{i+1}$ are forced. Thus, since Γ' is a minimum weight drawing, we conclude that Γ is a minimum weight triangulation of P.

We now consider Case (ii). Suppose that a_0 is adjacent to v_0, v_1, and v_2, that a_1 is adjacent to both v_1 and v_2, and that a_2 is adjacent only to v_2. A drawing of T is constructed in three steps: first a_0 and its incident edges are added to Γ', then a_1 is added, and finally a_2 is added. The coordinates of the vertices is chosen so that a_1, a_2, and a_0 form the convex hull of the new set of points. Also, the following constraints are satisfied (see Figure 1(b)):

Constraint a: Vertex a_0 is connected to v_0, v_1, and v_2 by segments that do not intersect any edge of Γ'. The distance between a_0 and each vertex of Γ' is larger than the length of the longest edge of the outer face of Γ'.

Constraint b: Vertex a_1 is connected to v_1, and v_2 by segments that do not intersect any edge of Γ'. A segment connecting a_1 to v_0 crosses only shorter segments.

Constraint c: Vertex a_2 is connected to v_2, by segments that do not intersect any edge of Γ'. Segments connecting a_2 to either v_0 or v_1 cross only shorter segments.

Observe that *Constraints a, b,* and *c* imply that the distance between a_i and each vertex of Γ' is larger than length of the longest edge of the outer face of Γ'. The reasoning to prove that the drawing defined with this construction is minimum weight is similar to that described for Case (i). We omit details for brevity.

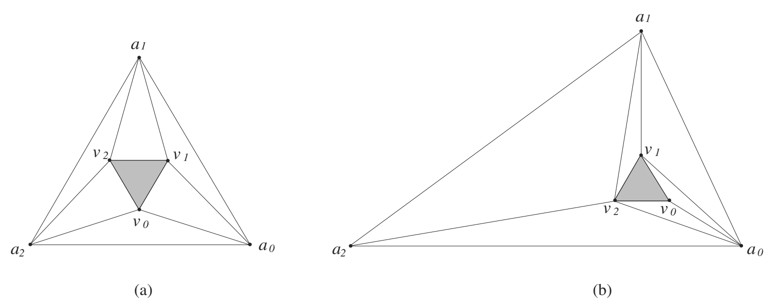

Fig. 1. Illustration for Theorem 3. The shaded grey regions represents Γ'. (a) A minimum weight drawing in which each a_i $(i = 0, 1, 2)$ is adjacent to two vertices of the outer face of Γ'. (b) A minimum weight drawing in which a vertex of the outer face of T adjacent to three vertices of the outer face of Γ'.

Theorem 3 provides a basic tool for constructing minimum weight drawable triangulations. One such triangulation, obtained after one step of the recursion, is depicted in Figure 2: Theorems 3 and 2 imply that the triangulation is minimum weight drawable. In the figure, some vertices of the triangulation are drawn as white circles: Removing the white vertices breaks the graph into four disconnected components. This means that the graph violates one of necessary conditions that all Delaunay drawable triangulations must satisfy [8]. Similarly, it can be verified that none of the triangulations recursively drawn by the above procedure are Delaunay drawable.

Lemma 5. *There exists an infinite family of triangulations that admit a minimum weight drawing, are not Delaunay drawable, and have skeletons that are maximal triangulations.*

We remark that the only families of minimum weight but not Delaunay drawable triangulations known so far had a considerably simpler combinatorial structure, since their skeletons were forced to be acyclic [13].

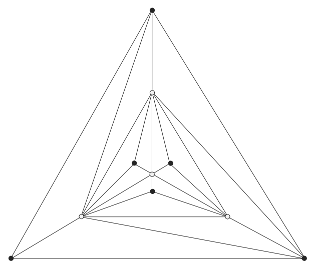

Fig. 2. A minimum weight drawable triangulation constructed by the recursive procedure of Theorem 3. The skeleton of the triangulation is a maximal triangulation. The triangulation is not Delaunay drawable.

5.2 Skeletons That Are Maximal Outerplanar Graphs

In [11,12] it is proved that all maximal outerplanar graphs admit minimum weight drawings and a linear time algorithm to compute these drawings is presented. In this section we prove that every maximal triangulation whose skeleton is a maximal outerplanar graph is minimum weight drawable. Our proof relies on the following approach:

- A special type of minimum weight drawing of $S(T)$ is computed by means of a variant of the algorithm in [11,12].
- Such a drawing of $S(T)$ is used as a building block to compute a minimum weight drawing of T.

