Surface Digitizations by Dilations Which Are Tunnel-Free

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Abstract. In this article we study digital topology with methods from mathematical morphology. We introduce reconstructions by dilations with appropriate continuous structural elements and prove that notions known from digital topology can be defined by continuous properties of this reconstruction. As a consequence we determine the domains for tunnel-free surface digitizations. It will be proven that the supercover and the grid-intersection digitization of every surface with or without boundary is always tunnel-free.

1 Introduction

Various approaches have been made to study geometrical and topological properties of binary digital images. Discussing the advantages and disadvantages of them would be far beyond this paper. However, there is a growing interest in relating these approaches to each other in order to develop a foundation for a mathematically consistent theory.

The most well-known approach, known as *digital topology* [KR89], is derived from graph theory. Elements of $\mathbb{Z}^n$ are interpreted as vertices. Edges are defined by different adjacency relations between object and background points. This approach serves well for two-dimensional image analysis. The 3D case [MR81] is far more complicated and a generalization to higher dimensions has not been made yet.

A *cellular approach* has been applied by Kovalevsky [Kov89] in 2D and by Herman et al. [HW83] in 3D. Voss [Vos88] studied a dual cell-structure in $\mathbb{Z}^n$ and Khalimsky [KKM90] developed a topological approach based on *connected ordered topological spaces*. The structure studied in these approaches is the *discrete* or *Alexandrov topology*. It is equivalent to a tessellation of $\mathbb{R}^n$ by n-dimensional unit cubes. Each approach maps $\mathbb{Z}^n$ onto different elements of the structure. In the first case an element of $\mathbb{Z}^n$ is associated with an open n-dimensional unit-cube, whereas Voss interprets $\mathbb{Z}^n$ as the set of the vertices of these cubes. In the third approach, every element of that structure is associated with an element of $\mathbb{Z}^n$.

Bertrand and Couprie [BC99] proposed a model for digital geometry that associates two orders to each subset of $\mathbb{Z}^n$. These order relations correspond to

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the different adjacencies as used in digital geometry. Moreover, the authors have proven that the notion of surfaces and simple points in their model correspond exactly to that very notions in digital topology. This justifies the original graph based approach for 2D and 3D space.

Digital images can also be investigated using a *digitization approach* [Ser82]. A discrete object has a certain property if it is a digitization of an appropriate continuous object with that property. Dual to this is the *embedding approach* in which continuous analogs [KR85] of discrete objects are studied. Both approaches define properties of discrete objects by well-known continuous, usually Euclidean, notions.

Reveillès [Rev92] introduced the *arithmetical geometry* approach. He defined *discrete analytical objects* as discrete objects which are the integer solution of a finite set of inequalities. Recently, Andres studied the supercover digitization of m-flats in the context of discrete analytical objects [And99]. In [LW00b] we generalized these results to linear analytical objects.

The digitization approach has been related to digital topology. Various researchers [Pav82,Ser82,GL95] studied the preservation of topological features of continuous objects under digitization. Bærentzen et al. [BŠC00] presented a criterion for determining whether a 3D solid is suitable for digitization at a given resolution. Although these articles make different assumptions, the common idea is to consider objects that are morphologically open and closed by a closed ball whose radius depends on the grid resolution.

We applied a similar approach to study the digitizations of surfaces without boundary [LW00a]. Since the opening of a simple surface is the empty set, we employed morphologically closed surfaces with respect to a ball of a radius r. We called these objects r*-surfaces*. Contrary to Bærentzen et al. [BŠC00] our approach did not take a reconstruction kernel into account, but this article explains that properties from digital geometry, such as connectivity and separability, can defined by a dilation of the discrete object with a continuous structural element.

An extension of these results for surfaces without boundary to surfaces with boundary is essential, because many real-life objects can be described as the union of surface patches. To evaluate the quality of digitizations of surfaces with boundary Cohen-Or et al. introduced the notion *tunnel-free* [CK95]. This notion has been applied successfully to polygons and polyhedra [ANF97]. In this paper we develop a theoretical framework for digitizations of surfaces with and without boundary which is based on mathematical morphology [Ser82,Hei94]. The same theoretical background has been used by Schmitt to study digitizations and connectivity [Sch98].

