

# MINIMUM S-T CUT OF A PLANAR UNDIRECTED NETWORK IN $O(n \log^2(n))$ TIME

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## SUMMARY

Let  $N$  be a planar undirected network with distinguished vertices  $s, t$ , a total of  $n$  vertices, and each edge labeled with a positive real (the edge's cost) from a set  $L$ . This paper presents an algorithm for computing a minimum (cost)  $s$ - $t$  cut of  $N$ . For general  $L$ , this algorithm runs in time  $O(n \log^2(n))$  time on a (uniform cost criteria) RAM. For the case  $L$  contains only integers  $\leq n^{O(1)}$ , the algorithm runs in time  $O(n \log(n) \log \log(n))$ . Our algorithm also constructs a minimum  $s$ - $t$  cut of a planar graph (i.e., for the case  $L = \{1\}$ ) in time  $O(n \log(n))$ .

The fastest previous algorithm for computing a minimum  $s$ - $t$  cut of a planar undirected network [Gomory and Hu, 1961] and [Itai and Shiloach, 1979] has time  $O(n^2 \log(n))$  and the best previous time bound for minimum  $s$ - $t$  cut of a planar graph [Cheston, Probert, and Saxton, 1977] was  $O(n^2)$ .

## 1. Introduction

The importance of computing a minimum  $s$ - $t$  cut of a network is illustrated by Ford and Fulkerson's [1962] Theorem which states that the value of the minimum  $s$ - $t$  flow of a network is precisely the minimum  $s$ - $t$  cut. The best known algorithms [Galil, Naamad, 1979; Shiloach, 1978] for computing the max flow or minimum  $s$ - $t$  cut of a *sparse directed or undirected network* (with  $n$  vertices and  $O(n)$  edges) has time  $O(n^2 \log^2(n))$ . This paper is concerned with a *planar undirected network*  $N$ , which occurs in many practical applications.

Ford and Fulkerson [1956] have an elegant minimum  $s$ - $t$  cut algorithm for the case  $N$  is  $(s, t)$ -*planar* (both  $s$  and  $t$  are on the same face) which efficiently implemented by Gomory and Hu [1961] and Itai and Shiloach [1979] has time  $O(n \log(n))$ . Moreover,  $O(n)$  executions of their algorithm suffices to compute the minimum  $s$ - $t$  cut of an *arbitrary planar network* in total time  $O(n^2 \log(n))$ . Also, Cheston, Probert, Saxton [1977] have an  $O(n^2)$  algorithm for the minimum  $s$ - $t$  cut of a planar graph.

Let  $Q_L(n)$  be the time to maintain a queue of  $O(n)$  elements with costs from a set  $L$  of nonnegative reals, and with  $O(n)$  insertions and deletions. For

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simplicity, we assume  $Q_L(O(n)) = O(Q_L(n))$ . For the general case,  $Q_L(n) = O(n \log(n))$  (see [Hopcroft and Ullman, 1974]). For the special case  $L$  is a set of positive integers  $\leq n^{O(1)}$  [Boas, Kaas and Zijlstra, 1977],  $Q_L(n) = O(n \log \log(n))$ . It is obvious that if  $L = \{1\}$ ,  $Q_L(n) = O(n)$ .

A key element of the [Ford and Fulkerson, 1956] algorithm for  $(s,t)$ -planar networks was an efficient reduction to finding a minimum cost path between two vertices in a sparse network. Dijkstra [1959] gives an algorithm for a generalization of this problem (to find a minimum cost path from a fixed "source" vertex  $s$  to each other vertex). Dijkstra's algorithm may be implemented (see [Aho, Hopcroft and Ullman, 1974]) in time  $O(Q_L(n))$  for a sparse network with  $n$  vertices, and  $L$  is the set of nonnegative reals labeling the edges.

Our algorithm for computing the minimum  $s$ - $t$  cut of a planar undirected network has time  $O(Q_L(n) \log(n))$ . This algorithm also utilizes an efficient reduction to minimum cost path problems. Our fundamental innovation is a divide and conquer approach for cuts on the plane.

The paper is organized as follows: The next section gives preliminary definitions of graphs, networks, min cuts, and duals of planar networks. Section 3 gives the Ford-Fulkerson Algorithm for  $(s,t)$ -planar graphs. Section 4 gives an efficient algorithm for minimum cut graphs containing a given face. Our divide and conquer approach is described and proved in Section 5. Section 6 presents our algorithm for minimum  $s$ - $t$  cuts of planar networks. Finally, Section 7 concludes the paper.

## 2. Preliminary Definitions

**2.1 Graphs.** Let a *graph*  $G = (V, E)$  consists of a *vertex set*  $V$  and a collection of *edges*  $E$ . Each edge  $e \in E$  connects two vertices  $u, v \in V$  (edge  $e$  is a *loop* if it connects identical vertices). We let  $e = \{u, v\}$  denote edge  $e$  connects  $u$  and  $v$ . Edges  $e, e'$  are *multiple* if they have the same connections. Let a *path* be a sequence of edges  $p = e_1, \dots, e_k$  such that  $e_i = \{v_{i-1}, v_i\}$  for  $i = 1, \dots, k$  (we say  $p$  *traverses* vertices  $v_0, \dots, v_k$ ). Let  $p$  be a *cycle* if  $v_0 = v_k$  (cycles containing the same edges are considered identical). A path  $p'$  is a *subpath* of  $p$  if  $p'$  is a subsequence of  $p$ . Let  $G$  be a *standard graph* if  $G$  has no multiple edges nor loops. Generally we let  $\underline{n}$  be the number of vertices of graph  $G$ .  $G$  is *sparse* if the number of edges is  $O(n)$ . If  $G$  is planar, then by Euler's formula  $G$  is sparse and contains at most  $6n - 12$  edges.

