CANONICAL DECOMPOSITIONS OF SYMMETRIC SUBMODULAR SYSTEMS

S. Fujishige
Institute of Socio-Economic Planning
University of Tsukuba
Sakura, Ibaraki, Japan 305

Abstract. Let E be a finite set, R the set of real numbers and $f: 2^E \to R$ a symmetric submodular function. The pair (E,f) is called a symmetric submodular system. We examine the structures of symmetric submodular systems and provide a decomposition theory of symmetric submodular systems. The theory is a generalization of the decomposition theory of 2-connected graphs developed by W. T. Tutte.

1. Introduction

A decomposition theory of graphs is developed by W. T. Tutte [9]. A connected graph G is decomposed into a set of 2-connected subgraphs of G and the incidence relation of these 2-connected subgraphs is represented by a tree. Moreover, a 2-connected graph G is decomposed into a set of 3-connected graphs, bonds and polygons, and their structural relation is represented by a tree. Also R. E. Gomory and T. C. Hu [7] derived a tree structure of the set of minimum cuts of a capacitated undirected (or symmetric) multi-terminal network. In extracting these tree structures, symmetric submodular functions play a crucial role. Related tree representation of a collection of sets was examined by J. Edmonds and R. Giles [4].

Let E be a finite set and $f: 2^E \to R$ a symmetric submodular function, whose precise definition will be given in Section 2. The pair (E,f) is called a symmetric submodular system. We shall consider symmetric submodular systems and provide a theory of decomposition of symmetric submodular systems, which is a generalization of the decomposition theory of 2-connected graphs by Tutte [9]. The decomposition theory can be applied to any systems with submodular functions such as graphs [9], capacitated networks [7], matroids [10], communication networks [5] etc., where if necessary the underlying submodular functions should be symmetrized (see Section 5).

2. Definitions and Assumptions

Let E be a finite set, R the set of real numbers and $f: 2^E \to R$ a submodular function, i.e.,

$$f(A) + f(B) \geq f(A \cup B) + f(A \cap B) \qquad (2.1)$$

for any $A, B \subseteq E$. The pair (E,f) is called a submodular system [6] and if the submodular function f is symmetric, i.e.,

$$f(A) = f(E-A) \qquad (2.2)$$

for any $A \subseteq E$, then (E,f) is called a symmetric submodular system.

If $C \subseteq E$ satisfies $|C| \geq k$ and $|E-C| \geq k$ for a positive integer k, we call C a k-cut of (E,f). Let $e_A \notin E$ be a new element corresponding to a nonempty subset A of E and define

$$E' = (E-A) \cup \{e_A\}, \qquad (2.3)$$

$$f'(B) = f(B) \quad \text{if } e_A \notin B \subseteq E', \qquad (2.4a)$$

$$= f((B-\{e_A\}) \cup A) \quad \text{if } e_A \in B \subseteq E'. \qquad (2.4b)$$

Then we call the submodular system (E',f') an aggregation of (E,f) by A and we denote it by $(E,f)//A$. Let $P = \{A_0, A_1, \cdots, A_k\}$ be a partition of E, i.e., $A_i \neq \emptyset$ $(i=0,1,\cdots,k)$, $A_i \cap A_j = \emptyset$ $(i \neq j; i,j=0,1,\cdots,k)$ and $A_0 \cup A_1 \cup \cdots \cup A_k = E$. For the partition P, let us define

$$(E,f)//P = (\cdots(((E,f)//A_0)//A_1)\cdots)//A_k. \qquad (2.5)$$

Note that $(E,f)//P$ does not depend on the order of the A_i's in (2.5). If subsets C_1 and C_2 of E satisfy $C_1 \cup C_2 \neq E$, $C_1 \cap C_2 \neq \emptyset$, $C_1 - C_2 \neq \emptyset$ and $C_2 - C_1 \neq \emptyset$, then we say C_1 and C_2 cross. We define a partial order $\leqq$ on the set of partitions of E as follows. For partitions P and P' of E, $P \leqq P'$ if and only if for each $A \in P$ there is an element $A' \in P'$ such that $A \subseteq A'$.

Throughout the present paper, we assume that (E,f) is a symmetric submodular system and

$$\min\{f(C) \mid C \text{ is a 1-cut of } (E,f)\} = \lambda^*. \qquad (2.6)$$

We denote by $\mathcal{C}_f$ the set of 2-cuts C such that $f(C) = \lambda^*$. We shall examine the structure of the set $\mathcal{C}_f$ and decompose (E,f) based on $\mathcal{C}_f$. It should be noted that $\mathcal{C}_f$ is complemented, i.e., if $C \in \mathcal{C}_f$ then $E-C \in \mathcal{C}_f$.

