Transformation ordering

F. BELLEGARDE and P. LESCANNE

CRIN CENTRE DE RECHERCHE EN INFORMATIQUE DE NANCY
BP 239
54506 Vandoeuvre les Nancy CEDEX
FRANCE

ABSTRACT

We define an ordering called transformation ordering which is useful for proving termination of rewriting systems. A transformation ordering is defined using two relations: a relation which transforms terms and a relation which ensures the well-foundedness of the ordering. A property between these two relations called cooperation is required. Cooperation is similar to confluence and thus may be localized. Therefore, if relations are rewrite relations, it is possible to decide the cooperation by looking at critical pairs. Transformation orderings prove termination of rewriting systems that cannot be proved by the classical methods.

Introduction

We describe an ordering called *transformation ordering* for proving termination of rewriting systems. Let ϕ be a transformation, i.e. a mapping of terms onto terms. In order to prove that a rewriting system R terminates one proves that when t1 rewrites to t2 , ϕ(t1) related to ϕ(t2) by a well-founded relation. Other termination proof methods [7] are based on the same idea but the mapping is usually required to be a morphism which is not the case in this paper. Indeed since ϕ is obtained by a rewriting system T many other transformation systems T are allowed usually. The well-founded relation used to prove termination is also a rewriting system S. A similar idea appeared recently in proofs of termination of rewriting modulo equational theories such as associative and commutative theories [1,2,8]. Simple proofs of termination based on transformation techniques can also be found in [5].

We define transformation orderings in the first section. The transitivity and the well-foundedness of these orderings come from properties mentioned in [1]. In the second section, we show that the properties that are necessary to have a well-founded ordering may be localized and checked by looking at critical pairs between S and T. We extend the ordering in the fourth section.

Notations

We suppose the reader is familiar with the basic features and notations of term rewriting systems [9]. Let F be a set of operators symbols and V be a set of variables, T(F,V) is the set of terms with symbols in F and variables in V.

The relations on terms $\rightarrow^{-1}$, or $\leftarrow$ denote the inverse of the relations $\rightarrow$ between two terms. $\rightarrow^{*}$ and $\rightarrow^{+}$ respectively denote respectively the transitive closure and the strict transitive closure of $\rightarrow$. We write $\rightarrow_{R1} o \rightarrow_{R2}$ for the composition of the two relations $\rightarrow_{R2}$ and $\rightarrow_{R1}$. We write $\rightarrow_{R1} \subseteq \rightarrow_{R2}$ if $\{(x,y) | x \rightarrow_{R1} y\} \subseteq \{(x,y) | x \rightarrow_{R2} y\}$ as sets.

$\rightarrow$ is *noetherian* if and only if there is no infinite sequence of terms t1,t2,..., such that t1$\rightarrow$t2$\rightarrow$.... The relation $\rightarrow$ is *confluent* if for all terms t, t1 and t2 such that t$\rightarrow^{*}$t1 and t$\rightarrow^{*}$t2, there exists a term t' such that t1$\rightarrow^{*}$t' and t2$\rightarrow^{*}$t'. In terms of inclusion we have $\leftarrow^{*} o \rightarrow^{*} \subseteq \rightarrow^{*} o \leftarrow^{*}$. If $\rightarrow_R$ is noetherian and confluent the normal-form of a term t, written $t\downarrow_R$, exists and is unique.

A rewriting system is a set R of rules that are ordered pairs of terms, written l$\rightarrow$r, such that $V(r) \subseteq V(l)$ (V(t) is the set of variables occurring in t). The rewriting relation is written $\rightarrow_R$ or $\rightarrow$ if there is no ambiguity. We say that the rewriting system R is confluent or noetherian when the rewriting relation $\rightarrow_R$ is confluent or noetherian. A noetherian ordering which is F-compatible, i.e. s$\rightarrow$t implies f(...s...)$\rightarrow$f(...t...) for the symbols of F, is called a *reduction ordering*.

1. TRANSFORMATION ORDERING

Two relations $\rightarrow_S$ and $\rightarrow_T$ are considered. $\rightarrow_S$ or $\rightarrow_T$ are not always rewriting relations. This means that when we say $\rightarrow_T$ is confluent or $\rightarrow_S$ is noetherian, we state properties on abstract relations.