**2.2 Networks.** Let an *undirected network*  $N = (G, c)$  consists of a graph  $G = (V, E)$  and a mapping  $c$  from  $E$  to the positive reals. For each edge  $e \in V$ ,  $c(e)$  is the *cost* of  $e$ . For any edge set  $E' \subseteq E$ , let  $c(E') = \sum_{e \in E'} c(e)$ . Let the *cost* of path  $p = e_1, \dots, e_k$  be  $c(p) = \sum_{i=1}^k c(e_i)$ . Let a path  $p$  from vertex  $u$  to vertex  $v$  be *minimum* if  $c(p) \leq c(p')$  for all paths  $p'$  from  $u$  to  $v$ . Let  $N = (G, c, s, t)$  be a *standard network* if  $(G, c)$  is an undirected network, with

$G = (V, E)$  a standard graph, and  $s, t$  are distinguished vertices of  $V$  (the *source*, *sink*, respectively).

2.3 Min Cuts and Flows in Networks. Let  $N = (G, c, s, t)$  be a standard network with  $G = (V, E)$ . An edge set  $X \subseteq E$  is a *s-t cut* if  $(V, E - X)$  has no paths from  $s$  to  $t$ . Let s-t cut  $X$  be *minimum* if  $c(X) \leq c(X')$  for each s-t cut  $X'$ . A function  $f$  mapping  $E$  to the nonnegative reals is a *flow* if

- (i)  $\forall e \in E, f(e) \leq c(e)$ , and
- (ii)  $\forall v \in V - \{s, t\}, \text{IN}(f, v) = \text{OUT}(f, v)$

$$\text{where } \text{IN}(f, v) = \sum_{\substack{e \in E \\ v \in e}} f(e) \quad \text{and} \quad \text{OUT}(f, v) = \sum_{\substack{e \in E \\ v \in e}} f(e).$$

The *value* of the flow  $f$  is  $\text{OUT}(f, s) - \text{IN}(f, t)$ . The following motivates our work on minimum s-t cuts:

THEOREM 1. [Ford and Fulkerson, 1962]. *The maximum value of any flow is the cost of a minimum s-t cut.*  $\square$

2.4 Planar Networks and Duals. Let  $G = (V, E)$  be a planar standard graph, with a fixed embedding on the plane. Each connected region of  $G$  is a *face* and has a corresponding cycle of edges which it borders. For each edge  $e \in E$ , let  $D(e)$  be the corresponding *dual edge* connecting the two faces bordering  $e$ . Let  $D(G) = (\mathcal{F}, D(E))$  be the *dual graph* of  $G$ , with vertex set  $\mathcal{F}$  = the faces of  $G$ , and with edge set  $D(E) = \bigcup_{e \in E} D(e)$ . Note that the dual graph is not necessarily standard (i.e., it may contain multiple edges and loops), but is planar. Let a cycle  $q$  of  $D(G)$  be a *cut-cycle* if the region bounded by  $q$  contains exactly one of  $s$  or  $t$ .

PROPOSITION 1.  $D$  induces an 1-1 correspondence between the s-t cuts of  $G$  and the cut-cycles of  $D(G)$ .  $\square$

Let  $N = (G, c, s, t)$  be a *planar* standard network, with  $G = (V, E)$  planar. Let the *dual network*  $D(N) = (D(G), D(c))$  have edge costs  $D(c)$ , where  $D(c)(D(e)) = c(e)$  for all edges  $e \in E$ . (Generally we will use just  $c$  in place of  $D(c)$  where no confusion will result.) For each face  $F \in \mathcal{F}$ , let a cut-cycle  $q$  in  $D(N)$  be  $F_i$ -*minimum* if  $q$  contains  $F_i$  and  $c(q) \leq c(q')$  for all cut-cycles  $q'$  containing  $F_i$ .

PROPOSITION 2. *A minimum s-t cut has the same cost as a minimum cost cut-cycle of  $D(G)$ .*  $\square$

### 3. Ford and Fulkerson's Min s-t Cut Algorithm for (s,t)-Planar Networks

Let  $N = (G, c, s, t)$  be a planar standard network.  $G$  (and also  $N$ ) is *(s,t)-planar* if there exists a face  $F_0$  containing both  $s$  and  $t$ . Let planar network  $N'$  be derived from  $N$  by adding on edge  $e_0$  connecting  $s$  and  $t$  with cost  $\infty$ . Let  $e_0$  be embedded onto a line segment from  $s$  to  $t$  in  $F_0$ , which separates  $F_0$

into two new faces  $F_1$  and  $F_2$ . Ford and Fulkerson [1956] have an elegant characterization of the minimum s-t cut of (s,t)-planar network  $N$ .

