A STATUS ON THE LINEAR ARBORICITY

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Abstract. In a linear forest, each component is a path. The linear arboricity $\Xi(G)$ of a graph G is defined in Harary [8] as the minimum number of linear forests whose union is G. This invariant first arose in a study [10] of information retrieval in file systems. A quite similar covering invariant which is well known to the linear arboricity is the arboricity of a graph, which is defined as the minimum number of forests whose union is G. Nash-Williams [11] determined the arboricity of any graph, however only few results on the linear arboricity are known. We shall present these discoveries and an open problem on this new invariant.

1. Introduction

In a linear forest, each component is a path. The linear arboricity $\Xi(G)$ of a graph G is defined as the minimum number of linear forests whose union is G. All other definitions and terminology employed in this paper can be found in Behzad, Chartrand and Lesniak-Foster [6] or Harary [9]. We now present a few fundamental results for specified families of graphs.

Theorem 1. If T is a tree with maximum degree ΔT, then

(1) $\Xi(T) = \{\Delta T/2\}$.

Proof. The lower bound $\Xi(T) \geqq \{\Delta T/2\}$ is obvious. Since tree T has maximum degree ΔT, its edge chromatic number $\chi'(T)$ is equal to ΔT. Each subgraph induced by subsets of edges with two colors is a linear forest. Thus we obtain the upper bound:

$$\Xi(T) \leqq \{\chi'(T)/2 = \{\Delta T/2\}. \blacksquare$$

The linear arboricity of the complete graph coincides with its path number, which was determined by Stanton, Cowan and James [14].

Theorem 2. (Stanton, Cowan and James) For the complete graph K_p,

$$\Xi(K_p) = \{p/2\}. \blacksquare$$

We also calculate this for complete bipartite graphs in [2], but we omit the proof since it is rather long.

The notation $\delta(m,n)$ is the conventional Kronecker delta.

Theorem 3. For the complete bipartite graph $K_{m,n}$ with $m \geqq n$, the linear arboricity is given by:

(2) $$\Xi(K_{m,n}) = \{(m + \delta(m,n)/2\}. \blacksquare$$

2. The linear arboricity for cubic graphs

We now turn our attention to cubic graphs G and find that the linear arboricity of G is 2. This result was proved by finding an avoidable set for cubic graphs by Akiyama, Exoo and Harary [2], but the following proof which applies Kempe chain arguments is due to Akiyama and Chvátal [1].

Recall that $\chi'(G)$ stands for the edge chromatic number of G.

Theorem 4. The linear arboricity for a cubic graph G is two;

$$\Xi(G) = 2$$

Proof. By Vizing's Theorem [16], we have the inequalities;

$$3 = \Delta G \leqq \chi'(G) \leqq \Delta G + 1 = 4.$$

We first color all the edges of G with 4 distinct colors, say, a, b, c and d, such that no adjacent edges have the same color. We replace the color of the edges as follows:

The edges colored with a or b are replaced with color 1.

The edges colored with c or d are replaced with color 2.

The subgraph G_1 (or G_2) induced by the edges with color 1 (or 2) has degree at most two, i.e., $\Delta G_i \leqq 2$, $i = 1, 2$. If neither G_1 or G_2 contains a cycle, the theorem is true. We now assume that G_1 or G_2 contains a cycle. Our purpose is to show the possibility that we can replace the color of some edges on each monochromatic cycle with the other color so that no monochromatic cycles are left. Let C_1 be a cycle induced by the edges with color 1, and take three successive vertices on C_1, say v_1, v_2, v_3. We denote the edges, outside of C_1, incident to v_i by e_i, $i = 1, 2, 3$, respectively as illustrated in Figure 1.