Definition 1 [1]

$=>_{S,T}$ is the relation $\rightarrow_T^* o \rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^* o \leftarrow_T^*$.

Two terms u and v are related by $=>_{S,T}$ if there exists u' such that u $\rightarrow_T^*$ u' and v' such that v $\rightarrow_T^*$ v' which are related by any sequence of $\rightarrow_S$ and $\rightarrow_T$ containing at least one $\rightarrow_S$.

Definition 2

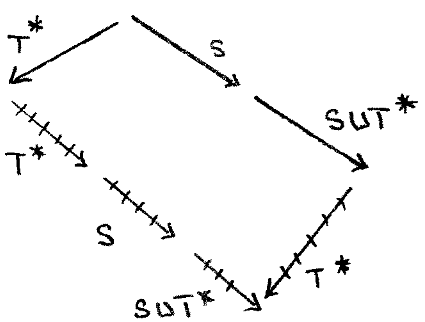
☞$_{S,T}$ is the relation $\leftarrow_T^* o \rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^*$.

As we have seen the confluence of $\rightarrow_T$ is the property $\leftarrow_T^* o \rightarrow_T^* \subseteq \rightarrow_T^* o \leftarrow_T^*$. We now define the *cooperation* of $\rightarrow_S$ with $\rightarrow_T$ which is a kind of confluence.

Definition 3

$\rightarrow_S$ **cooperates with** $\rightarrow_T$ if and only if ☞$_{S,T} \subseteq =>_{S,T}$ (Fig. 1).

Figure 1:$\rightarrow_S$ **cooperates with** $\rightarrow_T$



Basically $\rightarrow_T$ is noetherian and confluent and $\rightarrow_S$ cooperates with T then two terms u and v are related by $=>_{S,T}$ if their transformations by T, namely $u\downarrow_T$ and $v\downarrow_T$ are related by any sequence of $\rightarrow_S$ and $\rightarrow_T$ containing at least one $\rightarrow_S$.

Definition 4

$>_{S,T}$ is the relation $=>_{S,T} \cup \rightarrow_T^+$.

Lemma 1

If $\rightarrow_S \cup \rightarrow_T$ is noetherian, $\rightarrow_T$ is confluent and $\rightarrow_S$ cooperates with $\rightarrow_T$ then $>_{S,T}$ is a **partial ordering on terms**.

Proof:

• $\rightarrow_S$ and $\rightarrow_T$ are noetherian, thus they are irreflexive. Then $\rightarrow_T^+$ is irreflexive and $=>_{S,T}$ is irreflexive. Thus $>_{S,T}$ is irreflexive.

• $>_{S,T}$ is transitive. Recall that a relation $\rightarrow$ is transitive if and only if $\rightarrow \circ \rightarrow \subseteq \rightarrow$. We get the result by the confluence of $\rightarrow_T$, by the hypothesis of cooperation and by definition of $>_{S,T}$.

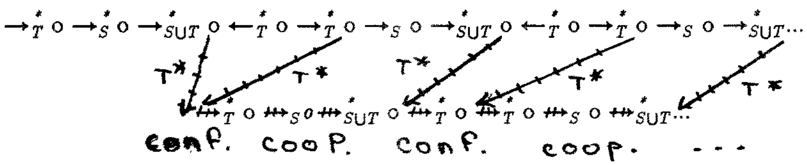
Lemma 2

If $\rightarrow_S \cup \rightarrow_T$ is noetherian, $\rightarrow_T$ is confluent and $\rightarrow_S$ cooperates with $\rightarrow_T$, then $>_{S,T}$ is a **well-founded** partial ordering.

Proof:

$\rightarrow_T^+$ is noetherian since $\rightarrow_S \cup \rightarrow_T$ is noetherian. Therefore, if an infinite sequence t1 $>_{S,T}$ t2 $>_{S,T}$ ... exists, then an infinite sequence $=>_{S,T} o =>_{S,T} o =>_{S,T} o$ ... which is, by definition, an infinite sequence $\rightarrow_T^* o \rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^* o \leftarrow_T^* o$ ... exists. Thus we use the confluence of $\rightarrow_T$ and the cooperation of $\rightarrow_S$ with $\rightarrow_T$ and show (Fig. 3) that an infinite sequence of rewriting with $\rightarrow_T^* o \rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^* o \rightarrow_T^*$ exists which is a contradiction with the well-foundedness of $\rightarrow_S \cup \rightarrow_T$.