**THEOREM 2.** *There is an 1-1 correspondence between the s-t cuts of  $N$  and the paths of  $D(N')$  from  $F_2$  to  $F_1$  and avoiding  $D(e_0)$ . Furthermore, this correspondence preserves edge costs. Therefore, the minimum s-t cuts of  $N$  corresponds to the minimum cost paths in  $D(N')$  from  $F_2$  to  $F_1$  (which avoid  $D(e_0)$ ).*  $\square$

**COROLLARY 2.** *A minimum cost cut of (s,t)-planar  $N$  with  $n$  vertices may be computed in time  $O(Q_L(n))$ , where  $L = \text{range}(c)$ .*  $\square$

Note that this implies the  $O(n \log(n))$  time minimum s-t cut algorithm of Gomory and Hu [1961] and Itai and Shiloach [1979] for (s,t)-planar undirected networks, and the  $O(n)$  time minimum s-t cut algorithm of Cheston, Probert, and Saxton [1977] for (s,t)-planar graphs.

#### 4. An $O(n \log(n))$ Algorithm for F-minimum Cut Cycles

Let  $N = (G, c, s, t)$  be a planar standard network, with  $G = (V, E)$  and  $L = \text{range}(c)$ . Our algorithm for minimum s-t cuts will require efficient construction of F-minimum cut cycles for certain given faces  $F$ . Let  $s$  be the set of faces bordering  $s$  and let  $t$  be the faces bordering  $t$ . Let a  $\mu(s, t)$  path be a minimum cost path in  $D(N)$  from a face of  $s$  to a face of  $t$ .

**PROPOSITION 3.** *Let  $\mu$  be a  $\mu(s, t)$  path traversing faces  $F_1, \dots, F_d$ . Let  $q_i$  be a  $F_i$ -minimum cut-cycle of  $D(N)$  for  $i = 1, \dots, d$ . Then  $D^{-1}(q_{i_0})$  is a minimum s-t cut of  $N$ , where  $c(q_{i_0}) = \min\{c(q_i) \mid i = 1, \dots, d\}$ .*  $\square$

(NOTE: It is easy to compute a  $\mu(s, t)$  path in time  $O(Q_L(n))$ . Let  $M$  be the planar network derived from  $D(N)$  by adding new vertices  $v_s, v_t$  and an edge connecting  $v_s$  to each face in  $s$  and an edge connecting each face in  $t$  to  $v_t$ . Let the cost of each of these edges be 1. Let  $p$  be a minimum cost path in  $M$  from  $v_s$  to  $v_t$ . Then  $p$ , less its first and last edges, is a  $\mu(s, t)$  path.)

Let  $\mu$  be a  $\mu(s, t)$  path traversing faces  $F_1, \dots, F_d$ . By viewing  $\mu$  as a horizontal line segment with  $s$  on the left and  $t$  on the right, each edge of  $D(N)$  connected to a face  $F_i$  may be considered to be connected to  $F_i$  from the *below* or *above* (or *both*). Let  $\mu'$  be a copy of  $\mu$  traversing new vertices  $x_1, \dots, x_d$ . Let  $D'$  be the network derived from  $D(N)$  by reconnecting to  $x_i$  each edge entering  $F_i$  from above. If  $p$  is a path of  $D'$ , then a corresponding path  $\hat{p}$  in  $D(N)$  is constructed by replacing each edge and face appearing in  $\mu'$  with the corresponding edge or face of  $\mu$ . Clearly,  $c(p) = c(\hat{p})$ .

**THEOREM 3.** *If  $p$  is a minimum cost path connecting  $F_i$  and  $x_i$  in  $D'$ , then  $\hat{p}$  is a  $F_i$ -minimum cut cycle of  $D(N)$ .*  $\square$

**Proof.** Clearly,  $\hat{p}$  is a cut-cycle of  $D(N)$ . Suppose  $\hat{p}$  is not  $F_i$ -minimum. Let  $q$  be a  $F_i$ -minimum cut-cycle of  $D(N)$ , with  $c(q) < c(\hat{p})$ . Then there must be a

subpath  $q_1$  of  $q$  connecting faces  $F_j, F_k$  of  $\mu$  but otherwise disjoint from  $\mu$  and such that the edges of  $q_1$  together with  $\mu$  form a cut-cycle of  $D(N)$ , else we can show  $q$  is not a cut-cycle (see Figure 1). Let  $\mu_1$  be the minimal subpath of  $\mu$  containing faces  $F_i, F_j$ , and  $F_k$ . Observe that the edges of  $q_1$  together with  $\mu_1$  form a  $F_i$ -minimum cut-cycle, else  $\mu$  is not a  $\mu(s,t)$  path. Let  $q'_1$  be derived from  $q_1$  by reconnecting the last edge to  $x_k$  instead of  $F_k$ . Let  $\mu_2$  be the subpath of  $\mu_1$  connecting  $F_i$  and  $F_j$  and let  $\mu_3$  be the subpath of  $\mu_1$  connecting  $F_i$  and  $F_k$ . Also, let  $\mu'_3$  be the subpath of  $\mu'$  in  $D'$  corresponding to  $\mu_3$ . Then the edges of  $\mu_2, q'_1$ , and  $\mu'_3$  form a path from  $F_i$  to  $x_i$  in  $D'$  and with cost  $c(q)$ . But  $c(q) < c(\hat{p}) = c(p)$  is a contradiction with the assumption that  $p$  is a minimum cost path from  $F_i$  to  $x_i$ . (See Figure 2.)