This article is outlined as follows: Section 2 states the basic definitions from differential geometry, digital topology and mathematical morphology. In section 3 important results about surface digitizations and digitizations by dilation will be recalled. In the following section *reconstructions by dilation* will be introduced and in section 5 this notion is employed to prove a condition under which a surface digitization is tunnel-free. We conclude with a summary and remarks on future work.

2 Basic Definitions

2.1 Differential Geometry

In differential geometry continuous objects are studied as their parametrization. Curves are basically 1-dimensional and surfaces are $(n-1)$-dimensional parametrizations in $\mathbb{R}^n$. Thus, in $\mathbb{R}^2$ curves can be considered as surfaces.

A set of points $C \subseteq \mathbb{R}^n$ $(n \geq 2)$ is said to be a $C^r(I)$-*curve* $(r \geq 1)$ if there exists an open interval $I \subseteq \mathbb{R}$ and an r times continuously differentiable function $\gamma : I \to \mathbb{R}^n$ such that $C = \gamma(I)$. The function γ is called *parametrization*. A curve γ is *smooth* if, for all $t \in I$, the first derivative exists and is non-zero. A curve is *simple* if it has no self-intersection.

From now on all curves will be considered to be smooth and simple. A curve $C = \gamma([a, b])$ with end points $\gamma(a)$ and $\gamma(b)$ is a subset of curve $\gamma(I)$ that is defined on an appropriate open interval I that contains $[a, b]$. Let $C = \gamma((a, b))$ be a simple curve and let $\gamma(a) = \gamma(b)$, then $C = \gamma([a, b])$ is a *simple closed curve*.

A set of points $S \subseteq \mathbb{R}^n$ $(n \geq 2)$ is said to be a $C^r(U)$-*surface* $(r \geq 1)$ if there exists a non-empty open set $U \subseteq \mathbb{R}^{n-1}$ and an r-times continuously differentiable function $f : U \to \mathbb{R}^n$ such that $S = f(U)$.

Again, only *simple, smooth* surfaces with or without boundary are considered. A simple *surface without boundary* is either a closed surface, such as a sphere, or an infinite object homeomorphic to a hyperplane. These notions are intuitively clear and similar to those for curves. For a detailed definition the reader is referred to text books on differential geometry such as [LV85].

2.2 Digital Topology

We define a *discrete object* A as a subset of $\mathbb{Z}^n$. Its complementary set $A^C = \mathbb{Z}^n \setminus A$ is called the *background*. We think of $\mathbb{Z}^n$ as a subset of n-dimensional Euclidean space $\mathbb{R}^n$. An element $z \in \mathbb{Z}^n$ is called a *grid point*.

There are various equivalent ways to introduce the basic notions of digital topology. We define the neighborhood of grid points through Voronoi sets [Kle85,Wüt98]. Other definitions are based on distances or differences in the coordinates of these points.

The *Voronoi set* $\mathbb{V}(z)$ of a grid point z is the set of all points in $\mathbb{R}^n$ which are at least as close to z as to any other grid point. $\mathbb{V}(z)$ is a closed axes-aligned n-dimensional unit cube with center z. The Voronoi sets of a 2D and 3D grid point are known as *pixel* and *voxel*, respectively. Neighboring n-dimensional Voronoi sets can share a point, a straight line segment, up to an $(n-1)$-dimensional cube.

Two grid points $z, z' \in \mathbb{Z}^n$ are said to be k-*neighbors* $(0 \leq k \leq n-1)$ if their Voronoi sets share a point set of dimension k or higher, i.e. if $\dim(\mathbb{V}(z) \cap \mathbb{V}(z')) \geq k$.

A sequence $(z_0, \ldots, z_l)$ of points of an object $A \subseteq \mathbb{Z}^n$ is said to be a k-*arc* from z_0 to z_l in A if successive elements are k-neighbors. $K \subseteq \mathbb{Z}^n$ is a *(simple closed)* k-*curve* if each point of K has exactly two k-neighbors.