3. Main Theorems

The following lemma is fundamental for the symmetric submodular system (E,f) satisfying (2.6).

Lemma 1: Suppose that subsets C_1 and C_2 of E cross and satisfy

$$f(C_1) = f(C_2) = \lambda^*. \tag{3.1}$$

Then we have

$$f(C_1 \cup C_2) = f(C_1 \cap C_2) = f(C_1 - C_2) = f(C_2 - C_1) = \lambda^*. \tag{3.2}$$

(Proof) Since

$$f(C_1) + F(C_2) \geqq f(C_1 \cup C_2) + f(C_1 \cap C_2) \tag{3.3}$$

and C_1 and C_2 cross, we have from (2.6)

$$f(C_1 \cup C_2) = f(C_1 \cap C_2) = \lambda^*. \tag{3.4}$$

Because of the symmetry of f, Lemma 1 follows from (3.4). Q.E.D.

Lemma 2: Let e_1, e_2, e_3 and e_4 be four distinct elements of E. If $\{e_1,e_2\}$, $\{e_1,e_3\}$, $\{e_1,e_4\} \in \mathcal{C}_f$, then $\{e_2,e_3\}$, $\{e_2,e_4\}$, $\{e_3,e_4\} \in \mathcal{C}_f$.
(Proof) Since $\{e_1,e_2\}$ and $\{e_1,e_3\}$ in $\mathcal{C}_f$ cross, we have from Lemma 1

$$f(\{e_1,e_2,e_3\}) = \lambda^*. \tag{3.5}$$

If $E = \{e_1,e_2,e_3,e_4\}$, then $\{e_2,e_3\} = E - \{e_1,e_4\} \in \mathcal{C}_f$. Therefore, suppose $E \neq \{e_1,e_2,e_3,e_4\}$. Then, since $\{e_1,e_2,e_3\}$ and $\{e_1,e_4\}$ cross, we have from (3.5) and Lemma 1

$$\{e_2,e_3\} = \{e_1,e_2,e_3\} - \{e_1,e_4\} \in \mathcal{C}_f. \tag{3.6}$$

Because of the symmetry among the elements e_2, e_3 and e_4, this completes the proof of Lemma 2. Q.E.D.

Now, let R_f be a collection of two-element subsets of E defined by

$$R_f = \{C \mid C \in \mathcal{C}_f, |C|=2\}. \tag{3.7}$$

Theorem 1: Let $G = (E,R_f)$ be a graph with the vertex set E and the edge set R_f defined by (3.7). If G is connected, then G is a complete graph or an elementary closed path.
(Proof) By definition, connectedness of G implies that $|E| = 1$ or $|E| \geqq 4$ and thus we assume $|E| \geqq 4$. It follows from Lemma 2 that G can be a complete graph, an elementary closed path or an elementary nonclosed path. Therefore, let us assume that $E = \{e_1,e_2,\cdots,e_n\}$ $(n \geqq 4)$ and that $\{e_i,e_{i+1}\} \in \mathcal{C}_f$ $(i=1,2,\cdots,n-1)$. Then $\{e_1,e_n\}$ must be in $\mathcal{C}_f$ because from Lemma 1 we have $\{e_2,e_3,\cdots,e_{n-1}\} \in \mathcal{C}_f$. Consequently, G cannot be an elementary nonclosed path. Q.E.D.

Suppose that the graph $G = (E,R_f)$ has at least four vertices. If G is a complete graph or an elemenary closed path, then we say (E,f) is

<u>of bond type</u> or <u>of polygon type</u>, respectively. We call (E,f) <u>irreducible</u> if $\mathcal{C}_f$ is empty or (E,f) is of bond type or of polygon type. In particular, if $\mathcal{C}_f$ is empty, we call (E,f) <u>absolutely irreducible</u>.

Suppose that, for $e^* \varepsilon E$, a partition $P(e^*) = \{\{e^*\},A_1,A_2,\cdots,A_k\}$ of E satisfies

(i) $(E,f)//P(e^*)$ is irreducible,

(ii) for each $i = 1, 2, \cdots, k$, if $|A_i| \geqq 2$, then $A_i \varepsilon \mathcal{C}_f$.

Then $P(e^*)$ is called an <u>irreducibility partition associated with</u> e^* ε E. Let us denote by $\mathcal{P}(e^*)$ the set of all irreducibility partitions associated with $e^* \varepsilon E$. Note that $\mathcal{P}(e^*)$ is nonempty for every e^* ε E.

For partitions P and P' of E given by $P = \{A_0,A_1,\cdots,A_k\}$ and $P' = \{A_0',A_1',\cdots,A_h'\}$, let us define a partition $P \wedge P'$ of E by

$$P \wedge P' = \{A_i \cap A_j' \mid i=0,1,\cdots,k; j=0,1,\cdots,h; A_i \cap A_j' \neq \emptyset\}. \quad (3.8)$$

We shall show Theorems 2 - 5 from which follows the fact that, for every $e^* \varepsilon E$, $\mathcal{P}(e^*)$ is closed with respect to the operation $\wedge$ (Theorem 6). We need some preliminary lemmas.