**COROLLARY 3.** *There is an  $O(Q_L(n))$  time algorithm to compute a  $F_i$ -minimum cut cycle for any face  $F_i$  of a  $\mu(s,t)$  path in  $D(N)$ .*  $\square$

### 5. A Divide and Conquer Approach

Let  $\mu$  be a  $\mu(s,t)$  path of  $D(N)$  traversing faces  $F_1, \dots, F_d$  as in Section 4. Note that any  $s$ - $t$  cut of planar network  $N$  must contain an edge bounding on a face  $F_1, \dots$ , or  $F_d$ . Thus an obvious algorithm for computing a minimum  $s$ - $t$  cut of  $N$  is to construct a  $F_i$ -minimum cut cycle  $q_i$  in  $D(N)$  for each  $i=1, \dots, d$ . This may be done by  $d$  executions of the  $O(Q_L(n))$  time algorithm of Corollary 3. Then by Proposition 3,  $D^{-1}(q_{i_0})$  is a minimum  $s$ - $t$  cut where  $c(q_{i_0}) = \min\{c(q_1), \dots, c(q_d)\}$ . In the worst case, this requires  $O(Q_L(n) \cdot n)$  total time. This section presents a divide and conquer approach which requires only  $\log(d)$  recursive executions of a  $F_i$ -minimum cut algorithm.

**LEMMA 1.** *Let  $F_i, F_j$  be distinct faces of  $\mu$ , with  $i < j$ . Let  $p$  be any  $F_j$ -minimum cut-cycle of  $D(N)$  such that the closed region  $R$  bounded by  $p$  contains  $s$ . Then there exists an  $F_i$ -minimum cut-cycle  $q$  contained entirely in  $R$ . (See Figure 3.)*  $\square$

Proof. Let  $q$  be any  $F_i$ -minimum cut-cycle. Let  $q'$  be the cut-cycle derived from  $q$  by repeatedly replacing subpaths connecting faces traversed by  $\mu$  with the appropriate subpaths of  $\mu$  (only apply replacements for which the resulting  $q'$  is cut-cycle). Observe  $c(q') \leq c(q)$  (else we can show  $\mu$  is not a  $\mu(s,t)$  path). Let  $R'$  be the closed region bounded by  $q'$ . Suppose  $R \not\subseteq R'$ . Then there must be a subpath  $q_1$  of  $q'$  connecting faces  $F^a, F^b$  of  $p$  such that  $q_1$  only intersects  $R$  at  $F^a$  and  $F^b$ . Let  $p_1$  be the subpath of  $p$  connecting  $F^a$  and  $F^b$  in  $R'$ . We claim  $c(p_1) \leq c(q_1)$ . Suppose  $c(p_1) > c(q_1)$ . By our construction of  $q'$ , either  $q_1$  avoids  $F_j$ ,  $F_j = F^a$  or  $F_j = F^b$ . In any case, we may derive a cut-cycle  $p'$  from  $p$  by substituting  $q_1$  for  $p_1$ . But this implies  $c(p') < c(p)$ , contradicting our assumption that  $p$  is a  $F_i$ -minimum cut-cycle. Now substitute  $p_1$  for  $q_1$  in  $q'$ . The resulting cut-cycle is no more costly than  $q'$ , since  $c(p_1) \leq c(q_1)$ . (See

Figure 4.) The lemma follows by repeated application of this process.

The above lemma implies a method for dividing the planar standard network  $N$ , given an  $s$ - $t$  cut  $X$ . Let  $N_X$  be the network derived from  $N$  by deleting all edges of  $X$ .  $N_X$  can be partitioned into two networks  $N_s, N_t$ , where no vertex of  $N_s$  has a path to  $t$ , and no vertex of  $N_t$  has a path to  $s$ . Also, each edge  $e \in X$  must have connections to a vertex of  $N_s$  and a vertex of  $N_t$ . Let  $N'_s$  be the planar network consisting of  $N_s$ , a new vertex  $t'$ , and for each  $e \in X$ , add a new edge with cost  $c(e)$  connecting  $t'$  to the vertex of  $e$  contained in  $N_s$ . Similarly, let  $N'_t$  be the planar network consisting of  $N_t$ , a new vertex  $s'$ , and adding a new edge of cost  $c(e)$  connecting  $s'$  to the vertex of  $e$  contained in  $N_t$ , for each  $e \in X$  (see Figure 5). Note that  $N'_s$  and  $N'_t$  are not necessarily standard since they may contain multiple edges connecting a given vertex to  $s$  or  $t$ . Let  $\text{DIVIDE}(N, X, s)$  and  $\text{DIVIDE}(N, X, t)$  be the planar standard networks derived from  $N'_s, N'_t$  respectively by merging multiple edges and setting the cost of each resulting edge to be the sum of the costs of the multiple edges from which it was derived (see Figure 6).