Figure 3.



Lemma 3

If $\rightarrow_T$ and $\rightarrow_S$ are F-compatible and stable by substitutions then $>_{S,T}$ is F-compatible and stable by substitutions.

Theorem 1

If $\rightarrow_S$ cooperates with $\rightarrow_T$, $\rightarrow_S \cup \rightarrow_T$ is noetherian and $\rightarrow_T$ is confluent then $>_{S,T}$ is a well-founded ordering and moreover when $\rightarrow_S$ and $\rightarrow_T$ are F-compatible and stable by substitution, $>_{S,T}$ is F-compatible and stable by substitution.

Since rewriting relations on T(F,V) are F-compatible and stable by substitutions, we may state the following result:

Corollary 1

Let S and T be two rewriting systems. Suppose S cooperates with T, S$\cup$T is noetherian and T is confluent then $>_{S,T}$ is a reduction ordering stable by substitution.

Fact: With the condition of Theorem 1, a rewriting system that satisfies $l >_{S,T} r$ for all rules $l \rightarrow r$ is noetherian.

Example 1

The following example comes from [4,5].

a: (x*y)*z → x*(y*z)
b: f(x)*f(y) → f(x*y)
c: f(x)*(f(y)*z) → f(x*y)*z.

a, b and c are the rules of a rewriting system R. Proving that R is noetherian is not easy since the classical methods namely simplification orderings [7] such as recursive path ordering (RPO) or recursive decomposition ordering (RDO) methods fails. We choose T to be

r1: $f(x)*y \rightarrow f(x*y)$
r2: $x*f(y) \rightarrow f(x*y)$
a: $(x*y)*z \rightarrow x*(y*z)$

in order to push up f and put down *. We choose S to be

$f(f(x)) \rightarrow f(x)$.

T ∪ S is noetherian. S and T satisfy the condition of the forthcoming Theorem 3, we will see that this implies S and T cooperate. Thus we may use $>_{S,T}$ to prove the termination of R. We have $l >_{S,T} r$ for all rules of R:

proof

• $(x*y)*z \rightarrow_T^+ x*(y*z)$ (by a ∈ T)

$(x*y)*z >_{S,T} x*(y*z)$ (by definition)

• $f(x)*f(y){\downarrow_T}=f(f(x*y))$ and

$f(f(x*y)) \rightarrow_S f(x*y)$. Thus

$f(x)*f(y) \Rightarrow_{S,T} f(x*y)$ (by definition) and

$f(x)*f(y) >_{S,T} f(x*y)$ (by definition)

• $f(x)*(f(y)*z){\downarrow_T}=f(f(x*(y*z)))$,

$f(x*y)*z{\downarrow_T}=f(x*(y*z))$ and

$f(f(x*(y*z))) \rightarrow_S f(x*(y*z))$. Thus

$f(x)*(f(y)*z) \Rightarrow_{S,T} f(x*y)*z$ (by definition) and

$f(x)*(f(y)*z) >_{S,T} f(x*y)*z$ (by definition)

2. LOCALIZATION OF THE COOPERATION

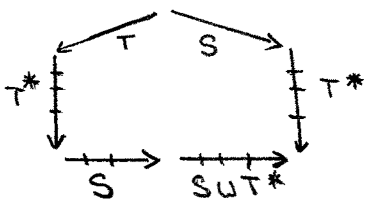
Thus if we have two rewriting systems S and T such that S cooperates with T, T is confluent and S∪T is noetherian and $l >_{S,T} r$ for all rules $l \rightarrow r$ of R, then R is noetherian. The confluence of T may be tested using the Knuth-Bendix procedure. The termination of S ∪ T may be tested using other well-founded orderings [7]. Only the cooperation of S with T has to be checked with appropriate methods, for instance using the solution proposed in this section.

Like confluence, cooperation may be localized, and we are going to prove a Newman-like theorem for cooperation.

Definition 5

$\rightarrow_S$ **locally cooperates with** $\rightarrow_T$ if and only if $\leftarrow_T o \rightarrow_S \subseteq =>_{S,T}$ (Fig. 4).