<u>Lemma 3</u>: Suppose $P \equiv \{A_0,A_1,\cdots,A_k\}$ $(k \geqq 4)$ is a partition of E and define

$$A_\ell^* = \bigcup\{A_j \mid j=\ell,\ell+1,\cdots,k\} \quad (3.9)$$

and

$$P' = \{A_0,A_1,\cdots,A_{\ell-1},A_\ell^*\}, \quad (3.10)$$

where $3 \leqq \ell < k$. Then the following (i) and (ii) hold.

(i) If $(E,f)//P$ is of polygon type and $f(A_i \cup A_{i+1}) = \lambda^*$ $(i=0,1,\cdots,k)$, where $A_{k+1} = A_0$, then $(E,f)//P'$ is also of polygon type and $f(A_{\ell-1} \cup A_\ell^*) = f(A_\ell^* \cup A_0) = \lambda^*$.

(ii) If $(E,f)//P$ is of bond type, then $(E,f)//P'$ is also of bond type.

(Proof) From Lemma 1 we have $f(A_\ell^*) = \lambda^*$ and $f(A_{\ell-1} \cup A_\ell^*) = f(A_\ell^* \cup A_0) = \lambda^*$. Because of the assumption and Theorem 1 this implies that $(E,f)//P'$ is of polygon type or of bond type according as $(E,f)//P$ is of polygon type or of bond type. Q.E.D.

<u>Lemma 4</u>: Suppose $P \equiv \{A_0,A_1,\cdots,A_k\}$ $(k \geqq 3)$ is a partition of E such that $(E',f') \equiv (E,f)//P$ is of polygon type and that $f(A_i \cup A_{i+1}) = \lambda^*$ $(i=0,1,\cdots,k)$, where $A_{k+1} = A_0$. Also suppose $B \varepsilon \mathcal{C}_f$ and $A_0 \cap B = \emptyset$ and define

$$J = \{j \mid j=1,2,\cdots,k; A_j \cap B \neq \emptyset\}. \quad (3.11)$$

Then, for any integer i^* such that $\min J < i^* < \max J$, we have $A_{i^*} \subseteqq B$, where min J and max J denote the minimum integer and the maximum integer in J, respectively.

(Proof) Suppose there were an integer i^* such that $\min J < i^* < \max J$ and $A_{i^*} - B \neq \emptyset$. Put

$$J_1 = \{j \mid j \in J, j < i^*\}, \quad (3.12)$$
$$J_2 = \{j \mid j \in J, j > i^*\}. \quad (3.13)$$

Also define

$$A_1^* = \bigcup\{A_j \mid \min J_1 \leqq j \leqq \max J_1\}, \quad (3.14)$$
$$A_2^* = \bigcup\{A_j \mid \min J_2 \leqq j \leqq \max J_2\}, \quad (3.15)$$
$$P' = (P - \{A_j \mid A_j \subseteqq A_1^* \cup A_2^*;\ j=1,2,\cdots,k\}) \cup \{A_1^*, A_2^*\}. \quad (3.16)$$

It follows from Lemma 3 that the aggregation $(E'', f'') \equiv (E,f)//P'$ is of polygon type. Furthermore, put $B^* = B - A_{i^*}$. Then $f(B^*) = \lambda^*$ and we have from Lemma 1 and the definition of A_1^* and A_2^*

$$f(A_1^* \cup A_2^*) = f((A_1^* \cup B^*) \cup (A_2^* \cup B^*)) = \lambda^*. \quad (3.17)$$

This contradicts the assertion that (E'', f'') is of polygon type. Q.E.D.

<u>Lemma 5</u>: Under the assumption of Lemma 4, if B and A_{j^*} with $j^* = \min J$ cross, then $(E,f)//P'$ is of polygon type, where

$$P' = \{A_0, A_1, \cdots, A_{j^*-1}, A_{j^*}-B, A_{j^*} \cap B, A_{j^*+1}, \cdots, A_k\}. \quad (3.18)$$

Furthermore, we have

$$f(A_{j^*-1} \cup (A_{j^*}-B)) = f((A_{j^*} \cap B) \cup A_{j^*+1}) = \lambda^*. \quad (3.19)$$

(Proof) Since $A_{j^*-1} \cap B = \emptyset$ and either $A_{j^*-1} \cup A_{j^*} \cup B = E$ or $A_{j^*-1} \cup A_{j^*}$ and B cross, we have $f(A_{j^*-1} \cup (A_{j^*}-B)) = f(A_{j^*} \cap B) = \lambda^*$. Therefore, from the assumption and Theorem 1 $(E,f)//P'$ must be of polygon type and the remaining part follows. Q.E.D.

