Homotopy in Digital Spaces*

Rafael Ayala[1], Eladio Domínguez[2], Angel R. Francés[2], and Antonio Quintero[1]

[1] Dpt. de Geometría y Topología, Facultad de Matemáticas. Universidad de Sevilla
Apto. 1160. E-41080, Sevilla, Spain
quintero@cica.es

[2] Dpt. de Informática e Ingeniería de Sistemas, Facultad de Ciencias
Universidad de Zaragoza. E-50009, Zaragoza, Spain
ccia@posta.unizar.es

Abstract. The main contribution of this paper is a new "extrinsic" digital fundamental group that can be readily generalized to define higher homotopy groups for arbitrary digital spaces. We show that the digital fundamental group of a digital object is naturally isomorphic to the fundamental group of its continuous analogue. In addition, we state a digital version of the Seifert–Van Kampen theorem.

Keywords: Digital homotopy, digital fundamental group, lighting functions, Seifert–Van Kampen theorem.

1 Introduction

Thinning is an important pre-processing operation in pattern recognition whose goal is to reduce a digital image into a "topologically equivalent skeleton". In particular, thinning algorithms must preserve "tunnels" when processing three–dimensional digital images. As it was pointed out in [4], this requirement can be correctly established by means of an appropriate digital counterpart of the classical fundamental group in algebraic topology; see [11].

The first notion of a digital fundamental group (and even of higher homotopy groups) is dued to Khalimsky ([3]). He gave an "extrinsic" definition of this notion for a special class of digital spaces based on a topology on the set $\mathbb{Z}^n$, for every positive integer n. However, this approach is not suitable for other kinds of digital spaces often used in image processing, as the (α, β)-connected spaces, where $(\alpha, \beta) \in \{(4, 8), (8, 4)\}$ if $n = 2$ and $(\alpha, \beta) \in \{(6, 26), (26, 6), (6, 18), (18, 6)\}$ if $n = 3$. Within the graph-theoretical approach to Digital Topology, Kong solved partially this problem in [4] by defining an "intrinsic" digital fundamental group for the class of *strongly normal digital picture spaces* (SN-DPS), which include as particular cases both the (α, β)-connected spaces and the 2- and 3-dimensional Khalimsky's spaces. Nevertheless, Kong's definition seems not be general enough to give higher homotopy groups.

* This work has been partially supported by the projects DGICYT PB96-1374 and DGES PB96-0098C04-01.

G. Borgefors, I. Nyström, and G. Sanniti di Baja (Eds.): DGCI 2000, LNCS 1953, pp. 3–14, 2000.

The goal of this paper is to introduce, via the framework of the multilevel architecture for Digital Topology in [2], a new notion of digital fundamental group (denoted by π_1^d) that, at least from a theoretical point of view, presents certain advantages over the notions of Khalimsky and Kong. Firstly, the group π_1^d is defined by using an "extrinsic" setting that can be readily generalized to define higher digital homotopy groups (see Section 3). Secondly, this group is available on a larger class of digital spaces than Khalimsky's and Kong's digital fundamental groups, since the digital spaces described in the multilevel architecture quoted above include as examples all Khalimsky's spaces and the (α, β)-connected spaces (see [1]), and even most of the SN-DPS. And finally, our digital fundamental group of a digital object O turns out to be naturally isomorphic to the fundamental group of its continuous analogue; that is, of the continuous object perceived when one looks at O (see Section 4). This isomorphism shows that the group π_1^d is an appropriate counterpart of the ordinary fundamental group in continuous topology. In particular, this fact leads us to obtain a (restricted) digital version of the Seifert–Van Kampen Theorem (see Section 5). Although this theorem provides a powerful theoretical tool to obtain the group π_1^d for certain digital objects, it remains as an open question to find an algorithm that computes this group for arbitrary objects; that is, to resemble in our framework the well–known algorithm for the fundamental group of polyhedra ([8]). This problem could be tackled by adapting to our multilevel architecture the algorithm recently developed by Malgouyres in [7], which computes a presentation of the digital fundamental group of an object embedded in an arbitrary graph.

2 The Multilevel Architecture

In this section we briefly summarize the basic notions of the multilevel architecture for digital topology developed in [2] as well as the notation that will be used through all the paper.

In that architecture, the spatial layout of pixels in a digital image is represented by a *device model*, which is a homogeneously n-dimensional locally finite polyhedral complex K. Each n-cell in K is representing a pixel, and so the digital object displayed in a digital image is a subset of the set $\mathrm{cell}_n(K)$ of n-cells in K. A *digital space* is a pair (K, f), where K is a device model and f is *weak lighting function* defined on K. The function f is used to provide a continuous interpretation, called *continuous analogue*, for each digital object $O \subseteq \mathrm{cell}_n(K)$. Next we recall the notion of weak lighting function. For this we need the following notation.

Let K be a device model and γ, σ cells in K. We shall write $\gamma \leq \sigma$ if γ is a face of σ, and $\gamma < \sigma$ if in addition $\gamma \neq \sigma$. If $|K|$ denotes the underlying polyhedron of K, a centroid-map is a map $c : K \to |K|$ such that $c(\sigma)$ belongs to the interior of σ; that is, $c(\sigma) \in \sigma - \partial\sigma$, where $\partial\sigma = \cup\{\gamma; \gamma < \sigma\}$ stands for the boundary of σ. Given a cell $\alpha \in K$ and a digital object $O \subseteq \mathrm{cell}_n(K)$, the *star of* α *in* O and *the extended star of* α *in* O are respectively the digital objects

$\mathrm{st}_n(\alpha; O) = \{\sigma \in O; \alpha \leq \sigma\}$ and $\mathrm{st}^*_n(\alpha; O) = \{\sigma \in O; \alpha \cap \sigma \neq \emptyset\}$. The *support of* O, $\mathrm{supp}(O)$, is the set of all cells $\alpha \in K$ such that $\alpha = \cap\{\sigma; \sigma \in \mathrm{st}_n(\alpha; O)\}$. To ease the writing, we shall use the following notation: $\mathrm{supp}(K) = \mathrm{supp}(\mathrm{cell}_n(K))$, $\mathrm{st}_n(\alpha; K) = \mathrm{st}_n(\alpha; \mathrm{cell}_n(K))$ and $\mathrm{st}^*_n(\alpha; K) = \mathrm{st}^*_n(\alpha; \mathrm{cell}_n(K))$. Finally, we shall write $\mathcal{P}(A)$ for the family of all subsets of a given set A.