Before giving more technical details, we briefly recall the basic idea behind the algorithm of [11,12]. Let G be a maximal outerplanar graph and let $D(G)$ be its dual; observe that $D(G)$ is a tree such that all non-leaf vertices have degree three. The vertices of G are drawn as cocircular points chosen to be a subset of the vertices of regular polygon Π. Π is defined so that the dual graph of its minimum weight triangulation is a complete tree having $D(G)$ as its subtree (in [11,12] it is shown that the minimum weight triangulation of a regular polygon coincides with its greedy triangulation). The minimum weight triangulation of G can be obtained by deleting vertices of degree 2 from the minimum weight triangulation of Π, until the dual of the remaining triangulation becomes identical to $D(G)$ (deleting a vertex of degree 2 and its incident edges from the minimum weight triangulation of Π corresponds to deleting a leaf from its dual tree). Since Π is defined in such a way that it may have exponentially many more vertices than G, additional tools are devised [11,12] by which a time complexity proportional to the number of vertices of G is achieved. Intuitively, the algorithm does not explicitly construct the minimum weight drawing of Π, but it uses the knowledge of the topology of its dual tree to directly compute the coordinates of the vertices of the minimum weight drawing of G.

We are now ready to prove the main result of this section. Let T be a maximal triangulation whose skeleton $S(T)$ is a maximal outerplanar graph. Let a_0, a_1, and a_2 be the three vertices of the outer face of T. We distinguish between those vertices of $S(T)$ that are adjacent to exactly one of a_0, a_1, and a_2 and those vertices of $S(T)$ that are adjacent to at least two of a_0, a_1, and a_2. Vertices of $S(T)$ of this second type are called *transition vertices*. Since T is a maximal triangulation, $S(T)$ has either two or three transition vertices. An example where $S(T)$ has two transition vertices is given in Figure 3 (a); an example where $S(T)$ has three transition vertices is given in Figure 3 (b).

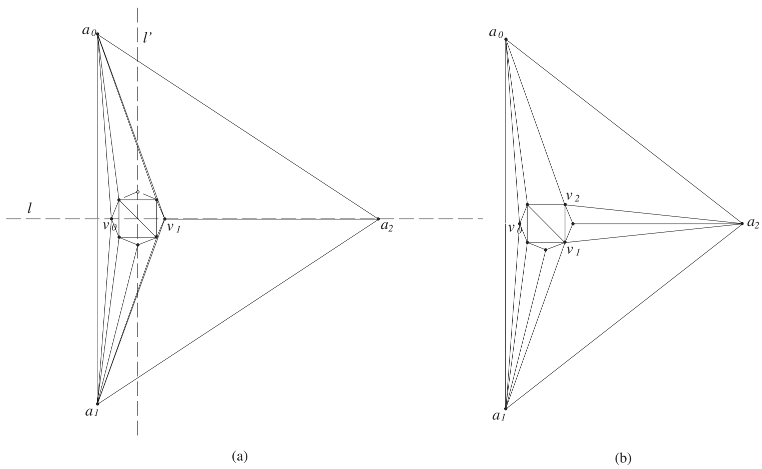

Fig. 3. (a) A minimum weight drawing of a triangulation whose skeleton is a maximal outerplanar graph with two transition vertices. (b) A minimum weight drawing of a triangulation whose skeleton is a maximal outerplanar graph with three transition vertices. Figure (a) also shows lines l and l' and the minimum weight triangulation of a regular polygon Π that are used by the drawing algorithm of Lemma 6.

Lemma 6. *It T is a maximal triangulation whose skeleton $S(T)$ is a maximal outerplanar graph with exactly two transition vertices, then T is minimum weight drawable.*

Sketch of Proof. Since $S(T)$ has exactly two transition vertices, one of them is adjacent to all of the outer vertices a_0, a_1, and a_2 of T, while the other is adjacent to only two of them. Let us label the two transition vertices as v_0 and v_1 and assume that v_0 is adjacent only to to a_0 and a_1. Let n_0 be the number of non-transition vertices of $S(T)$ that are adjacent to a_0, let n_1 be the number of non-transition vertices of $S(T)$ that are adjacent to a_1 and let $n = \max\{n_1, n_2\}$.

We modify the drawing algorithm of [11,12] as follows. A polygon Π is defined such that: (i) The dual tree of a minimum weight triangulation of Π is a complete tree having the dual of $S(T)$ as its subtree (we recall that the minimum weight triangulation of a regular polygon coincides with its greedy triangulation); (ii) Π has at least $2n+2$ vertices. A minimum weight drawing of $S(T)$ is now computed by vertex deletion from the minimum weight triangulation of Π. Vertices v_0 and v_1 are chosen to be antipodal points of the polygon; therefore, there are n vertices on both sides of the polygon between v_0 and v_1. This guarantees that we can delete vertices of degree 2 from the minimum weight triangulation of Π in such a way that: (i)the dual graph of the resulting triangulation after the deletions coincides with the dual graph of $S(T)$ (i.e. the resulting triangulation is a drawing of $S(T)$), and (ii) the two paths from v_0 to v_1 along the drawing of $S(T)$ consist of n_0 and of n_1 vertices, respectively. Let Γ' be the drawing of $S(T)$ obtained by this procedure. The fact that Γ' is a minimum weight drawing of $S(T)$ is a consequence of the property that deleting a vertex of degree 2 and its incident edges from a minimum weight triangulation of a set of cocircular points gives as a result a minimum weight triangulation of the remaining points.