<u>Theorem 2</u>: Suppose $P, P' \in \mathcal{P}(e^*)$ and $|P| \geqq 4$. If $(E,f)//P$ is of polygon type, then $(E,f)//P \wedge P'$ is of polygon type and, therefore, $P \wedge P' \in \mathcal{P}(e^*)$. Moreover, if $|P'| \geqq 4$, $(E,f)//P'$ is also of polygon type.

(Proof) Suppose $P = \{\{e^*\}=A_0, A_1, \cdots, A_k\}$ $(k \geqq 3)$ and $P' = \{\{e^*\}=A_0', A_1', \cdots, A_h'\}$. If $A_i \in P$ and $A_j' \in P'$ cross, then for the partition P_1 obtained from P by dividing A_i into $A_i - A_j'$ and $A_i \cap A_j'$, $(E,f)//P_1$ is irreducible and of polygon type due to Lemma 5. By repeating this process we obtain a partition $P^* = \{\{e^*\}=A_0^*, A_1^*, \cdots, A_{k^*}^*\}$ which is minimal, with respect to the partial order $\lesssim$, with the property: " $P^* \lesssim P$ and A_i^* and A_j' do not cross for any $A_i^* \in P^*$ and $A_j' \in P'$." The obtained $(E,f)//P^*$ is of polygon type.

If there is no A_i^* in P^* such that A_i^* contains at least two

A_j' 's, then $P^* = P \wedge P'$ and this completes the proof. Therefore, suppose that some $A_{i_0}^*$ is expressed as $A_{i_0}^* = \bigcup \{A_j' \mid j=t_1,t_2,\cdots,t_p\}$ $(p \geqq 2)$. Since $(E,f)//P^*$ is of polygon type, $f(A_{i_0}^*) = \lambda^*$. It follows that $(E,f)//P'$ must be of polygon type or of bond type. In either case, from Theorem 1, for some $j^* \in \{t_1,t_2,\cdots,t_p\}$ and some $j' \in \{0, 1,\cdots,h\} - \{t_1,t_2,\cdots,t_p\}$ there holds $f(A_{j^*}' \cup A_{j'}') = \lambda^*$. Therefore, since $A_{i_0}^*$ and $A_{j^*}' \cup A_{j'}'$ cross, we see from Lemma 5 that $(E,f)//P_1^*$ is of polygon type, where P_1^* is the partition of E obtained from P^* by dividing $A_{i_0}^*$ into $A_{i_0}^* \cap (A_{j^*}' \cup A_{j'}') = A_{j^*}'$ and $A_{i_0}^* - (A_{j^*}' \cup A_{j'}') = A_{i_0}^* - A_{j^*}'$. By repeating this process we reach the partition $P \wedge P'$ for which $(E,f)//P \wedge P'$ is of polygon type.

Moreover, since $P \wedge P' \precneqq P'$, if $|P'| \geqq 4$, then $(E,f)//P'$ is of polygon type due to Lemma 3. Q.E.D.

<u>Lemma 6</u>: Suppose $P \equiv \{A_0,A_1,\cdots,A_k\}$ $(k \geqq 3)$ is a partition of E and $(E',f') \equiv (E,f)//P$ is of bond type. Also suppose $B \in \mathcal{C}_f$ and $A_{j^*} \in P$ cross and $A_0 \cap B = \emptyset$. Then $(E,f)//P'$ is of bond type, where $P' = \{A_0,A_1,\cdots,A_{j^*-1},A_{j^*}-B,A_{j^*}\cap B,A_{j^*+1},\cdots,A_k\}$.

(Proof) Since B and A_{j^*} cross, there is an $A_{i^*} \in P$ such that $A_{i^*} \cap B \neq \emptyset$ and $i^* \neq 0, j^*$. Put $B^* = A_{i^*} \cup B$. Then we have $f(B^*) = \lambda^*$. Since B and A_{j^*} cross and B^* and $A_{i^*} \cup A_{j^*}$ cross, we get

$$f(A_{j^*} \cap B) = f(A_{j^*}-B) = f(A_{i^*} \cup (A_{j^*} \cap B)) = \lambda^*. \qquad (3.20)$$

From (3.20) and Theorem 1 we see that $(E,f)//P'$ is of bond type.

Q.E.D.

<u>Theorem 3</u>: Suppose $P, P' \in \mathcal{P}(e^*)$ and $|P| \geqq 4$. If $(E,f)//P$ is of bond type, then $(E,f)//P \wedge P'$ is of bond type and, therefore, $P \wedge P' \in \mathcal{P}(e^*)$. Moreover, if $|P'| \geqq 4$, $(E,f)//P'$ is also of bond type.