Given a device model K, a *weak lighting function* (w.l.f.) on K is a map $f : \mathcal{P}(\mathrm{cell}_n(K)) \times K \to \{0, 1\}$ satisfying the following five properties for all $O \in \mathcal{P}(\mathrm{cell}_n(K))$ and $\alpha \in K$:

1. if $\alpha \in O$ then $f(O, \alpha) = 1$;
2. if $\alpha \notin \mathrm{supp}(O)$ then $f(O, \alpha) = 0$;
3. $f(O, \alpha) \leq f(\mathrm{cell}_n(K), \alpha)$;
4. $f(O, \alpha) = f(\mathrm{st}^*_n(\alpha; O), \alpha)$; and,
5. if $O' \subseteq O \subseteq \mathrm{cell}_n(K)$ and $\alpha \in K$ are such that $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O')$, $f(O', \alpha) = 0$ and $f(O, \alpha) = 1$, then the set of cells $\alpha(O'; O) = \{\beta < \alpha; f(O', \beta) = 0,\ f(O, \beta) = 1\}$ is not empty and connected in $\partial\alpha$; moreover, if $O \subseteq \overline{O} \subseteq \mathrm{cell}_n(K)$, then $f(\overline{O}, \beta) = 1$ for every $\beta \in \alpha(O'; O)$.

A w.l.f. f is said to be *strongly local* if $f(O, \alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$ for all $\alpha \in K$ and $O \subseteq \mathrm{cell}_n(K)$. Notice that this strong local condition implies both properties 4 and 5 in the definition above.

To define the continuous analogue of a given digital object O in a digital space (K, f), we need to introduce several other intermediate models (the levels of this multilevel architecture) as follows.

The *device level* of O is the subcomplex $K(O) = \{\alpha \in K; \alpha \leq \sigma,\ \sigma \in O\}$ of K induced by the cells in O. Notice that the map f_O given by $f_O(O', \alpha) = f(O, \alpha) f(O', \alpha)$, for all $O' \subseteq O$ and $\alpha \in K(O)$, is a w.l.f. on $K(O)$, and we call the pair $(K(O), f_O)$ the *digital subspace* of (K, f) induced by O.

The *logical level* of O is an undirected graph, $\mathcal{L}^f_O$, whose vertices are the centroids of n-cells in O and two of them $c(\sigma)$, $c(\tau)$ are adjacent if there exists a common face $\alpha \leq \sigma \cap \tau$ such that $f(O, \alpha) = 1$.

The *conceptual level of* O is the directed graph $\mathcal{C}^f_O$ whose vertices are the centroids $c(\alpha)$ of all cells $\alpha \in K$ with $f(O, \alpha) = 1$, and its directed edges are $(c(\alpha), c(\beta))$ with $\alpha < \beta$.

The *simplicial analogue* of O is the order complex $\mathcal{A}^f_O$ associated to the digraph $\mathcal{C}^f_O$. That is, $\langle x_0, x_1, \ldots, x_m\rangle$ is an m-simplex of $\mathcal{A}^f_O$ if $x_0, x_1, \ldots, x_m$ is a directed path in $\mathcal{C}^f_O$. This simplicial complex defines the simplicial level for the object O in the architecture and, finally, the *continuous analogue* of O is the underlying polyhedron $|\mathcal{A}^f_O|$ of $\mathcal{A}^f_O$.

For the sake of simplicity, we will usually drop "f" from the notation of the levels of an object. Moreover, for the whole object $\mathrm{cell}_n(K)$ we will simply write $\mathcal{L}_K$, $\mathcal{C}_K$ and $\mathcal{A}_K$ for its levels.

Example 1. In this paper it will be essential the role played by the archetypical device model R^n, termed the *standard cubical decomposition* of the Euclidean n-space $\mathbb{R}^n$. Recall that the device model R^n is the complex determined by the collection of unit n-cubes in $\mathbb{R}^n$ whose edges are parallel to the coordinate axes and whose centers are in the set $\mathbb{Z}^n$. The centroid-map we will consider in R^n associates to each cube σ its barycenter $c(\sigma)$. In particular, if $\dim \sigma = n$ then $c(\sigma) \in \mathbb{Z}^n$, where $\dim \sigma$ stands for the dimension of σ. So that, every digital object O in R^n can be identified with a subset of points in $\mathbb{Z}^n$. Henceforth we shall use this identification without further comment. In particular, we shall consider the family of digital spaces (R^n, g), for every positive integer n, where g is the w.l.f. given by $g(O, \alpha) = 1$ if and only if $\mathrm{st}_n(\alpha; R^n) \subseteq O$, for any digital object $O \subseteq \mathrm{cell}_n(R^n)$ and any cell $\alpha \in R^n$. Notice that the w.l.f. g induces in R^n the $(2n, 3^n - 1)$–connectedness (see [1, Def. 11]); that is, the generalization to arbitrary dimension of the $(4, 8)$–connectedness on $\mathbb{Z}^2$.

3 A Digital Fundamental Group

We next introduce an "extrinsic" digital fundamental group that readily generalizes to higher digital homotopy groups of arbitrary digital spaces. For this purpose, we will first define a digital-map (Def. 2), and then we will focus our interest in a special class of such digital-maps termed digital homotopies (Defs. 5 and 6).

Definition 1. *Let $S \subseteq cell_n(K)$ be a digital object in a digital space (K, f). The* light body of K shaded with S *is the set of cells*

$$Lb(K/S) = \{\alpha \in K; f(cell_n(K), \alpha) = 1, f(S, \alpha) = 0\};$$

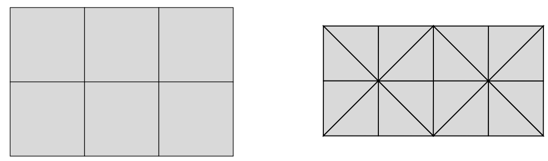
that is, $Lb(K/S) = \{\alpha \in K; c(\alpha) \in |\mathcal{A}_K| - |\mathcal{A}_S|\}$. Notice that if $S = \emptyset$ is the empty object then $Lb(K/\emptyset) = Lb(K) = \{\alpha \in K; f(cell_n(K), \alpha) = 1\}$. Moreover, $Lb(K/cell_n(K)) = \emptyset$.