Figure 4: S locally cooperates with T



Theorem 2

If $\rightarrow_S \cup \rightarrow_T$ is noetherian and $\rightarrow_T$ is confluent then the local cooperation implies the cooperation of $\rightarrow_S$ with $\rightarrow_T$.

Proof:

We use a noetherian induction on $\rightarrow_S \cup \rightarrow_T$. Let us have y ☞$_{S,T}$ z thus y $\leftarrow_T^m$ x $\rightarrow_S$ s $(\rightarrow_S \cup \rightarrow_T)^*$ z, we have to show that y $=>_{S,T}$ z.

• If m = 0, we have y=x $\rightarrow_S$ s $(\rightarrow_S \cup \rightarrow_T)^*$ z thus y $=>_{S,T}$ z.

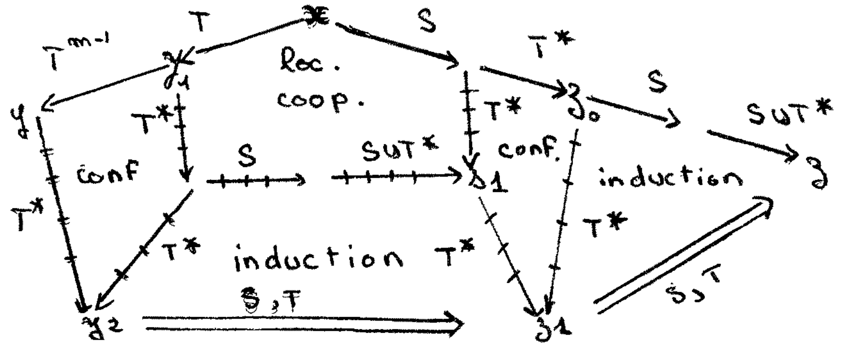
• If m>0, we have y1 such that y $\leftarrow_T^{m-1}$ y1 $\leftarrow_T$ x. From the local cooperation, we get y1 $=>_{S,T}$ s. Thus we have x1 and s1 such that y1 $\rightarrow_T^*$ x1, s $\rightarrow_T^*$ s1 and x1 $\rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^*$ s1. From the confluence of $\rightarrow_T$, there is an y2 such that y $\rightarrow_T^*$ y2 and x1 $\rightarrow_T^*$ y2. We notice that $(\rightarrow_S \cup \rightarrow_T)^* = \rightarrow_T^* \cup \rightarrow_T^* o \rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^*$, and thus we find two subcases.

(1) Suppose s $(\rightarrow_S \cup \rightarrow_T)^*$ z means s$\rightarrow_T^*$z. With the confluence of $\rightarrow_T$, there is a z1 such that s1$\rightarrow_T^*$z1 and z$\rightarrow_T^*$z1. Thus we get x1$\rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^* o \rightarrow_T^*$z1 and thus x1$\rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^*$z1 (by $\rightarrow_T^* \subseteq (\rightarrow_S \cup \rightarrow_T)^*$ and transitivity of $(\rightarrow_S \cup \rightarrow_T)^*$). Now we have y2☞$_{S,T}$z1. So by noetherian induction y2$=>_{S,T}$z1 and therefore, by transitivity of $\rightarrow_T^*$, we conclude that y$=>_{S,T}$z (Fig. 5).

(2) If $(\rightarrow_S \cup \rightarrow_T)^* = \rightarrow_T^* o \rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^*$, thus s$\rightarrow_T^*$ z0 $\rightarrow_S o (\rightarrow_S \cup \rightarrow_T)^*$z and the confluence of $\rightarrow_T$ provides a z1 such that s1 $\rightarrow_T^*$ z1 and z0 $\rightarrow_T^*$ z1. Now by noetherian

induction, we get z1 $=>_{S,T}$ z. Moreover by noetherian induction, we get y2 $=>_{S,T}$ z1. Therefore by transitivity of $\rightarrow_{S,T}$, we get y2 $=>_{S,T}$ z and by transitivity of $\rightarrow^*_T$, we conclude that y $=>_{S,T}$ z (Fig. 5).

Figure 5.



If S and T are rewriting systems, by looking at critical pairs between S and T it is possible to decide that S locally cooperates with T.

Definition 6

A critical pair $p \leftarrow_T o \rightarrow_S q$ between a rule of S and a rule of T is cooperative if and only if p $=>_{S,T}$ q.