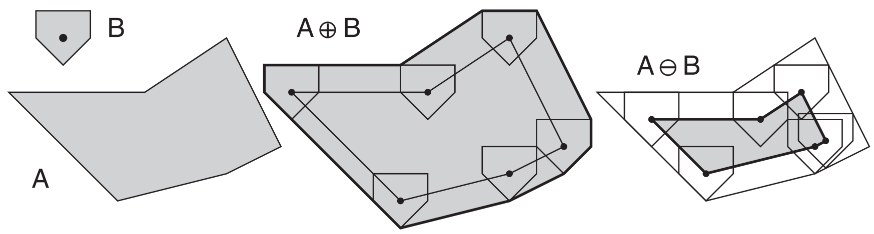

Fig. 1. Dilation $A \oplus B$ and erosion $A \ominus B$ of a set A by a structuring Element B

An object $A \subseteq \mathbb{Z}^n$ is *k-connected* if there exists a k-arc in A from z to z' for any points $z, z' \in A$. A *k-component* of $A \subseteq \mathbb{Z}^n$ is defined as a maximal k-connected non-empty subset of A.

A discrete object $A \subseteq \mathbb{Z}^n$ is said to be *k-separating* if the background $\mathbb{Z}^n \setminus A$ consists of exactly two k-components. A k-separating object A is called *k-minimal* if for any $z \in A$ $A \setminus \{z\}$ is not k-separating. A *k-separating surface* (without boundary) is a minimal k-separating object.

To avoid pathological situations in 2D, a 1-curve must consist of at least 8 points and an 0-curve of at least 4 points [KR89]. A discrete surface should have no touching points. Traditionally [KR89], in $\mathbb{Z}^2$ 1- and 0-neighbors are called 4-neighbors and 8-neighbors, respectively, and in $\mathbb{Z}^3$ 26-, 18- and 6-neighbors are common notions.

2.3 Morphological Definitions

In this article, morphological operations [Ser82,Hei94] on point sets will be required. Let A and B be two subsets of $\mathbb{R}^n$. Since $\mathbb{Z}^n \subseteq \mathbb{R}^n$, the following operations can be applied to continuous as well as to discrete point sets.

$A \oplus B = \{a + b : a \in A, b \in B\}$ is called *Minkowski addition* and $A \ominus B = \{p : b + p \in A \text{ for all } b \in B\}$ is the *Minkowski subtraction* of A and B. In mathematical morphology, $A \oplus B$ and $A \ominus B$ are known *dilation* and *erosion* of A by the *structuring element* B, respectively. Fig. 1 shows an example.

The operation $A \circ B = (A \ominus B) \oplus B$ is called the *opening* and $A \bullet B = (A \oplus B) \ominus B$ is called the *closing* of A with respect to B. A is *morphologically open (closed)* with respect to B if $A \circ B = A$ (resp. $A \bullet B = A$).

Finally, $A_z = A \oplus \{z\}$ is the translate of A by z and $\check{A} = \{-a : a \in A\}$ denotes the reflected set of A.

3 Surface Digitizations

In our previous work [LW00a] we studied a class of digitizations, commonly known as digitizations by dilations [Hei94]. The grid-intersection [Kle85] and the supercover [CK95] digitization schemes, which are common for surfaces, are special cases of digitizations by dilations.

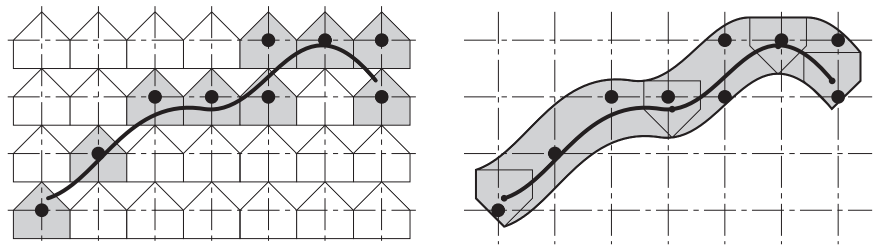

Fig. 2. Digitization of a curve as the set of translated basic domains D_z hit by A (left) and as the set of grid points contained in $A \oplus \check{D}$

A *digitization by dilation* with *domain* $D \subseteq \mathbb{R}^n$ is a function $\Delta_{\oplus}^{D} : \wp(\mathbb{R}^n) \rightarrow \wp(\mathbb{Z}^n)$ that is defined as $\Delta_{\oplus}^{D}(A) = \{z \in \mathbb{Z}^n : A \cap D_z \neq \emptyset\}$ for every continuous object $A \subseteq \mathbb{R}^n$.