Definition 2. *Let $(K_1, f_1), (K_2, f_2)$ be two digital spaces, with $\dim K_i = n_i$ $(i = 1, 2)$, and let $S_1 \subset cell_{n_1}(K_1)$ and $S_2 \subset cell_{n_2}(K_2)$ be two digital objects. A map $\phi : Lb(K_1/S_1) \to Lb(K_2/S_2)$ is said to be a* (digital) (S_1, S_2)-map from (K_1, f_1) into (K_2, f_2) *(or, simply, a* d-map *denoted $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$) provided*

1. *$\phi(cell_{n_1}(K_1) - S_1) \subseteq cell_{n_2}(K_2) - S_2$; and,*
2. *for $\alpha, \beta \in Lb(K_1/S_1)$ with $\alpha < \beta$ then $\phi(\alpha) \leq \phi(\beta)$.*

Example 2. (1) Let $S' \subset S \subseteq \mathrm{cell}_n(K)$ be two digital objects and $(K(S), f_S)$ the digital subspace of (K, f) induced by S. Then, the inclusion $\mathrm{Lb}(K(S)/S') \subseteq \mathrm{Lb}(K/S')$ is a (S', S')-map from $(K(S), f_S)$ into (K, f). And, similarly, the inclusion $\mathrm{Lb}(K/S') \subseteq \mathrm{Lb}(K/\emptyset)$ defines a $(S', \emptyset)$-map from (K, f) into itself.

(2) Let $S_1 \subset \mathrm{cell}_{n_1}(K_1)$ and $\sigma \in \mathrm{cell}_{n_2}(K_2)$. For any digital object $S_2 \subseteq \mathrm{cell}_{n_2}(K_2) - \{\sigma\}$, the constant map $\phi^\sigma : \mathrm{Lb}(K_1/S_1) \to \mathrm{Lb}(K_2/S_2)$, given by $\phi^\sigma(\alpha) = \sigma$, for all $\alpha \in \mathrm{Lb}(K_1/S_1)$, defines a (S_1, S_2)-map from (K_1, f_1) into (K_2, f_2) .



$V_{(2,1)}$ $\mathcal{A}_{V_{(2,1)}}$

Fig. 1. A $(2,1)$-window in R^2 and its simplicial analogue

(3) The composition of digital maps is a digital map. Namely, if

$$\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2) \text{ and } \Phi_{S_2,S_3} : (K_2, f_2) \to (K_3, f_3)$$

are d-maps, then their composite $\Phi_{S_1,S_2} \circ \Phi_{S_2,S_3}$ is also a d-map from (K_1, f_1) into (K_3, f_3).

A d-map from (K_1, f_1) into (K_2, f_2) naturally induces a simplicial map between the simplicial analogues of K_1 and K_2. More precisely, if $L_1, L_2 \subseteq L$ are simplicial complexes and $L_1 \setminus L_2 = \{\alpha \in L_1; \alpha \cap |L_2| = \emptyset\}$ denotes the simplicial complement of L_2 in L_1, then it is straightforward to show

Proposition 1. *Any d-map $\Phi_{S_1,S_2} : (K_1, f_1) \to (K_2, f_2)$ induces a simplicial map $\mathcal{A}(\Phi_{S_1,S_2}) : \mathcal{A}_{K_1} \setminus \mathcal{A}_{S_1} \to \mathcal{A}_{K_2} \setminus \mathcal{A}_{S_2}$.*

Given two points $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{R}^m$, we write $x \preceq y$ if $x_i \leq y_i$, for all $1 \leq i \leq m$, while $x+y$ will stand for the point $(x_1+y_1, \ldots, x_m+y_m) \in \mathbb{R}^m$.

Definition 3. *Given two points $r, x \in \mathbb{Z}^m$, with $r_i \geq 0$ for $1 \leq i \leq m$, we call a* window of size r *(or* r-window*) of* R^m based at x *to the digital subspace V_r^x of (R^m, g) induced by the digital object $O_r^x = \{\sigma \in cell_m(R^m); x \preceq c(\sigma) \preceq x+r\}$, where (R^m, g) is the digital space defined in Example 1.*

Notice that the simplicial analogue of an r-window V_r^x of R^m is (simplicially isomorphic to) a triangulation of a unit n-cube, where n is the number of non-zero coordinates in r (see Figure 1). Moreover, the set $\{y \in \mathcal{Z}^m; x \preceq y \preceq x + r\}$ are the centroids of the cells in $\mathrm{Lb}(V_r^x/\emptyset)$ which actually span the simplicial analogue of V_r^x. Here $\mathcal{Z} = \frac{1}{2}\mathbb{Z}$ stands for the set of points $\{z \in \mathbb{R}; z = y/2, y \in \mathbb{Z}\}$.

For the sake of simplicity, we shall write V_r to denote the r-window of R^m based at the point $x = (0, \ldots, 0) \in \mathbb{Z}^m$. Moreover, if V_r is an r-window of R^1, then $\mathrm{Lb}(V_r/\emptyset) = \{\sigma_0, \sigma_1, \ldots, \sigma_{2r-1}, \sigma_{2r}\}$ consists of $2r+1$ cells such that $c(\sigma_i) = i/2$.

With this notation we are now ready to give “extrinsic” notions of walks and loops, in a digital object, which will lead us to the definition of a digital fundamental group.