Definition 7

A rewriting system is variable preserving if and only if all rules are variable preserving i.e., variables that occur on the left-hand side l do not disappear on the right-hand side r and thus V(l)=V(r).

Definition 8

A rewriting system is left-linear if and only if all rules are left-linear i.e. , variables occur only once on the left-hand side.

Theorem 3

Suppose T is a left-linear rewriting system and a variable preserving rewiting system. A rewriting system S cooperates locally with T if and only if all the critical pairs between S and T are cooperative. The proof looks like the proof of the similar theorem on confluent critical pairs.

3. EXTENDED TRANSFORMATION ORDERING

Results of Section 1 are useful in many cases like Example 1 and relations T and S can be easily found. In this section we want to go again further and to show that T can be extended by using any ordering that contains S and T. This way, we expect to prove termination of more rewriting systems The problem with T usually arises when both sides of a rewrite rule are transformed by T into the same term.

We now use a well-founded ordering $>>$ such that $\rightarrow_S \subseteq >>$ and $\rightarrow_T \subseteq >>$ to define a relation between terms, written $\rightarrow_{EXT(T)}$. This relation extends $\rightarrow_T$ in the sense that $\rightarrow_{EXT(T)} \subseteq >>$ and $\rightarrow_{EXT(T)} \subseteq =_T$. The last condition is necessary to ensure the confluence of $\rightarrow_{EXT(T)}$ if $\rightarrow_T$ is confluent and the cooperation of $\rightarrow_{EXT(T)}$ with $\rightarrow_S$ if $\rightarrow_T$ cooperates with $\rightarrow_S$. Therefore, we define $\rightarrow_{EXT(T)}$ as $=_T \cap >>$.

Definition 9

Let $\rightarrow_T$ be a confluent and noetherian and let $>>$ be a well-founded ordering on terms that contains $\rightarrow_S \cup \rightarrow_T$. $s \rightarrow_{EXT(T)} t$ if and only if $s\downarrow_T = t\downarrow_T$ and $s>>t$.

Proposition 1

Suppose $\rightarrow_S$ cooperates with a confluent and noetherian relation $\rightarrow_T$, $>>$ is a well-founded ordering on terms that contains $\rightarrow_S \cup \rightarrow_T$ then $\rightarrow_{EXT(T)}$ is confluent, $\rightarrow_S \cup \rightarrow_{EXT(T)}$ is noetherian, $\rightarrow_S$ cooperates with $\rightarrow_{EXT(T)}$.

Fact: Therefore $>_{S,EXT(T)}$ can be used to prove termination.

Lemma 4:

Suppose that $\rightarrow_{T1} \subseteq \rightarrow_{T2}$ and $\rightarrow_{S1} \subseteq \rightarrow_{S2}$ then $=>_{S,T1} \subseteq =>_{S,T2}$ and $>_{S,T1} \subseteq >_{S,T2}$, $=>_{S1,T} \subseteq =>_{S2,T}$ and $>_{S1,T} \subseteq >_{S2,T}$, $=>_{S1,T1} \subseteq =>_{S2,T2}$ and $>_{S1,T1} \subseteq >_{S2,T2}$.

Lemma 5:

$$\rightarrow_T \subseteq \rightarrow_{EXT(T)}$$

Fact: $>_{S,T} \subseteq >_{S,EXT(T)}$ (direct consequence of Lemma 4 and 5).

Proof of Proposition 1:

• $\rightarrow_S \cup \rightarrow_{EXT(T)}$ is noetherian:

obvious since $\rightarrow_S \subseteq >>$ (by hypothesis) and $\rightarrow_{EXT(T)} \subseteq >>$ by definition.

• $\rightarrow_{EXT(T)}$ is confluent:

If t$\rightarrow_{EXT(T)}$t1 and t $\rightarrow_{EXT(T)}$t2, by definition, we have $t1\downarrow_T = t\downarrow_T = t2\downarrow_T$ = t'. Since $\rightarrow_T \subseteq \rightarrow_{EXT(T)}$ by Lemma 5, we get t1 $\overset{*}{\rightarrow}_{EXT(T)}$ t' and t2 $\overset{*}{\rightarrow}_{EXT(T)}$ t' thus $\rightarrow_{EXT(T)}$ is locally confluent. Since $\rightarrow EXT(T) \subseteq >>$ by definition, $\rightarrow_{EXT(T)}$ is noetherian. Now $\rightarrow_{EXT(T)}$ is locally confluent and noetherian. Therefore it is confluent.