(Proof) Theorem 3 can be shown by using Lemmas 3 and 6 and Theorem 1 in a way similar to the proof of Theorem 2. Q.E.D.

<u>Theorem 4</u>: Suppose $e^* \in E$, $P = \{\{e^*\},A_1,A_2\} \in \mathcal{P}(e^*)$ and $P' = \{\{e^*\}, A_1',A_2'\} \in \mathcal{P}(e^*)$. Then $P \wedge P' \in \mathcal{P}(e^*)$. If $|P| = 3$ for any $P \in \mathcal{P}(e^*)$, then $|\mathcal{P}(e^*)| = 1$.

(Proof) Suppose $P \neq P'$.

First, suppose $A_1 \subsetneqq A_1'$. Then $|A_2| \geqq 2$ and $f(\{e^*\} \cup A_1) = f(E -A_2) = \lambda^*$. Therefore, for the partition $P \wedge P' \equiv \{\{e^*\},A_1,A_2 \cap A_1',A_2-A_1'\}$, $(E,f)//P \wedge P'$ is of bond type or of polygon type and $P \wedge P' \in \mathcal{P}(e^*)$.

Next, suppose A_1 and A_1' cross and A_2 and A_1' cross. Then $f(\{e^*\} \cup (A_1-A_1')) = f(A_1 \cap A_1') = f(A_2 \cap A_1') = f(A_2-A_1') = \lambda^*$. It follows that, for $P \wedge P' \equiv \{\{e^*\},A_1-A_1',A_1 \cap A_1',A_2 \cap A_1',A_2-A_1'\}$, $(E,f)//P \wedge P'$ is of bond type or of polygon type and $P \wedge P' \in \mathcal{P}(e^*)$.

The remaining part of the theorem follows from the fact that, if

$P, P' \in \mathcal{P}(e^*)$, $P \neq P'$ and $|P| = |P'| = 3$, then $P \wedge P' \in \mathcal{P}(e^*)$ and $|P \wedge P'| \geqq 4$. Q.E.D.

Lemma 7: Suppose that $P \equiv \{A_0, A_1, \cdots, A_k\}$ $(k \geqq 3)$ is a partition of E and that $(E,f)//P$ is absolutely irreducible. Then, for any $B \in \mathcal{C}_f$ such that $A_0 \cap B = \emptyset$, B and any of $A_1, \cdots, A_k$ do not cross.
(Proof) Suppose B and A_1 cross. Let us define

$$I = \{i \mid A_i \cap B \neq \emptyset,\ i=1,2,\cdots,k\}. \tag{3.21}$$

Then $|I| \geqq 2$ and, from Lemma 1, $A^* \equiv \bigcup\{A_i \mid i \in I\}$ satisfies $f(A^*) = \lambda^*$. It follows that $I = \{1,2,\cdots,k\}$, since $(E,f)//P$ is absolutely irreducible. Put

$$B^* = (B \cup (\bigcup\{A_i \mid i=2,\cdots,k\})) - A_1. \tag{3.22}$$

From Lemma 1 we have $f(B^*) = \lambda^*$. Consequently, $f(A_0 \cup A_1) = \lambda^*$, since $B^* = E - (A_0 \cup A_1)$. This contradicts the absolute irreducibility of $(E,f)//P$. Q.E.D.

Theorem 5: Suppose that, for some $P \in \mathcal{P}(e^*)$ such that $|P| \geqq 4$, $(E,f)//P$ is absolutely irreducible. Then $|\mathcal{P}(e^*)| = 1$.
(Proof) Suppose $P = \{\{e^*\}, A_1, \cdots, A_k\}$ and there is another $P' = \{\{e^*\}, A_1', \cdots, A_h'\}$ in $\mathcal{P}(e^*)$. It follows from Lemma 7 and the absolute irreducibility of $(E,f)//P$ that each $A_j' \in P'$ is included in some $A_i \in P$. Suppose that, for some distinct indices $j_1, j_2 \in \{1,2,\cdots,h\}$, $A_{j_1}' \cup A_{j_2}'$ is included in some A_i. Then $(E,f)//P'$ must be of polygon type or of bond type. This contradicts Theorem 2 or 3. Therefore, $P = P'$. Q.E.D.

It should be noted that, if $|E| \leq 3$, (E,f) is absolutely irreducible. Therefore, from Theorems 2 - 5 we have the following.

Theorem 6: For any $e^* \in E$, there is a unique minimal element of the partially ordered set $(\mathcal{P}(e^*), \leqq)$.