Definition 4. *Let (K,f) be a digital space and $S,O \subseteq cell_n(K)$ two disjoint digital objects in (K,f). A* S-walk in O of length $r \in \mathbb{Z}$ from σ to τ *is a digital $(\emptyset,S)$-map $\phi_r : Lb(V_r/\emptyset) \rightarrow Lb(K(O\cup S)/S)$ such that $\phi_r(\sigma_0)=\sigma$ and $\phi_r(\sigma_{2r})=\tau$. A* S-loop in O based at σ *is a S-walk ϕ_r such that $\phi_r(\sigma_0)=\phi_r(\sigma_{2r})=\sigma$.*

The juxtaposition *of two given S-walks ϕ_r,ϕ_s in O, with $\phi_r(\sigma_{2r})=\phi_s(\sigma_0)$, is the S-walk $\phi_r * \phi_s : Lb(V_{r+s}/\emptyset) \rightarrow Lb(K(O\cup S)/S)$ of length $r+s$ given by*

$$\phi_r * \phi_s(\sigma_i) = \begin{cases} \phi_r(\sigma_i) & \text{if } 0 \leq i \leq 2r \\ \phi_s(\sigma_{i-2r}) & \text{if } 2r \leq i \leq 2(r+s) \end{cases}$$

Notice that the notion of a S-walk is compatible with the definition of S-path given in [1, Def. 5]. Actually, each S-walk ϕ_r defines a S-path given by the sequence $\varphi(\phi_r)=(\phi_r(\sigma_{2i}))_{i=0}^r$. And, conversely, a S-path $(\tau_i)_{i=0}^r$ in O can be associated with a family Φ_r of S-walks such that $\phi_r \in \Phi_r$ if and only if $\phi_r(\sigma_{2i})=\tau_i$ $(0 \leq i \leq r)$ and $\phi_r(\sigma_{2i-1}) \in \{\alpha \leq \tau_{2i-2} \cap \tau_{2i}; f(O\cup S,\alpha)=1, f(S,\alpha)=0\}$ $(1 \leq i \leq r)$. However, this "extrinsic" notion of S-walk will be more suitable to define the digital fundamental group of an object since, together with the notion of r-window, it allows us to introduce the following definition of digital homotopy.

Definition 5. *Let ϕ_r^1,ϕ_r^2 two S-walks in O of the same length $r \in \mathbb{Z}$ from σ to τ. We say that ϕ_r^1,ϕ_r^2 are* digitally homotopic *(or, simply,* d-homotopic*)* relative $\{\sigma,\tau\}$*, and we write $\phi_r^1 \simeq_d \phi_r^2$ rel. $\{\sigma,\tau\}$, if there exists an (r,s)-window $V_{(r,s)}$ in R^2 and a $(\emptyset,S)$-map $H : Lb(V_{(r,s)}/\emptyset) \rightarrow Lb(K(O\cup S)/S)$, called a* d-homotopy*, such that $H(i/2,0)=\phi_r^1(\sigma_i)$ and $H(i/2,s)=\phi_r^2(\sigma_i)$, for $0 \leq i \leq 2r$, and moreover $H(0,j/2)=\sigma$ and $H(r,j/2)=\tau$, for $0 \leq j \leq 2s$. Here we use the identification $H(x,y)=H(\alpha)$, where $c(\alpha)=(x,y) \in \mathcal{Z}^2$ is the centroid of a cell $\alpha \in Lb(V_{(r,s)}/\emptyset)$.*

Clearly, the previous definition of d-homotopy induces an equivalence relation between the S-walks in O from σ to τ of the same length. Moreover, it is easy to show that d-homotopies are compatible with the juxtaposition of S-walks.

The definition of d-homotopy between S-walks of the same length can be extended to arbitrary S-walks as follows.

Definition 6. *Let ϕ_r,ϕ_s two S-walks in O from σ to τ of lengths $r \neq s$. We say that ϕ_r is d-homotopic to ϕ_s relative $\{\sigma,\tau\}$, $\phi_r \simeq_d \phi_s$ rel. $\{\sigma,\tau\}$, if there exist constant S-loops $\phi_{r'}^\tau$ and $\phi_{s'}^\tau$ such that $r+r'=s+s'$ and $\phi_r * \phi_{r'}^\tau \simeq_d \phi_s * \phi_{s'}^\tau$ rel. $\{\sigma,\tau\}$.*

Proposition 2. *Let ϕ_r be a S-walk in O from σ to τ, and ϕ_s^σ, ϕ_s^τ two constant S-loops of the same length s. Then, $\phi_s^\sigma * \phi_r \simeq_d \phi_r * \phi_s^\tau$ rel. $\{\sigma,\tau\}$.*

The proof of this proposition, although it is not immediate, can be directly obtained from definitions by means of an inductive argument. Moreover, from Proposition 2 and the remarks above, it can be easily derived that d-homotopy defines an equivalence relation in the set of S-walks in O of arbitrary length. So, we next introduce the digital fundamental group as follows.

Definition 7. *Let O be a digital object in a digital space (K, f). The* digital fundamental group *of O at σ is the set $\pi_1^d(O, \sigma)$ of d-homotopy classes of $\emptyset$-loops in O based at σ provided with the product operation $[\phi_r] \cdot [\psi_s] = [\phi_r * \psi_s]$.*

Remark 1. Definition 7 can be easily extended to the definition of a digital fundamental group for the complement of an object S in a digital space using the notion of S-loop. Moreover, this last notion readily generalizes to give higher digital homotopy groups by replicating the same steps as above but starting with a suitable notion of m-dimensional S-loop. More explicitly, let $r \in \mathbb{Z}^m$ be a point with positive coordinates, and call boundary of an r-window V_r to the set of cells $\partial V_r = \{\alpha \in \mathrm{Lb}(V_r/\emptyset); c(\alpha) \in \partial \mathcal{A}_{V_r}\}$. Notice that the boundary $\partial \mathcal{A}_{V_r}$ is well-defined since $\mathcal{A}_{V_r}$ triangulates the unit m-cube. Then define an *m-dimensional S-loop* in O of size r at σ as any $(\emptyset, S)$-map $\phi_r : \mathrm{Lb}(V_r/\emptyset) \to \mathrm{Lb}(K(O \cup S)/S)$ such that $\phi_r|_{\partial V_r} = \sigma$.