By virtue of this definition, a grid point z belongs to digitization $\Delta_{\oplus}^{D}(A)$, if and only if D_z, the domain translated to $z \in \mathbb{Z}^n$, hits the continuous object A. $\Delta_{\oplus}^{D}(A)$ is called digitization by dilation because it is the set of grid points contained in the dilation of A by the reflected domain $\check{D}$, i.e. $\Delta_{\oplus}^{D}(A) = (A \oplus \check{D}) \cap \mathbb{Z}^n$ [Hei94].

If the domain of a digitization is not specified then $\Delta(A)$ is any subset of $\mathbb{Z}^n$ that is intended to serve as a discrete approximation of a continuous object $A \subseteq \mathbb{R}^n$. It does not need to be a digitization by dilation when we focus on criteria for the quality of these approximations.

The criterion "k-separating"can only be applied to discrete surfaces without boundaries. To overcome this limitation the notion of a k-tunnel-free digitization of a surface has been introduced [CK95].

Let $(z_0, \ldots, z_l)$ be a *k-arc*. Then the continuous polygonal arc consisting of the straight line segments $[z_0, z_1]$, $[z_1, z_2]$, ..., $[z_{l-1}, z_l]$ is called a *continuous k-path*. A continuous path π *hits* a surface $S \subseteq \mathbb{R}^n$ in a point $p \in S$ if $p \in \pi \cap S$. A continuous path π *crosses* a surface $S \subseteq \mathbb{R}^n$ in $p \in S$ if there exists an $\epsilon > 0$ such that π hits two different components of $B_\epsilon(p) \setminus S$. $B_\epsilon(p)$ denotes the closed ball of radius ϵ with center p.

A digitization $\Delta(S) \subseteq \mathbb{Z}^n$ of a continuous surface $S \subseteq \mathbb{R}^n$ is *k-tunnel-free* $(0 \leq k \leq n-1)$ if every continuous k-path in $(\Delta(S))^C = \mathbb{Z}^n \setminus \Delta(S)$ does not cross S. A continuous k-path in $(\Delta(S))^C$ that crosses S is called *k-tunnel*.

If $\Delta(S) \subseteq \mathbb{Z}^n$ is a k-tunnel-free digitization of S then $\Delta(S) \cup A$ is also k-tunnel-free for every $A \subseteq \mathbb{Z}^n$. As illustrated in Fig. 3, a k-tunnel-free digitization of a continuous surface without boundary is not necessarily a k-separating discrete object.

4 Reconstructions by Dilation

The foundation of digital topology is the notion "k-neighbor". In Section 2 k-neighborhood of grid points was defined by means of their Voronoi sets. In this

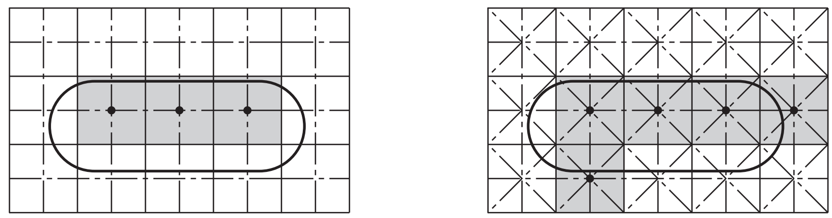

Fig. 3. A 1-tunnel-free (left) an 0-tunnel-free (right) digitization of a simple closed curve, that are no discrete curves

section will be shown that two point $z, z' \in \mathbb{Z}^n$ are neighbored if $R_z \cup R_{z'} = (\{z\} \oplus R) \cup (\{z'\} \oplus R)$ is a connected set in $\mathbb{R}^n$ for an appropriate structural element $R \subseteq \mathbb{R}^n$. Consequently, if $A \in \mathbb{Z}^n$ is a discrete object, we can think of the continuous set $(A \oplus R) \subseteq \mathbb{R}^n$ as its *reconstruction by dilation*.