Because of Theorem 6, for each $e^* \in E$, we call the unique minimal element of $\mathcal{P}(e^*)$ the minimal irreducibility partition of E associated with e^* and denote it by $\hat{P}(e^*)$. Moreover, we call $A \in \hat{P}(e^*)$ a minimal irreducibility component of (E,f) associated with e^*.

Lemma 8: For $e^*, e \in E$, if the set $\{e\}$ is a minimal irreducibility component of (E,f) associated with e^*, then $\hat{P}(e^*) = \hat{P}(e)$.
(Proof) From the assumption, $\hat{P}(e^*) \in \mathcal{P}(e)$. Therefore, $\hat{P}(e) \leqq \hat{P}(e^*)$ and $\hat{P}(e) \in \mathcal{P}(e^*)$. By the minimality of $\hat{P}(e^*)$, this means $\hat{P}(e^*) = \hat{P}(e)$. Q.E.D.

Theorem 7: Suppose a set $D \subseteq E$ is a minimal irreducibility component of (E,f) associated with $e^* \varepsilon E$ such that $|D| \geq 2$. Then, for any $e \varepsilon D$, $E - D$ is included in a minimal irreducibility component of (E,f) associated with e.

(Proof) Let $\hat{P}(e^*) = \{\{e^*\}=A_0, A_1, \cdots, A_k\}$ and $\hat{P}(e) = \{\{e\}=A_0', A_1', \cdots, A_h'\}$, where $e \varepsilon A_1 = D$ and $e^* \varepsilon A_1'$. Suppose that $A_1 \cup A_1' \neq E$. Then, since from Lemma 8 we have $\{e^*\} \subsetneqq A_1'$ and since from Lemmas 5, 6 and 7 for each $A_j' \varepsilon \hat{P}(e)$ A_j' and any of $A_1, \cdots, A_k$ do not cross, both A_1' and $E - A_1'$ are unions of at last two A_i's of $\hat{P}(e^*)$. Therefore, $(E,f)//\hat{P}(e^*)$ is of bond type or of polygon type, and, by the same argument, $(E,f)//\hat{P}(e)$ is also of bond type or of polygon type. Similarly as the proof of Theorem 2, this contradicts the minimality of $\hat{P}(e)$ and $\hat{P}(e^*)$. Therefore, $A_1 \cup A_1' = E$. Q.E.D.

4. Canonical Decomposition

Let us define an equivalence relation $\hat{R} \subseteqq E \times E$ as follows: For $e^*, e \varepsilon E$, $(e^*,e) \varepsilon \hat{R}$ if and only if $\hat{P}(e^*) = \hat{P}(e)$. Let $\Pi \equiv \{S_1, S_2, \cdots, S_p\}$ be the partition of E composed of the equivalence classes of E relative to $\hat{R}$. The partition Π is called the canonical 2-cut partition, of level 1, of E. For any $S_j \varepsilon \Pi$, define

$$\hat{P}(S_j) = \hat{P}(e) \tag{4.1}$$

for any $e \varepsilon S_j$, where note that $\hat{P}(e) = \hat{P}(e')$ for any $e, e' \varepsilon S_j$. Each $A \varepsilon \hat{P}(S_j)$ with $|A| \geq 2$ is called a minimal irreducibility component of (E,f) associated with S_j.

Suppose that, for each $i = 1, 2, \cdots, k$ $(k \geq 3)$, A_i is a minimal irreducibility component of (E,f) associated with $S_{j(i)} \varepsilon \Pi$ and that $P^* \equiv \{E-A_1, E-A_2, \cdots, E-A_k\}$ is a partition of E. Then we call the partition P^* a 2-cut aggregation partition, of level 1, of E. Denote by $\mathcal{A}$ the set of all 2-cut aggregation partitions, of level 1, of E. Moreover, we call the aggregation $(E,f)//P^*$ $(P^* \varepsilon \mathcal{A})$ a 2-cut aggregation, of level 1, of (E,f) by P^*.

Let $G_1^* = (V_1^*, E_1^*)$ be a graph with a vertex set V_1^* and an edge set E_1^* defined as follows:

$$V_1^* = V_\Pi \cup V_{\mathcal{A}}, \tag{4.2}$$

where $V_\Pi = \{v_S \mid S \varepsilon \Pi\}$ and $V_{\mathcal{A}} = \{v_P \mid P \varepsilon \mathcal{A}\}$, and

$$E_1^* = A_1^* \cup B_1^*, \tag{4.3}$$

where

(i) $a \varepsilon A_1^*$ if and only if $a = \{v_S, v_{S'}\}$ such that $S, S' \varepsilon \Pi$

and $E - A = A'$ for minimal irreducibility components A and A' associated with S and S', respectively,

and

(ii) $a \in B_1{}^*$ if and only if $a = \{v_S, v_P\}$ such that $S \in \Pi$, $P \in A$ and $E - A = B$ for a minimal irreducibility component A associated with S and a component B of the 2-cut aggregation partition P.