4 Isomorphism with the Continuous Fundamental Group

As usual the fundamental group of a topological space X, $\pi_1(X, x_0)$, is defined to be the set of homotopy classes of paths $\xi : I = [0,1] \to X$ that send 0 and 1 to some fixed point x_0 (*loops* at x_0). The set $\pi_1(X, x_0)$ is given the structure of a group by the operation $[\alpha] \cdot [\beta] = [\alpha * \beta]$, where $\alpha * \beta$ denotes the juxtaposition of paths. However, for a polyhedron $|K|$ there is an alternative definition of the fundamental group $\pi_1(|K|, x_0)$ that is more convenient for our purposes, so we next explain it briefly. Recall that an *edge–path* in $|K|$ from a vertex v_0 to a vertex v_n is a sequence α of vertices $v_0, v_1, \ldots, v_n$, such that for each $k = 1, 2, \ldots, n$ the vertices v_{i-1}, v_i span a simplex in K (possibly $v_{i-1} = v_i$). If $v_0 = v_n$, α is called an *edge–loop*.

Given another edge–path $\beta = (v_j)_{j=n}^{m+n}$ whose first vertex is the same as the last vertex of α, the *juxtaposition* $\alpha * \beta = (v_i)_{i=0}^{m+n}$ is defined in the obvious way. The *inverse* of α is $\alpha^{-1} = (v_n, v_{n-1}, \ldots, v_0)$.

Two edge–paths α and β are said to be *equivalent* if one can be obtained from the other by a finite sequence of operations of the form:

(a) if $v_{k-1} = v_k$, replace $\ldots, v_{k-1}, v_k, \ldots$ by $\ldots, v_k, \ldots$, or conversely replace $\ldots, v_k, \ldots$ by $\ldots, v_{k-1}, v_k, \ldots$; or
(b) if v_{k-1}, v_k, v_{k+1} span a simplex of K (not necessarily 2-dimensional), replace $\ldots, v_{k-1}, v_k, v_{k+1}, \ldots$ by $\ldots, v_{k-1}, v_{k+1}, \ldots$, or conversely.

This clearly sets up an equivalence relation between edge–paths, and the set of equivalence classes $[\alpha]$ of edge–loops α in K, based at a vertex v_0, forms a group $\pi_1(K, v_0)$ with respect to the juxtaposition of edge–loops. This group will be called the *edge–group* of K. Moreover it can be proved

Theorem 1. (Maunder; 3.3.9) *There exists an isomorphism $\pi_1(|K|, v_0) \to \pi_1(K, v_0)$ which carries the class $[f]$ to the class $[\alpha_f]$, where α_f is an edge–loop defined by a simplicial approximation of f.*

Corollary 1. *Let O, S be two disjoint digital objects of a digital space (K, f). Then $\pi_1(\mathcal{A}_{O\cup S} \setminus \mathcal{A}_S, c(\sigma)) \cong \pi_1(|\mathcal{A}_{O\cup S}| - |\mathcal{A}_S|, c(\sigma))$ for any $\sigma \in O$.*

The proof of this corollary is a consequence of Theorem 1 and next lemma.

Lemma 1. *Let $K, L \subseteq J$ be two full subcomplexes. Then $|K \setminus L| = |K \setminus K \cap L|$ is a strong deformation retract of $|K| - |L| = |K| - |K \cap L|$.*

This lemma is actually Lemma 72.2 in [9] applied to the full subcomplex $K \cap L \subseteq K$ (this fact is shown using that L is full in J).

We are now ready to prove the main result of this paper. Namely,

Theorem 2. *Let O be a digital object in the digital space (K, f). Then there exists an isomorphism $h : \pi_1^d(O, \sigma) \to \pi_1(|\mathcal{A}_O|, c(\sigma))$.*

The function h is defined as follows. Let ϕ_r be any $\emptyset$-loop in O based at σ. Then the sequence $c(\phi_r) = (c(\phi_r(\sigma_i)))_{i=0}^{2r}$ defines and edge–loop in $\mathcal{A}_O$ based at $c(\sigma)$, and so we set $h([\phi_r]) = [c(\phi_r)]$. Lemma 2 below, and the two immediate properties (1), (2), show that the function h is a well defined homomorphism of groups.

(1) $c(\phi_r * \phi'_s) = c(\phi_r) * c(\phi'_s)$
(2) if ϕ_r is a constant $\emptyset$-loop then $c(\phi_r)$ is also a constant edge–loop.

Lemma 2. *If $\phi_r \simeq_d \phi'_s$ are equivalent $\emptyset$-loops then $c(\phi_r)$ and $c(\phi'_s)$ define both the same element in $\pi_1(|\mathcal{A}_O|, c(\sigma))$.*

Proof. According to (2) above and the definition of equivalence between two $\emptyset$-loops it will be enough to show that any d-homotopy $H : \mathrm{Lb}(V_{(r,s)}/\emptyset) \to \mathrm{Lb}(K(O)/\emptyset)$ between two $\emptyset$-loops ϕ_r, ϕ'_r of the same length r induces a continuous homotopy $\tilde{H} : [0,1] \times [0,1] \to |\mathcal{A}_O|$ between $c(\phi_r)$ and $c(\phi'_r)$. This fact is readily checked since Proposition 1 yields a simplicial map $\mathcal{A}(H) : \mathcal{A}_{V_{(r,s)}} \to \mathcal{A}_O$, and $\mathcal{A}_{V_{(r,s)}}$ is (simplicially isomorphic to) a triangulation of the unit square (see Figure 1). Moreover $\mathcal{A}(H)$ restricted to the top and the bottom of that unit square define $c(\phi_r)$ and $c(\phi'_r)$ respectively, and the result follows.

Now, let γ be any loop in $|\mathcal{A}_O|$ based at $c(\sigma)$. By Corollary 1 we can assume that $\gamma = (c(\gamma_i))_{i=0}^{k}$ is an edge–loop based at $c(\sigma)$ in $\mathcal{A}_O$. After applying equivalence operations (a) and (b) above we can reduce the edge–loop γ to a new edge–loop $\overline{\gamma} = (c(\overline{\gamma}_i))_{i=0}^{2r}$ equivalent to γ and such that $\overline{\gamma}_{2i-1}$ are k-cells in K with $k < n$ and $\overline{\gamma}_{2i-1} \leq \overline{\gamma}_{2i-2} \cap \overline{\gamma}_{2i}$ $(1 \leq i \leq r)$. By the use of $\overline{\gamma}$ we define the following set $F(\gamma)$ of $\emptyset$-loops at σ of length r.