Let  $E$  be the edges of network  $N$ , and let  $Y$  be a set of edges of  $N_s$  (or  $N_t$ ). Also, let  $E(Y)$  be the set of edges of  $E$  derived from  $Y$  by substituting for any edge  $e$  connecting  $t'$  (or  $s'$ ) the corresponding edges of  $X$  from which  $e$  was derived. The following theorem follows immediately from the above lemma and Proposition 3.

**THEOREM 4.** *Let  $X$  be an  $s$ - $t$  cut of planar standard network  $N$  such that  $D(X)$  is a  $F$ -minimum cut-cycle, for some face  $F$  in a  $\mu(s, t)$  path of  $D(N)$ . Let  $X_s$  be a minimum  $s$ - $t'$  cut of  $\text{DIVIDE}(N, X, s)$  and let  $X_t$  be a minimum  $s'$ - $t$  cut of  $\text{DIVIDE}(N, X, t)$ . Then  $E(X_s)$  or  $E(X_t)$  is a minimum  $s$ - $t$  cut of  $N$ .  $\square$*

## 6. The Min $s$ - $t$ Cut Algorithm for Planar Networks

Theorem 4 of the previous Section 4 yields a very simple, but efficient, "divide and conquer" algorithm for computing minimum  $s$ - $t$  cut of a planar standard network. We assume the [Ford and Fulkerson, 1956] Algorithm (given in Section 3):

(i)  $(s, t)$ -PLANAR-MIN-CUT( $N$ ) which computes a minimum  $s$ - $t$  of  $(s, t)$ -planar standard network  $N$  in time  $O(Q_L(n))$ .

We also assume algorithms (given in Section 4):

(ii)  $\mu(s, t)$  PATH( $D(N)$ ) computes a  $\mu(s, t)$  path of  $D(N)$  in time  $O(Q_L(n))$ .

(iii)  $F$ -MIN-CUT-CYCLE( $N, F_i, \mu$ ) computes a  $F_i$ -minimum cycle of  $N$  (for  $F_i$  in  $\mu(s, t)$  path  $\mu$ ), in time  $O(Q_L(n))$ .

Recursive Algorithm PLANAR-MIN-CUT(N,  $\mu$ )

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 planar standard network  $N = (G, c, s, t)$ , where  $G = (V, E)$ , and  $\mu(s, t)$  path  $\mu$ .
begin
  Let  $F_1, \dots, F_d$  be the faces traversed by  $\mu$ .
  if  $d=1$  then return  $(s, t)$ -PLANAR-MIN-CUT(N);
  else begin
     $X \leftarrow D^{-1}(F\text{-MIN-CUT-CYCLE}(N, F_{\lfloor d/2 \rfloor}, \mu))$ ;
     $N_0 \leftarrow \text{DIVIDE}(N, X, s)$ ;  $N_1 \leftarrow \text{DIVIDE}(N, X, t)$ ;
    Let  $\mu_0$  and  $\mu_1$  be the subpaths of  $\mu$  contained in  $N_0$  and  $N_1$ , respectively
     $X_1 \leftarrow \text{PLANAR-MIN-CUT}(N_1, \mu_1)$ ;  $X_0 \leftarrow \text{PLANAR-MIN-CUT}(N_0, \mu_0)$ 
    if  $c(E(X_0)) \leq c(E(X_1))$  then return  $E(X_0)$  else return  $E(X_1)$ ;
  end;
end

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For any  $\omega \in \{0, 1\}^r$ ,  $r \geq 0$ , inductively let  $N_\omega = (G_\omega, c_\omega, s_\omega, t_\omega)$  be the planar standard network and let  $\mu_\omega$  be the  $\mu(s_\omega, t_\omega)$ -path in  $N_\omega$  defined by recursive calls to PLANAR-MIN-CUT. Let  $n_\omega$  and  $m_\omega$  be the number of vertices and edges of  $N_\omega$  (let  $n$  and  $m$  be the number of vertices and edges of  $N$ ). Suppose PLANAR-MIN-CUT( $N_\omega, \mu_\omega$ ) is called. If  $\mu_\omega$  contains only one face, then let  $N_{\omega 0}$  and  $N_{\omega 1}$  be empty networks, and let  $\mu_{\omega 0}$  and  $\mu_{\omega 1}$  be empty paths. Else let  $X_\omega$  be the  $s_\omega$ - $t_\omega$  cut of  $N_\omega$  computed by the call to  $D(F\text{-MIN-CUT-CYCLE}(-))$  and let  $N_{\omega 0}, N_{\omega 1}$  be the planar standard networks constructed by the calls to DIVIDE, and let  $\mu_{\omega 0}, \mu_{\omega 1}$  be the subpaths of  $\mu$  contained in  $N_{\omega 0}, N_{\omega 1}$ . Then it is easy to verify that  $\mu_{\omega 0}$  is a  $\mu(s_{\omega 0}, t_{\omega 0})$ -path in  $N_{\omega 0}$  and  $\mu_{\omega 1}$  is a  $\mu(s_{\omega 1}, t_{\omega 1})$ -path in  $N_{\omega 1}$ . Furthermore, if  $d$  is the length of  $\mu$  (the  $\mu(s, t)$  path of  $N$ ), there can be no more than  $\log(d) \leq \log(n)$  recursive calls. The following theorem provides an upper bound on the sum of the series of graphs derived by  $r$  recursive calls to PLANAR-MIN-CUT.