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two points in $\mathbb{R}^n$ then $[x, y]$ and $[(x_1, \ldots, x_n), (y_1, \ldots, y_n)]$ denotes the straight line segment between these points. $R_k^{(n)} \in \mathbb{R}^n$ is the union of all line segments $[(0, \ldots, 0), (x_1, \ldots, x_n)]$ with the property that at least k $(0 \leq k < n)$ of the coordinates x_i are 0, while the others are either $\frac{1}{2}$ or $-\frac{1}{2}$.

For simplicity the index for the dimension will be omitted if it is clear by the context. Fig. 4 shows the sets R_0 and R_1 in $\mathbb{R}^2$ and R_0, R_1 and R_2 in $\mathbb{R}^3$. The following lemma is a simple conclusion of the definitions of R_k and k-neighbors.

Lemma 1. *Two grid points $z, z' \in \mathbb{Z}^n$ are k-neighbors $(0 \leq k \leq n-1)$, if and only if the reconstruction $\{z, z'\} \oplus R_k$ is a connected set in $\mathbb{R}^n$.*

There exists a straight line segment between k-neighbored two grid points in the reconstruction by R_k and, conversely, if there is a straight line segment between two grid points z and z' in $\{z, z'\} \oplus R_k$ then these points are k-neighbors. Notice that there may exist other continuous paths between these points. Only for $(n-1)$-neighbors this path is unique. A k-arc from z to z' in $A \subseteq \mathbb{Z}^n$ exists, iff there exists a continuous path between z and z' in $A \oplus R_k$.

Lemma 2. *An object $A \subseteq \mathbb{Z}^n$ is k-connected if and only if the reconstruction $A \oplus R_k$ is a connected set in $\mathbb{R}^n$.*

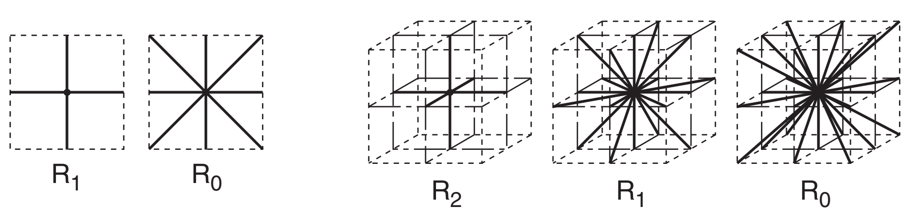

Fig. 4. R_1 and R_0 in $\mathbb{R}^2$ and R_2, R_1 and R_0 in $\mathbb{R}^3$

Proof. Assume $A \subseteq \mathbb{Z}^n$ is k-connected. Then, for any $z, z' \in A$, there exists a continuous path between z and z' in $A \oplus R_k$. R_k is a connected set. Hence, for every $z'' \in \mathbb{Z}^n$ there exist a continuous path between z'' and every point of $\{z''\} \oplus R_k$. Consequently, there exists a continuous path between any two points in $A \oplus R_k$, i.e. $A \oplus R_k$ is connected in $\mathbb{R}^n$.

Conversely, suppose $A \oplus R_k$ is connected in $\mathbb{R}^n$. By construction of R_k there exist a continuous k-path between any grid points in A and A is k-connected. q.e.d.

As a result of this lemma k-components can be defined as continuous components of the reconstruction by R_k. The proofs of the following two lemmas will be omitted. They are similarly simple.

Lemma 3. *A discrete object $A \subseteq \mathbb{Z}^n$ is k-separating iff $(\mathbb{Z}^n \setminus A) \oplus R_k$ consists of exactly two continuous components.*

Lemma 4. *A k-separating object A is k-minimal iff $((\mathbb{Z}^n \setminus A) \cup \{z\}) \oplus R_k$ is not separating for all $z \in A$.*

Let us conclude this section with some remarks. The reconstruction by dilation with a structural element R_k represents only the number of components of a discrete object. For example the reconstruction of a discrete curve or a surface is not a continuous curve or surface. It also does not reconstruct the number of background components.