The set $F(\gamma)$ consists of all $\emptyset$-loops ϕ_r for which $\phi_r(\sigma_0) = \phi(\sigma_{2r}) = \sigma$, $\phi_r(\sigma_{2i-1}) = \overline{\gamma}_{2i-1}$ $(1 \leq i \leq r)$ and $\phi_r(\sigma_{2i}) \in \mathrm{st}_n(\overline{\gamma}_{2i}; O)$ $(0 \leq i \leq r)$. Notice that $\mathrm{st}_n(\overline{\gamma}_{2i}; O) = \{\overline{\gamma}_{2i}\}$ if and only if $\overline{\gamma}_{2i} \in O$, while $\mathrm{st}_n(\overline{\gamma}_{2i}; O)$ contains at least two elements otherwise. This is clear since $c(\overline{\gamma}_{2i}) \in \mathcal{A}_O$ yields $\overline{\gamma}_{2i} \in \mathrm{supp}(O)$ by Axiom 2 of w.l.f.'s. Notice that $F(\gamma) = \{\overline{\gamma}\}$ if and only if $\overline{\gamma}_{2i} \in O$ for all $0 \leq i \leq r$; and moreover, in any case, $F(\gamma) \neq \emptyset$ is a non-empty set.

Lemma 3. *For each $\phi_r \in F(\gamma)$, $c(\phi_r)$ is homotopic to the loop γ. Hence h is onto.*

Proof. Let us consider the set $\overline{F}(\gamma)$ of edge–loops $\alpha = (\alpha_i)_{i=0}^{2r}$ at $c(\sigma)$ such that $\alpha_0 = \alpha_{2r} = c(\sigma)$, $\alpha_{2i-1} = \overline{\gamma}_{2i-1}$ $(1 \leq i \leq r)$ and $\alpha_{2i} \in \mathrm{st}_n(\overline{\gamma}_{2i}; O) \cup \{\overline{\gamma}_{2i}\}$. Notice that $\{c(\phi_r); \phi_r \in F(\gamma)\} \cup \{\overline{\gamma}\} \subseteq \overline{F}(\gamma)$. Since $\overline{\gamma}$ was obtained from γ by transformations of types (a) and (b), they are equivalent edge–loops by Theorem 1. So, it will suffice to show that α is homotopic to $\overline{\gamma}$, for any $\alpha \in \overline{F}(\gamma)$. This will be done by induction on the number $t(\alpha)$ of vertices $\alpha_{2i} \neq \overline{\gamma}_{2i}$ in α.

For $t(\alpha) = 0$ we get $\alpha = \overline{\gamma}$. Assume that $\alpha \in \overline{F}(\gamma)$ is equivalent to $\overline{\gamma}$ if $t(\alpha) \leq t-1$. Then, for an edge–loop $\alpha \in \overline{F}(\gamma)$ with $t(\alpha) = t$ let α_{2i} any vertex in α such that $\alpha_{2i} \neq \overline{\gamma}_{2i}$. Then we have $\overline{\gamma}_{2i+1}, \overline{\gamma}_{2i-1} < \overline{\gamma}_{2i} < \alpha_{2i}$, and we obtain a new edge–loop $\tilde{\alpha} \in \overline{F}(\gamma)$, replacing α_{2i} by $\overline{\gamma}_{2i}$ in α, with $t(\tilde{\alpha}) = t-1$ and two equivalence transformations of type (b) relating α and $\tilde{\alpha}$. Hence α is an edge–loop equivalent to $\tilde{\alpha}$ and, by induction hypothesis, to $\overline{\gamma}$.

Lemma 4. *Any two $\emptyset$-loops ϕ_r^1, ϕ_r^2 in $F(\gamma)$ are d-homotopic.*

Proof. It is enough to observe that the map $H : \mathrm{Lb}(V_{(r,1)}/\emptyset) \to \mathrm{Lb}(K(O)/\emptyset)$ given by $H(i/2, 0) = \phi_r^1(\sigma_i)$, $H(i/2, 1) = \phi_r^2(\sigma_i)$ and $H(i/2, 1/2) = \overline{\gamma}_i$, for $0 \leq i \leq 2r$, and $H(i - 1/2, k) = \overline{\gamma}_{2i-1}$, for $1 \leq i \leq 2r$ and $k \in \{0, 1/2, 1\}$, is a d-homotopy relating ϕ_r^1 and ϕ_r^2. Here we use again the identification $H(x, y) = H(\alpha)$, where $c(\alpha) = (x, y) \in \mathcal{Z}^2$ is the centroid of a cell $\alpha \in \mathrm{Lb}(V_{(r,1)}/\emptyset)$.

Lemma 5. *Let γ^1 and γ^2 be two edge–loops at $c(\sigma)$ in $\mathcal{A}_O$ such that they are related by an equivalence transformation of type (a) or (b). Then there exist $\emptyset$-loops $\phi_{r_i}^i \in F(\gamma^i)$ $(i = 1, 2)$ and a d-homotopy such that $\phi_{r_1}^1 \simeq_d \phi_{r_2}^2$ rel. σ. Hence h is injective.*

Proof. In case γ^1 is related to γ^2 by a transformation of type (a), it is readily checked that $\overline{\gamma}^1 = \overline{\gamma}^2$. Hence $F(\gamma^1) = F(\gamma^2)$ and the result follows. And it suffices to check the essentially distinct twelve ways for deriving γ^2 from γ^1 by transformations of type (b) to complete the proof.