THEOREM 5. For each  $r \geq 0$ ,  $\sum_{\omega \in \{0, 1\}^r} m_\omega \leq 2m + 2^r$ . □

Proof. Note that by definition of DIVIDE, each of the edges of  $N_\omega$  are derived from disjoint sets of edges of  $N_\omega$ . Fix an edge  $e$  of  $N$ . Let  $e_\omega$  be the edge (if it exists) of  $N_\omega$  derived from a set of edges of  $N$  containing  $e$ . For each  $r \geq 0$ , let  $B_r(e) = \{e_\omega \mid e_\omega \neq \{s_\omega, t_\omega\} \text{ and } \omega \in \{0, 1\}^r\}$ .

We require a technical lemma:

LEMMA 2.  $|B_r(e)| \leq 2$ , and furthermore if  $B_r(e) = \{e_\omega, e_z\}$  for  $\omega < z$ , then edge  $e_\omega$  is connected to  $t_\omega$  and edge  $e_z$  is connected to  $s_z$ .

Proof by induction. Suppose for some fixed  $r_0$ , this lemma holds for all  $r \leq r_0$ . If  $B_{r_0}(e) = \emptyset$  then clearly  $B_{r_0+1}(e) = \emptyset$ . Suppose  $|B_{r_0}(e)| \geq 1$  and consider any  $e_\omega \in B_{r_0}(e)$ . If  $e_\omega \notin X_\omega$  then by definition of DIVIDE, either  $e_\omega = e_{\omega 0}$  appears in  $N_{\omega 0}$  or  $e_\omega = e_{\omega 1}$  appears in  $N_{\omega 1}$ , but not both. On the other hand, if  $e_\omega \in X_\omega$ , then  $e_{\omega 0}$  appears in  $N_{\omega 0}$  connected to  $t_{\omega 0}$  and also  $e_{\omega 1}$  appears in  $N_{\omega 1}$  connected to  $s_{\omega 1}$ . In either case, if  $|B_{r_0}(e)| = 1$ , then  $|B_{r_0+1}(e)| \leq 2$ . Else suppose there exists some  $e_z \in B_{r_0}(e)$  with  $\omega < z$ . By the induction hypothesis,

$e_w$  is connected to  $t_w$  and  $e_z$  is connected to  $s_z$ . Thus for  $j=0,1$  edge  $e_{wj}$  (if it exists) is connected to  $t_{wj}$  and edge  $e_{zj}$  (if it exists) is connected to  $s_{zj}$ . Hence if  $e_w \in X_w$  then  $e_{z1} = \{s_{z1}, t_{z1}\}$ . In each case,  $|B_{r_0+1}(e)| \leq 2$ .  $\square$

To complete the proof of Theorem 5, observe that  $|\{(s_w, t_w) | w \in \{0,1\}^r\}| = 2^r$ .

Hence

$$\sum_{w \in \{0,1\}^r} m_r \leq \left( \sum_{e \in E} |B_r(e)| \right) + |\{(s_w, t_w) | w \in \{0,1\}^r\}| \leq 2m + 2^r$$

by Lemma 2.  $\square$

**THEOREM 6.** *Given a planar standard network  $N = (G, c, s, t)$  with  $L = \text{range}(c)$ , and  $\mu$  is a  $\mu(s, t)$  path of  $N$  then  $\text{PLANAR-MIN-CUT}(N, \mu)$  computes a minimum  $s$ - $t$  cut of  $N$  in time  $O(Q_L(n) \log(n))$ .*

Proof. The total time cost is

$$\begin{aligned} \sum_{\substack{w \in \{0,1\}^r \\ r \leq \log n}} O(Q_L(m_w)) &= \sum_{r \leq \log n} O(Q_L(2m + 2^r)) \quad \text{by Theorem 5} \\ &= O(Q_L(n) \log n), \quad \text{since } 2m + 2^{\log n} = O(n) \quad \square \end{aligned}$$

By known upper bounds on the cost of maintaining queues (as discussed in the Introduction), we also have:

**COROLLARY 4.** *A minimum  $s$ - $t$  cut of  $N$  is computed in time  $O(n \log^2(n))$  for general  $L$  (i.e., a set of positive reals), in time  $O(n \log(n) \log \log(n))$  for the case  $L$  is a set of positive integers bounded by a polynomial in  $n$  and in time  $O(n \log(n))$  for the case  $L = \{1\}$  (in this case  $N$  is a graph with identically weighted edges).  $\square$*

## 7. Conclusion

We have presented an algorithm for computing a minimum  $s$ - $t$  cut of a planar undirected network. Our algorithm runs in an order of magnitude less time than previous algorithms for this problem. An additional attractive feature of this algorithm is its *simplicity*, as compared to other algorithms for computing minimum  $s$ - $t$  cuts for sparse networks [Galil, Naamad, 1979] and [Shiloach, 1978].