We can easily see from Theorem 7 that the graph $G_1{}^* = (V_1{}^*, E_1{}^*)$ is a tree. We call the tree $G_1{}^*$ the <u>canonical decomposition tree, of level 1, of</u> (E,f). It should be noted that for each vertex v of $G_1{}^*$, if v corresponds to an $S_j \in \Pi$, then the vertex v is associated with $(E,f)//P(S_j)$ and, if v corresponds to a 2-cut aggregation partition P^*, then v is associated with the 2-cut aggregation $(E,f)//P^*$. Also note that there may be more than one 2-cut aggregation partitions of E of (E,f).

If a 2-cut aggregation $(E,f)//P^*$ of (E,f) is reducible, then further construct the canonical decomposition tree, of level 1, of $(E,f)//P^*$ and repeat this decomposition process until the constructed canonical decomposition tree does not contain any vertex which corresponds to a reducible 2-cut aggregation. If a canonical decomposition tree is obtained after k-1 repeated 2-cut aggregations, then we call the tree the canonical decomposition tree, <u>of level k</u>, of (E,f).

In this way we can decompose (E,f) into irreducible aggregations of (E,f) and extract the tree structures of these aggregations of all levels and, at the same time, the hierarchical structure of the reducible 2-cut aggregations.

A canonical decomposition tree of level $k+1$ can be embedded into a canonical decomposition tree of level k as follows. Let $G_{k+1}{}^*$ and $G_k{}^*$ be canonical decomposition trees, of level 1, of $(E^{(k)}, f^{(k)})$ and $(E^{(k-1)}, f^{(k-1)})$, respectively, and

$$(E^{(k)}, f^{(k)}) = (E^{(k-1)}, f^{(k-1)})//P^{(k-1)}, \tag{4.4}$$

where $P^{(k-1)}$ is a 2-cut aggregation partition of $E^{(k-1)}$ of $(E^{(k-1)}, f^{(k-1)})$. Note that $E^{(k)} = \{e_A \mid A \in P^{(k-1)}\}$. Let v^* be the vertex in $G_k{}^*$ which corresponds to $P^{(k-1)}$. Also let $v_S{}^{(k)}$ be the vertex in $G_k{}^*$ which corresponds to a component S of the canonical 2-cut partition of $E^{(k-1)}$ such that $v_S{}^{(k)}$ is adjacent to v^* and $E - A = B$ for a minimal irreducibility component A associated with S and a component B of $P^{(k-1)}$. Furthermore, let S^* be a component of the canonical 2-cut partition of $E^{(k)}$ containing the element e_B. Then replace the edge $\{v_S{}^{(k)}, v^*\}$ by $\{v_S{}^{(k)}, v_{S^*}{}^{(k+1)}\}$, where $v_{S^*}{}^{(k+1)}$ is the vertex in G_{k+1}* which corresponds to S*. In this way we replace all the edges, in G_k*, incident to v* and then delete v*, which yields a tree composed of G_k* and G_{k+1}*.

All the canonical decomposition trees can thus be embedded into the canonical decomposition tree, of level 1, of (E,f) by repeatedly embedding canonical decomposition trees into canonical decomposition trees of lower levels. We call the tree composed of all the canonical decomposition trees the total decomposition tree of (E,f).

5. Examples of Symmetric Submodular Systems and Their Decompositions

Now, let us show some examples.

Example 1: Let G = (V,E) be a connected but not 2-connected graph and define

$$f(A) = |V(A)| + |V(E-A)| - |V| \tag{5.1}$$

for any $A \subsetneqq E$, where for $B \subsetneqq E$ V(B) is the set of end-vertices of edges in B. Then (E,f) is a symmetric submodular system and satisfies (2.6) with $\lambda^* = 1$. Any 2-cut aggregations, of level 1, of (E,f) are of bond type if the ground sets have the cardinality not less than 4, so that (E,f) is decomposed up to level 1.

The canonical decomposition tree, of level 1, of (E,f) is different from, but essentially the same as, the tree representing the incidence relation of 2-connected subgraphs of G which is described in [9]. See Figure 1.