Instead of R_k we could have used other structural elements such as the convex hull of R_k or the smallest closed ball that contains R_k. For these examples all results of this section would still hold.

We have chosen the term "reconstruction by dilation" to represent a dual or opposite operation of digitization by dilation. It should be pointed out that this term is also used in mathematical morphology with a different meaning. In the context of geodesic transformations there is a notion of "reconstruction by dilation of a mask image from a marker image" [Soi99] which is not related to our definition.

5 Tunnel-Free Surface Digitizations

In this section tunnel-free digitizations by dilations will be studied for surfaces with or without boundary. The first theorem establishes a sufficient condition such that a digitization by dilation is tunnel-free.

Theorem 1. *Let $\Delta(S)$ be a digitization of a surface $S \in \mathbb{R}^n$ such that the reconstruction of the background $(\Delta(S))^C \oplus R_k$ does not hit S. Then $\Delta(S)$ is k-tunnel-free.*

Proof. Let us assume $\Delta(S)$ is a digitization of $S \in \mathbb{R}^n$ such that $(\Delta(S))^C \oplus R_k$ does not hit the surface S. Suppose that there exists a k-tunnel in $\Delta(S)$. Then there exists a continuous path in $(\Delta(S))^C \oplus R_k$ that hits the surface S, which is a contradiction to our assumptions.

Note that the condition "the reconstruction of the background by R_k does not hit the surface" is not necessary for a k-tunnel-free digitization. Using this theorem, a relationship between existence of k-tunnels a digitization by dilation $\Delta_\oplus^D$ and the choice of the domain D can be proven.

Theorem 2. *Let $\Delta_\oplus^D$ be a digitization by dilation with the domain $D \supseteq R_k$. Then $\Delta_\oplus^D(S)$ is k-tunnel-free for every surface $S \subseteq \mathbb{R}^n$.*

Proof. Let $\Delta_\oplus^D$ be a digitization by dilation with the domain $D \supseteq R_k$. For every $S \subseteq \mathbb{R}^n$, $\Delta_\oplus^{R_k}(S)$ is a subset of $\Delta_\oplus^D(S)$. The digitization $\Delta_\oplus^{R_k}(S)$ contains all points $z \in \mathbb{Z}^n$ such that $z \oplus R_k$ hits S. The construction of its background $(\Delta_\oplus^{R_k}(S))^C \oplus R_k$ does not hit S. Hence, $\Delta_\oplus^{R_k}(S)$ is k-tunnel-free and so is $\Delta_\oplus^D(S)$.

This theorem justifies the grid-intersection [Kle85] and supercover digitization [CK95] as appropriate digitizations schemes for surfaces with or without boundary.

The grid-intersection digitization is a digitization by dilation whose domain is R_{n-1}. As a consequence of Theorem 2 every grid-intersection digitization of surfaces is always $(n-1)$-tunnel-free. The domain of the supercover digitization is $\mathbb{V}(0)$, which is a superset of R_0. Hence, the supercover of every surface is 0-tunnel-free.

6 Summary and Future Work

In this article we investigated digital topology with methods from mathematical morphology. We introduced reconstructions by dilations with a structural element R_k. We have proven that important notions from digital topology, such as k-neighbors, k-connected and k-separating objects, can be defined by continuous properties of the reconstruction dilation with R_k.

As a consequence the new notions have been used to prove that every digitization by dilation whose basic domain is a superset of R_k is k-tunnel-free. In particular the grid-intersection digitization and the supercover of every surface is always 0-tunnel-free and $(n-1)$-tunnel-free, respectively.

Currently we are relating our work on r-surfaces [LW00a] to the results of this paper in order to obtain a theoretical framework for the digitization of surfaces with boundary. Our future work includes also an algebraic study of the relationship between digitizations and reconstructions by dilations on the abstraction level of windowing functions.

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