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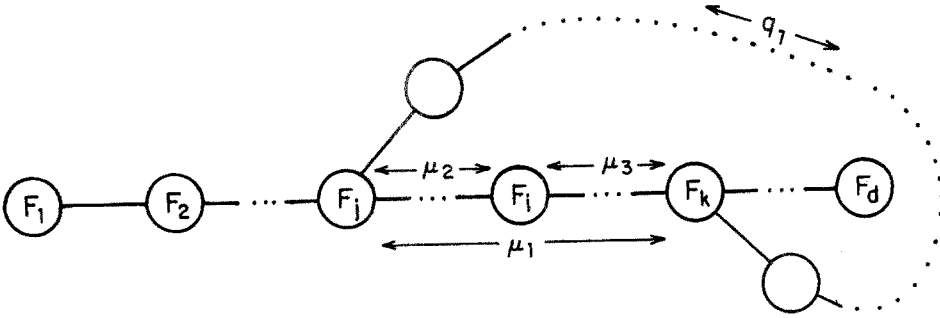


Fig. 1.  $F_i$ -minimum cut cycle  $\hat{p}$  in  $D(N)$  with  $\hat{p} = \mu_1 q_1$ .

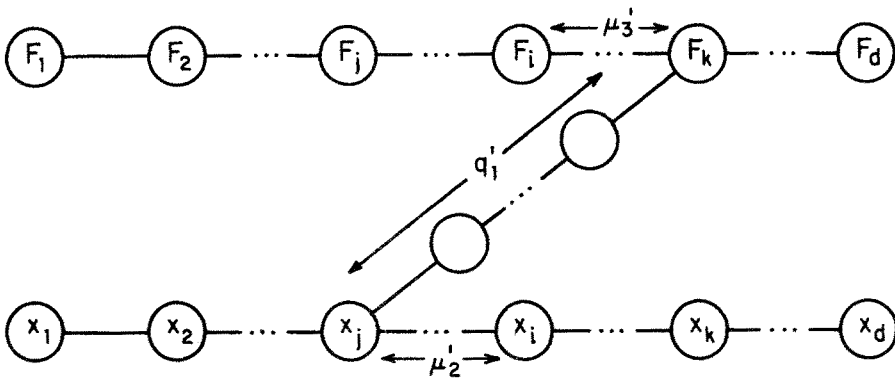


Fig. 2. Path  $p = \mu_2' \cdot q_1' \cdot \mu_3'$  from  $x_i$  to  $F_i$  in  $D'$ .

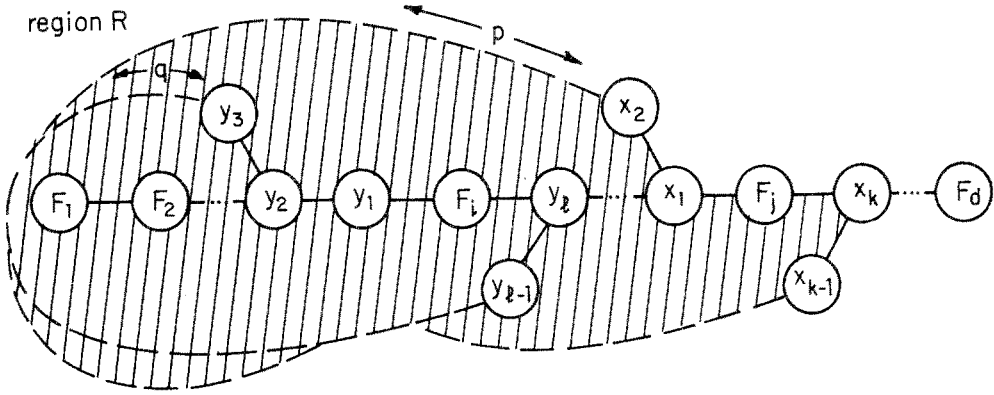


Fig. 3.  $F_1, F_2, \dots, F_d$  is a  $\mu(s, t)$  path in  $D(N)$ .  $p = (F_j, x_1, x_2, \dots, x_k)$  is a  $F_j$ -minimum cut-cycle enclosing region  $R$ . The  $F_i$ -minimum cut-cycle  $q = (F_i, y_1, y_2, \dots, y_l)$  is contained in  $R$ .