In order to construct a minimum weight drawing of T we now add to Γ' vertices a_0, a_1, a_2, and their incident edges. Refer to Figure 3 (a). Let ℓ be the line through v_0 and v_1 and let ℓ' be the perpendicular bisector of the segment having v_0 and v_1 as its endpoints. We assume that by construction Γ' has n_0 vertices in the half-plane above ℓ and thus there are n_1 vertices in the half-plane below ℓ and that ℓ is horizontal. The coordinates of the vertices are chosen so that a_1, a_2, and a_0 form the convex hull of the new set of points. The following additional constraints are satisfied

Constraint 1: Vertex a_0 is drawn on the left-hand side of line ℓ' and above line ℓ. Vertex a_0 is connected to v_0, v_1 and to the n_0 non-transition vertices above ℓ by segments that do not intersect any edge of Γ'. The distance between a_0 and each vertex of Γ' is larger than the length of the longest edge of Γ'.

Constraint 2: Vertex a_1 is drawn on the left-hand side of line ℓ' and below line ℓ. Vertex a_1 is connected to v_0, v_1 and to the n_1 non-transition vertices below ℓ by segments that do not intersect any edge of Γ'. The distance between a_1 and each vertex of Γ' is larger than the length of the longest edge of Γ'.

Constraint 3: Vertex a_2 is drawn on the right-hand side of ℓ' and on line ℓ. Vertex a_2 is connected to v_1 by a segment longer than those segments connecting a_0 and a_1 to vertices of $S(T)$.

Let Γ be the resulting drawing. Observe that the edges of the outer face and edge a_2v_1 are forced. Let E be the set consisting of the edges of Γ' and of the segments connecting a_0 and a_1 to $S(T)$. Clearly E is planar. Also, $I(E)$ consists of segments connecting either vertices of Γ' to vertices of the convex hull or pairs of vertices of Γ'. By Constraints 1, 2, and 3 and since Γ' is a minimum weight drawing, we have that each segment e of E is crossed by segments of $I(E)$ that are no shorter than e; hence E is light. Now, since $I(I(E)) - I(E) - E = \emptyset$, it follows by Lemma 1 that E is feasible. Therefore Γ is a minimum weight drawing of T.

Lemma 7. *If T is a maximal triangulation whose skeleton $S(T)$ is a maximal outerplanar graph with exactly two transition vertices, then T is minimum weight drawable.*

Sketch of Proof. A minimum weight drawing Γ of T is computed by a variant of the algorithm in the proof of Lemma 6. Namely, Γ' is computed in the same way as in the proof of the above lemma, and vertices a_0 and a_1 are added by the same strategy. The coordinates of vertex a_2 are now chosen so that: (i) all segments connecting a_2 to the vertices of Γ' are shorter that all segments that can possibly cross them and (ii) all segments starting at a_2 and that cross remaining edges of the drawing are longer than these edges. An example of a drawing computed by this strategy is given in Figure 3 (b). The proof that Γ is a minimum weight drawing relies on Lemma 1 and is similar to that of Lemma 6.

Theorem 4. *Let T be a maximal triangulation and let $S(T)$ be its skeleton. If $S(T)$ is a maximal outerplanar triangulation, then T is minimum weight drawable and its minimum weight drawing can be computed in linear time in the real RAM model of computation.*

6 Conclusions and Open Problems

In this paper we have presented new results on the minimum weight drawability problem by characterizing new families of maximal triangulations that admit a minimum weight drawing. The new results are based on a sufficient geometric condition for a set of line segments to be part of a minimum weight triangulation.

The general problem of determining which triangulations are minimum weight drawable is still far from solved. As intermediate steps toward solving the problem, we might suggest pursuing the following:

1. Devise other sufficient conditions of the type given in Lemma 1 that allow us to better understand the geometric properties of minimum weight triangulations of given sets of points.
2. Study the minimum weight drawability of k-outerplanar graphs (a graph is k-outerplanar when it has a planar embedding such that all vertices are on disjoint cycles properly nested at most k deep). The proof techniques of Lemmas 6 and 7 may be a good starting point.
3. Further investigate the relationship between Delaunay drawability and minimum weight drawability. An interesting class to study seems to be the set of 4-connected triangulations, for which a characterization in terms of Delaunay drawability is known.

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