# The Size of the Open Sphere of Influence Graph in $L_\infty$ Metric Spaces

## (Abstract)

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## 1  Introduction

Let $V$ be a set of distinct points in some metric space. For each point $x \in V$, let $r_x$ be the distance from $x$ to its nearest neighbour, and let $s_x$ be the open ball centered at $x$ with radius equal to the distance from $x$ to its nearest neighbour. We refer to these balls as the *spheres of influence* of the set $V$. The *open sphere of influence graph* on $V$ is defined as the graph where $(x, y)$ is an edge if and only if $s_x$ and $s_y$ intersect.

The maximum size of sphere of influence graphs (SIGs) in Euclidean space has been studied. Most notably, Avis and Horton proved that the number of edges of a SIG in the Euclidean plane is linear with respect to the number of vertices [1], and Guibas, Pach and Sharir extended the result to any fixed dimension [2]. However, little attention has been placed on SIGs in other metric spaces. In this abstract, we focus on the size of open SIGs in $d$-dimensional metric spaces induced by the $\infty$-order Minkowski metric, where the spheres of influence are open hypercubes. We refer to these metric spaces as $L_\infty^d$ and present upper and lower bounds for the maximum number of edges in open $L_\infty^d$-SIGs.

## 2  An Upper Bound on the Number of Edges

**Lemma 1.** *Let $S$ be a collection of open balls in $L_\infty^d$ such that no ball in $S$ contains the center of any other ball in $S$. Then for all points $p \in L_\infty^d$, $p$ is contained in no more than $2^d$ balls of $S$.*

With the aid of Lemma 1, we would like to make our arguments by examining intersections which include the corners of the spheres of influence. However, since the spheres of influence are open and thus do not contain their corners, we instead choose to examine points which are inside the spheres and "close enough" to the corners. We refer to these points as $\varepsilon$-corners and define them as follows.

Let $S$ be the spheres of influence of some point set, and let $P$ be the set of all polytopes determined by pairwise intersections of balls in $S$. Let $\varepsilon$ be one-half the smallest width over all polytopes in $P$. For each corner $c$ of a ball in $S$, we define an $\varepsilon$-corner as $c + \varepsilon \hat{v}_{c \to o}$ where $\hat{v}_{c \to o}$ is the unit vector in the direction from the corner to the center of the ball.

**Lemma 2.** *Let $X$ and $Y$ be two intersecting spheres of influence in $L_\infty^d$. Then the number of $\varepsilon$-corners of $X$ plus the number of $\varepsilon$-corners of $Y$ contained in $X \cap Y$ is at least two.*

If we have $n$ points, then there are $2^d n$ $\varepsilon$-corners. Each of these is involved in no more than $2^d - 1$ intersections, and by Lemma 2 each intersection contains at least two $\varepsilon$-corners. This leads us to Theorem 1.

**Theorem 1.** *No open sphere of influence graph of $n$ vertices in $L_\infty^d$ has $(2^{2d-1} - 2^{d-1})n$ edges or more.*

## 3   A Lower Bound on the Maximum Number of Edges

We construct a lattice in $L_\infty^d$ space such that each vertex has $(2^{2d+3} - 6d - 8)/9$ sphere of influence neighbours.

Let $\widehat{u_j}$ be the unit vector parallel to the $j^{th}$ dimensional axis. We generate a lattice with the basis $\mathcal{B} = \{\boldsymbol{b_i} : 1 \le i \le d\}$ such that

$$\boldsymbol{b_i} = \widehat{u_i} + \delta_i \sum_{j<i} \widehat{u_j} - \delta_i \sum_{j>i} \widehat{u_j}$$

where each $\delta_i$ is a small positive real number, such that $0 < \delta_1 \le 1/4$, and $0 < \delta_j \le \frac{\delta_{j-1}}{4}$ for all $2 \le j \le d$. Let this d-dimensional lattice be known as $T_d$.

**Theorem 2.** *In the infinite lattice $T_d$, each vertex has $(2^{2d+3} - 6d - 8)/9$ sphere of influence neighbours.*

Thus for finite SIGs of $n$ vertices which are subsets of $T_d$, we achieve an asymptotic bound of $((2^{2d+2} - 3d - 4)/9)n$ edges.

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## Note

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## References

[1] David Avis and Joe Horton. Remarks on the sphere-of-influence graph. *Discrete Geometry and Convexity*, 440:323–327, 1985.
[2] Leonidas Guibas, János Pach, and Micha Sharir. Sphere-of-influence graphs in higher dimensions. *Colloquia Mathematica Societatis János Bolyai*, 63:131–137, 1994.