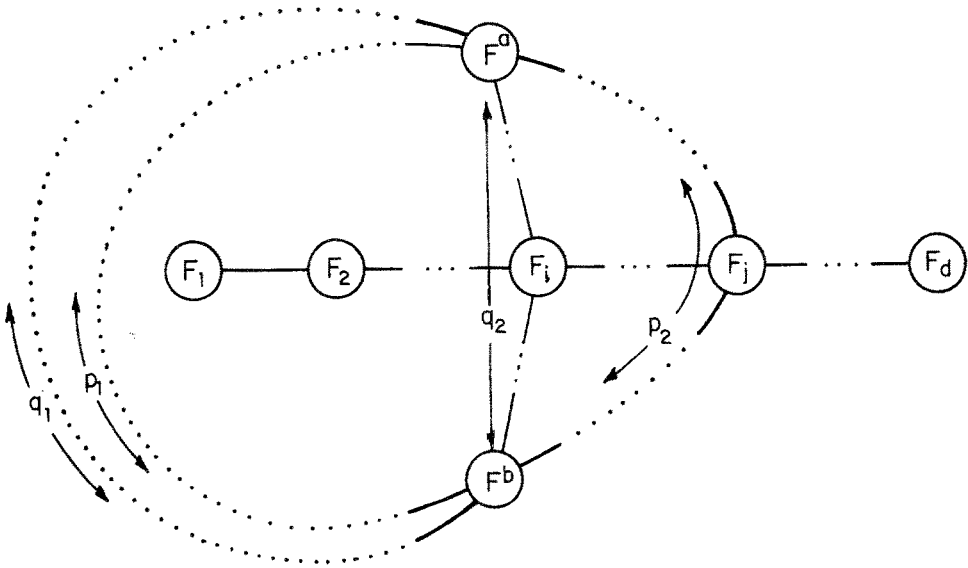


Fig. 4.  $F_1, F_2, \dots, F_d$  is a  $\mu(s, t)$ -path,  $p = p_1 \cdot p_2$  is a cut-cycle containing  $F_j$ .  $q = q_1 \cdot q_2$  is a cut-cycle containing  $F_i$ . If  $c(q_1) < c(p_1)$ , then  $p' = q_1 \cdot p_2$  is a cut-cycle containing  $F_j$  and with cost  $c(p') < c(p)$ .

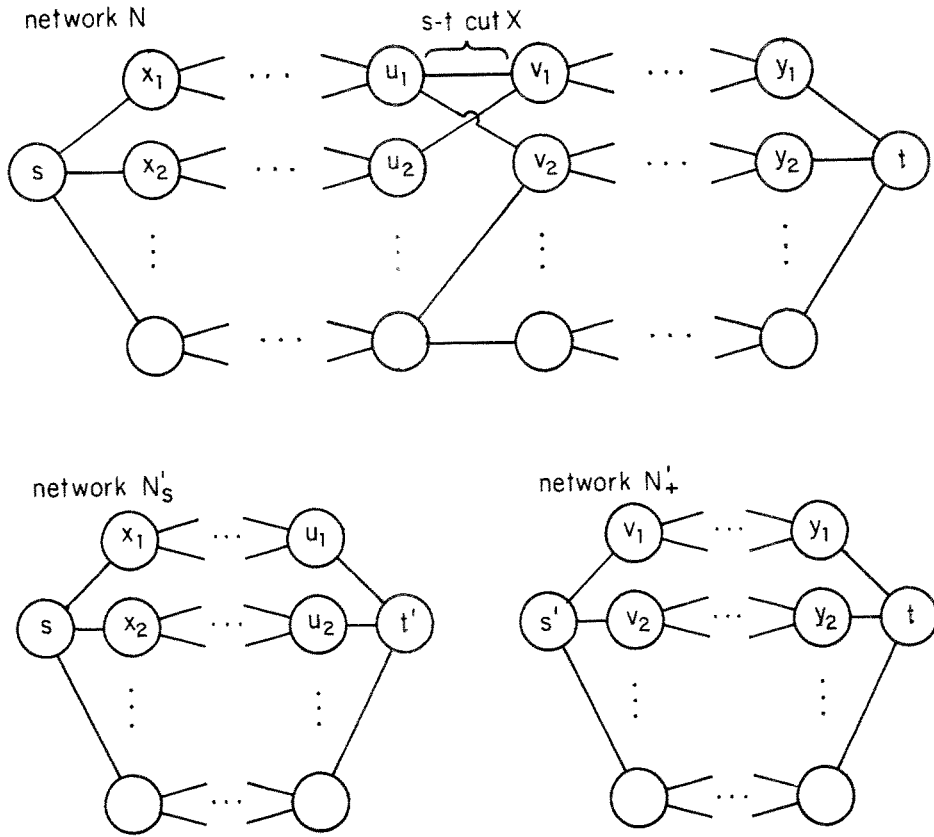


Fig. 5. The networks  $N'_s$  and  $N'_t$  derived from network N with s-t cut X.

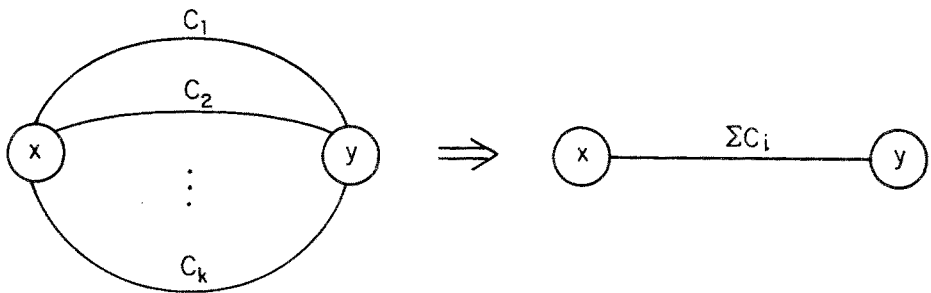


Fig. 6. The merging of multiple edges connected to vertex x and vertex y, into a single edge.