5 A Digital Seifert–Van Kampen Theorem

The Seifert–Van Kampen Theorem is the basic tool for computing the fundamental group of a space which is built of pieces whose groups are known. The statement of the theorem involves the notion of push–out of groups, so we begin by explaining this bit of algebra. A group G is said to be the push–out of the solid arrow commutative diagram

$$\begin{tikzcd}
& G_1 \arrow[dr, "\tilde{f}_1"] \arrow[drr, dashed, "\varphi_1"] & & \\
G_0 \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & G \arrow[r, dashed, "\varphi"] & H \\
& G_2 \arrow[ur, "\tilde{f}_2"'] \arrow[urr, dashed, "\varphi_2"'] & &
\end{tikzcd}$$

if for any group H and homomorphisms φ_1, φ_2 with $\varphi_1 f_1 = \varphi_2 f_2$ there exists a unique homomorphism φ such that $\varphi \tilde{f}_i = \varphi_i$ $(i = 1, 2)$. Then, the Seifert–Van Kampen Theorem is the following

Theorem 3. (Th. 7.40 in [10]) *Let K be a simplicial complex having connected subcomplexes K_1 and K_2 such that $K = K_1 \cup K_2$ and $K_0 = K_1 \cap K_2$ is connected. If $v_0 \in K_0$ is a vertex then $\pi_1(K, v_0)$ is the push–out of the diagram*

$$\begin{tikzcd} \pi_1(K_0, v_0) \arrow[r, "i_{1*}"] \arrow[d, "i_{2*}"'] & \pi_1(K_1, v_0) \arrow[d, "j_{1*}"] \\ \pi_1(K_2, v_0) \arrow[r, "j_{2*}"] & \pi_1(K, v_0) \end{tikzcd}$$

where i_{k*} and j_{k*} are the homomorphisms of groups induced the obvious inclusions.

By using explicit presentations of the groups $\pi_1(K_i, v_0)$ $(i = 0, 1, 2)$ the Seifert–Van Kampen Theorem can be restated as follows. Suppose there are presentations $\pi_1(K_i, v_0) \cong (x_1^i, x_2^i, \ldots; r_1^i, r_2^i, \ldots)$ $(i = 0, 1, 2)$ Then the fundamental group of K has the presentation

$$\begin{aligned}\pi_1(K, v_0) \cong (&x_1^1, x_2^1, \ldots, x_1^2, x_2^2, \ldots; \\ &r_1^1, r_2^1, \ldots, r_1^2, r_2^2, \ldots, i_{1*}(x_1^0) = i_{2*}(x_1^0), i_{1*}(x_2^0) = i_{2*}(x_2^0), \ldots).\end{aligned}$$

In other words, one puts together the generators and relations from $\pi_1(K_1, v_0)$ and $\pi_1(K_2, v_0)$, plus one relation for each generator x_i^0 of $\pi_1(K_0, v_0)$ which says that its images in $\pi_1(K_1, v_0)$ and $\pi_1(K_2, v_0)$ are equal.

The digital analogue of the Seifert–Van Kampen Theorem is not always true as the following example shows.

Example 3. Let O_1, O_2 be the two digital objets in the digital space (R^2, g) shown in Figure 2. It is readily checked that both $\pi_1^d(O_1, \sigma)$ and $\pi_1^d(O_2, \sigma)$ are trivial groups, but $\pi_1^d(O_1 \cup O_2, \sigma) = \mathbb{Z}$ despite of O_1, O_2 and $O_1 \cap O_2$ are connected digital objects.

However we can easily derive a Digital Seifert–Van Kampen Theorem for certain objects in a quite large class of digital spaces. Namely, the locally strong digital spaces; that is, the digital spaces (K, f) for which the lighting function f satisfies $f(O, \alpha) = f(\mathrm{st}_n(\alpha; O), \alpha)$. We point out that all the (α, β)-connected digital spaces on $\mathbb{Z}^3$ defined within the graph-theoretical approach to Digital Topology, for $\alpha, \beta \in \{6, 18, 26\}$, are examples of locally strong digital spaces; see [1, Example 2].

Theorem 4. (Digital Seifert–Van Kampen Theorem) *Let (K, f) be a locally strong digital space, and let $O \subseteq cell_n(K)$ be a digital object in (K, f) such that $O = O_1 \cup O_2$, where O_1, O_2 and $O_1 \cap O_2$ are connected digital objects. Assume in addition that $\mathcal{A}_{O_1 \cap O_2} \subseteq \mathcal{A}_{O_1} \cap \mathcal{A}_{O_2}$ and $\mathcal{A}_{O_i} \subseteq \mathcal{A}_O$ $(i = 1, 2)$. Moreover assume* *that for each cell $\sigma \in O_1 - O_2$ any cell $\tau \in O$ which is adjacent to σ in O lies in O_1. Then, for $\sigma \in O_1 \cap O_2$, $\pi_1^d(O, \sigma)$ is the push–out of the diagram*

$$\begin{array}{ccc} \pi_1^d(O_1 \cap O_2, \sigma) & \longrightarrow & \pi_1^d(O_1, \sigma) \\ \downarrow & & \downarrow \\ \pi_1^d(O_2, \sigma) & \longrightarrow & \pi_1^d(O, \sigma) \end{array}$$

where the homomorphisms are induced by the obvious inclusions.

The proof of this theorem is immediate consequence of Theorem 2 and Theorem 3 if we have at hand the equalities $|\mathcal{A}_{O_1 \cap O_2}| = |\mathcal{A}_{O_1}| \cap |\mathcal{A}_{O_2}|$ and $|\mathcal{A}_O| = |\mathcal{A}_{O_1}| \cup |\mathcal{A}_{O_2}|$. We devote the rest of this section to check these equalities.

Lemma 6. *If $f(O, \alpha) = 1$ then one of the following statements holds:*

(1) $st_n(\alpha; O) = st_n(\alpha; O_1 \cap O_2) = st_n(\alpha; O_1) = st_n(\alpha; O_2)$; or
(2) $st_n(\alpha; O) = st_n(\alpha; O_i)$ and $st_n(\alpha; O_j) = st_n(\alpha; O_1 \cap O_2)$, $\{i, j\} = \{1, 2\}$.