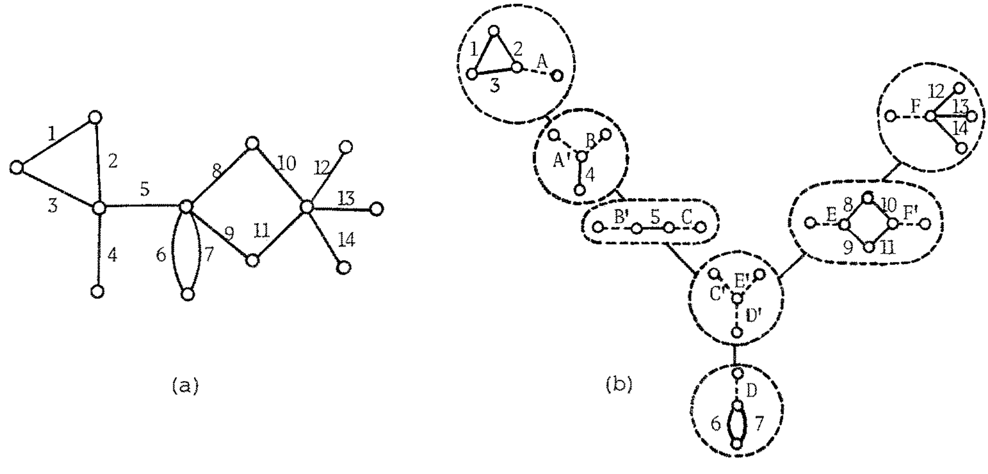

Figure 1. (a) A graph G; and (b) the canonical decomposition tree of (E,f) defined by (5.1).

Remark 1: The decomposition of a connected graph into 2-connected subgraphs [9] is determined by the structure of minimum 1-cuts of the symmetric submodular system (E,f) defined by (5.1). We can develop a decomposition theory based on the structure of minimum 1-cuts of symmetric submodular systems, which is similar to the theory, by Gomory and Hu [7], for representing the structure of the set of minimum cuts in a symmetric network by a tree.

Example 2: Let $G = (V,E)$ be a 2-connected graph and define $f: 2^E \to R$ by (5.1). Then (E,f) is a symmetric submodular system and satisfies (2.6) with $\lambda^* = 2$. The total decomposition tree of (E,f) is the same as the tree representing the structure of the set of two-terminal subgraphs of G described by Tutte [9], where the hierarchical structure of the set of two-terminal subgraphs is implicit.

Example 3: Let $M = (E,\rho)$ be a 2-connected matroid with a rank function ρ. Let us define

$$f(A) = \rho(A) + \rho(E-A) - \rho(E) + 1 \tag{5.2}$$

for any $A \subsetneqq E$. Then (E,f) is a symmetric submodular system and satisfies (2.6) with $\lambda^* = 2$ (cf. [10], [11]). Therefore, we can obtain the canonical decomposition trees of (E,f). Note that f defined by (5.2) is a symmetrization of the rank function ρ. It may also be noted that, if E with $|E| \geq 4$ is a circuit of the matroid (E,ρ), the corresponding (E,f) is not of polygon type but of bond type. Related works on matroid decompositions were made by R. E. Bixby [1] and W. H. Cunningham [3].

Remark 2: We have not discussed the algorithmic aspects of decompositions of symmetric submodular systems. Whether or not there exists an efficient algorithm for decomposing a symmetric submodular system depends on how the submodular system is represented. See [8] for decompositions of 2-connected graphs and [2] and [3] for decompositions of 2-connected matroids.

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References

[1] R.E. Bixby: Composition and Decomposition of Matroids and Related Topics. Ph.D. Thesis, Cornell University, 1972.
[2] R.E. Bixby and W.H. Cunningham: Matroids, graphs and 3-connectivity. Graph Theory and Related Topics (J.A. Bondy and U.S.R. Murty, eds.,

Academic Press, New York, 1979), pp. 91-103.
[3] W.H. Cunningham: A Combinatorial Decomposition Theory. Ph.D. Thesis, University of Waterloo, 1973; also W.H. Cunningham and J. Edmonds: A combinatorial decomposition theory. Canadian Journal of Mathematics, Vol. 32 (1980), pp. 734-765.
[4] J. Edmonds and R. Giles: A min-max relation for submodular functions on graphs. Annals of Discrete Mathematics, Vol. 1 (1977), pp. 185-204.
[5] S. Fujishige: Polymatroidal dependence structure of a set of random variables. Information and Control, Vol. 39 (1978), pp. 55-72.
[6] S. Fujishige: Principal structures of submodular systems. Discrete Applied Mathematics, Vol. 2 (1980), pp. 77-79.
[7] R.E. Gomory and T.C. Hu: Multi-terminal network flows. J. SIAM, Vol. 9 (1961), pp. 551-570.
[8] J.E. Hopcroft and R.E. Tarjan: Dividing a graph into triconnected components. SIAM Journal on Computing, Vol. 2 (1973), pp. 135-158.
[9] W.T. Tutte: Connectivity in Graphs. University of Toronto Press, Toronto, 1966.
[10] W.T. Tutte: Connectivity in matroids. Canadian Journal of Mathematics, Vol. 18 (1966), pp. 1301-1324.
[11] D.J.A. Welsh: Matroid Theory. Academic Press, London, 1976.