• $\rightarrow_S$ cooperates with $\rightarrow_{EXT(T)}$:

Since $\rightarrow_{EXT(T)}$ is confluent and $\rightarrow_S \cup \rightarrow_{EXT(T)}$ is noetherian, $\rightarrow_S$ cooperates with $\rightarrow_{EXT(T)}$ if it locally cooperates with $\rightarrow_{EXT(T)}$ (by Theorem 2). Suppose that t1$\leftarrow_{EXT(T)}$t$\rightarrow_S$t2. Since $t1\downarrow_T = t\downarrow_T$ (by definition of $\rightarrow_{EXT(T)}$) and $t\downarrow_T => _{S,T}$t2 (by cooperation of $\rightarrow_S$ with $\rightarrow_T$), we get t1$\overset{*}{\rightarrow}_T o => _{S,T}$t2. Then t1$=>_{S,T}$t2 (by definition) and t1$=>_{S,EXT(T)}$t2 (by Lemma 4 and 5).

Example 2 [10]

The termination of the rewriting system R

r1: f(s(x)) → f(p(s(x)))
r2: p(s(0)) → 0
r3: p(s(s(x))) → s(p(s(x)))

is not provable by simplification orderings since f(s(x)) is embedded in f(s(p(x))). But with the transformation rule

T': p(s(x)) → x

we get a rule

S': f(s(x)) → f(x)

T' is confluent, regular and left-linear. S'∪T' is noetherian. S' cooperates with T' (there is no critical pair). For the rules r2 and r3, we get $l{\downarrow}_{T'} = r{\downarrow}_{T'}$. So let us take a recursive path ordering based on the precedence p>s to extend T'. Then p(s(0) $>_{S,EXT(T')}$ 0 and R terminates.

4. Conclusion

The transformation orderings allow us to prove termination of rewrite systems where methods based on simplification orderings fail. We are currently looking way to implement it in REVE and to adapt it to the proofs of other systems.

References

1. L. Bachmair and N. Dershowitz, "Commutation, Transformation, and Termination," in *Proc. 8th Conf. on Automated Deduction* , Lecture Notes in Computer Science, vol. 230, Springer Verlag, Oxford (England), 1986.

2. L. Bachmair and D. Plaisted, "Associative Path Orderings," in *Proc. 1st Conference on Rewriting Techniques and Applications*, Lecture Notes in Computer Science, vol. 202, pp. 241-254, Springer Verlag, Dijon (France), 1985.

3. J. Backus, "Can Programming Be Liberated From the Von Neumann Style? A Functional Style And Its Algebra of Programs," *Comm. of ACM* , vol. 21, no. 8, pp. 613-641, 1978.

4. F. Bellegarde, "Utilisation des Systèmes de Réécriture d'Expressions Fonctionnelles comme outils de Transformation de Programmes Itératifs," Thèse de doctorat d'Etat, Université de Nancy I, Dept. Mathématiques Appliquées, 1985.

5. F. Bellegarde, "Rewriting Systems on FP Expressions to reduce the number of Sequences Yielded." *Science of Computer Programming* , vol. 6. pp. 11-34, North-Holland, 1986.

6. F. Bellegarde and P. Lescanne, "Termination Proofs Based On Transformation Techniques," Submitted To Information and Control, 1986.

7. N. Dershowitz, "Termination," in *Proc. 1rst Conf. Rewriting Techniques and Applications* , Lecture Notes in Computer Science, Vol. 202, pp. 180-224, Springer Verlag, Dijon (France), May 1985.

7. I. Gnaedig and P. Lescanne, "Proving Termination of Associative Commutative Rewriting Systems by Rewriting," *Proceedings 8th International Conference on Automated Deduction* , Oxford (England), 27-31 July 1986.

9. G. Huet and D. Oppen, "Equations and Rewrite Rules: A Survey," in *Formal Languages: Perspectives And Open Problems* , ed. Book R., Academic Press, 1980.

10. S. Kamin and J.J. Levy, "Attempts for Generalizing the Recursive Path Ordering," Inria, Rocquencourt, 1982, University of Illinois Report.