Proof. In case $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O_1 \cap O_2)$, we obtain (1) from the inclusions

$$\mathrm{st}_n(\alpha; O_1 \cap O_2) \subseteq \mathrm{st}_n(\alpha; O_i) \subseteq \mathrm{st}_n(\alpha; O),\ (i = 1, 2).$$

Otherwise, there exists $\sigma \in O - (O_1 \cap O_2)$ with $\alpha \leq \sigma$. Assume $\sigma \in O_1 - O_2$, then for all $\tau \in \mathrm{st}_n(\alpha; O)$ we have $\tau \in O_1$ by hypothesis and hence $\mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O_1)$. Moreover $\mathrm{st}_n(\alpha; O_2) \subseteq \mathrm{st}_n(\alpha; O) = \mathrm{st}_n(\alpha; O_1)$ yields $\mathrm{st}_n(\alpha; O_1 \cap O_2) = \mathrm{st}_n(\alpha; O_2) \subset \mathrm{st}_n(\alpha; O)$.

The case $\sigma \in O_2 - O_1$ is similar since then $\tau \in O_2$ ($\tau \notin O_2$ yields $\tau \in O_1 - O_2$ and hence $\sigma \in O_1$ by hypothesis).

Lemma 7. *$\mathcal{A}_{O_1} \cap \mathcal{A}_{O_2} \subseteq \mathcal{A}_{O_1 \cap O_2}$ and $\mathcal{A}_O \subseteq \mathcal{A}_{O_1} \cup \mathcal{A}_{O_2}$. And so the equalities follow by hypothesis.*

Proof. Let $c(\alpha) \in \mathcal{A}_{O_1} \cap \mathcal{A}_{O_2}$, then $f(O_i, \alpha) = 1$ for $i = 1, 2$ and hence $\mathrm{st}_n(\alpha; O_1 \cap O_2) = \mathrm{st}_n(\alpha; O_i)$ for some i by Lemma 6. Thus $f(O_1 \cap O_2, \alpha) = 1$ by the strong local condition of f, and so $c(\alpha) \in \mathcal{A}_{O_1 \cap O_2}$. Finally $\mathcal{A}_{O_1} \cap \mathcal{A}_{O_2} \subseteq \mathcal{A}_{O_1 \cap O_2}$ since $\mathcal{A}_{O_1 \cap O_2}$ is a full subcomplex.

Now let $\gamma = \langle c(\gamma_0), \ldots, c(\gamma_k) \rangle \in \mathcal{A}_O$. Then $\mathrm{st}_n(\gamma_k; O) \subseteq \mathrm{st}_n(\gamma_{k-1}; O) \subseteq \cdots \subseteq \mathrm{st}_n(\gamma_0; O)$. By Lemma 6 and the strong local condition we easily obtain $\gamma \in \mathcal{A}_{O_i}$ whenever $\mathrm{st}_n(\gamma_0; O) = \mathrm{st}_n(\gamma_0; O_i)$ $(i = 1, 2)$.

6 Future Work

The Digital Seifert–Van Kampen Theorem provides us a theoretical tool that, under certain conditions, computes the digital fundamental group of an object. Nevertheless, the effective computation of the digital fundamental group requires an algorithm to compute a presentation of this group directly at the object's logical level. In a near future we will intend to develop such an algorithm for general digital spaces, as well as to compare the digital fundamental group in Def. 7 with those already introduced by Khalimsky [3] and Kong [4].

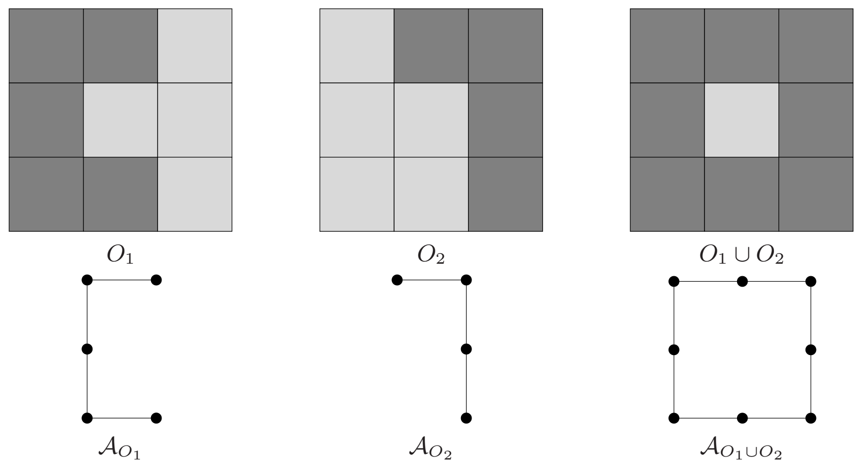


Fig. 2. A digital object for which the Digital Seifert–Van Kampen Theorem does not hold

References

1. R. Ayala, E. Domínguez, A. R. Francés, A. Quintero. Digital Lighting Functions. *Lecture Notes in Computer Science.* 1347(1997) 139-150. 4, 6, 8, 12
2. R. Ayala, E. Domínguez, A. R. Francés, A. Quintero. Weak lighting functions and strong 26-surfaces. To appear in *Theoretical Computer Science.* 4
3. E. Khalimsky. Motion, deformation and homotopy in finite spaces. *Proc. of the 1987 IEEE Int. Conf. on Systems., Man and Cybernetics, 87CH2503-1.* (1987) 227-234. 3, 13
4. T. Y. Kong. A Digital Fundamental Group. *Comput. & Graphics, 13(2).* (1989) 159-166. 3, 13
5. R. Malgouyres. Homotopy in 2-dimensional digital images. *Lecture Notes in Computer Science.* 1347(1997) 213-222.
6. R. Malgouyres. Presentation of the fundamental group in digital surfaces. *Lecture Notes in Computer Science.* 1568(1999) 136-150.
7. R. Malgouyres. Computing the fundamental group in digital spaces. *Proc. of the 7th Int. Workshop on Combinatorial Image Analysis IWCIA'00.* (2000) 103-115. 4
8. C. R. F. Maunder. *Algebraic Topology.* Cambridge University Press. 1980. 4
9. J. R. Munkres. *Elements of algebraic topology.* Addison-Wesley. 1984. 10
10. J. J. Rotman. *An introuduction to algebraic topology.* GTM, 119. Springer. 1988. 12
11. J. Stillwell. *Classical topology and combinatorial group theory.* Springer. 1995. 3