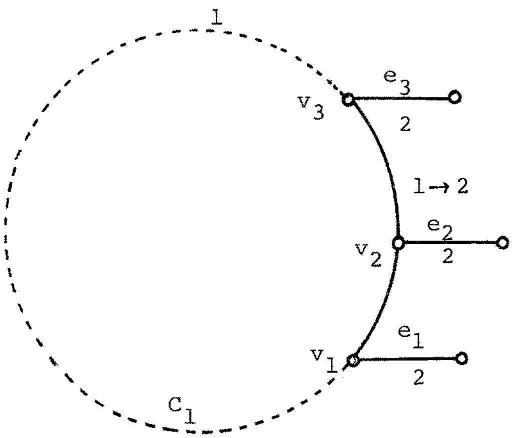

Figure 1. A step in the proof of Theorem 4.

It is obvious that the three edges e_i ($i = 1, 2, 3$) have color 2, since $\Delta G_i \leqq 2$ for $i = 1, 2$.

There are two essentially distinct cases:

Case 1. There is no path joining v_2 and v_3, consisting of edges with color 2. In this case, it is possible to replace the color 1 of the edge $\{v_2, v_3\}$ with color 2. As a consequence of the operation, we avoid the monochromatic cycle C_1 and produce no new monochromatic cycles.

Case 2. There is a path P, joining v_2 and v_3, consisting of edges with color 2. In this case, we show that there are no paths, joining v_2 and v_1, consisting of edges with color 2. Suppose that there exists a path P_1 consisting of edges with color 2 joining v_1 and v_2, see Figure 2.

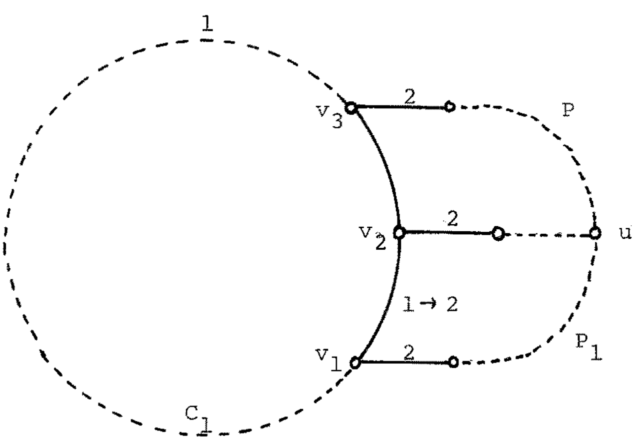

Figure 2. A step in the proof of Theorem 4.

Then there must be a vertex u on <u>both</u> P and P_1, which contradicts the fact that deg $u \leq 2$ in G_2. Thus we can replace the color 1 of the edge $\{v_1, v_2\}$ with color 2 so that no new monochromatic cycles are produced and the monochromatic cycle C_1 is broken. Repeating the operation above until no monochromatic cycle is left, we complete the proof.∎

3. 4-regular graphs

In the determination of the linear arboricity for 4-regular graphs, we found it impossible to apply either proof techniques applied in the proofs [1], [2], that is, to find an avoidable set for 4-regular graphs or to apply Kempe chain arguments. However, it was proved that every 4-regular graph has the linear arboricity 3 in [3] (later, independently by Enomoto [7] and Peroche [12]) by applying the classical results of Petersen [13] on the factorizations of regular graphs of even degree.

<u>Theorem 5.</u> The linear arboricity of every 4-regular graph is 3.

<u>Proof.</u> Let G be a 4-regular graph. Then Petersen showed that G has a 2-factorization. Let $C(1,1), \ldots, C(1,m_1)$ and $C(2,1), \ldots, C(2,m_2)$ be the cycles of two 2-factors of G comprising a 2-factorization.

We shall describe a edge-coloring of G, using the colors red, white and blue, such that each maximal monochromatic subgraph is a linear forest. This is done in three steps.

<u>Step 1.</u> Select one edge $e(1,i)$ from each cycle $C(1,i)$ in the first 2-factor. Color these edges blue and color all the other edges of the first 2-factor red.

<u>Step 2.</u> Select one edge $e(2,i)$ from each $C(2,i)$ of the cycles in the 2nd 2-factor. We will color these edges in Step 3. Now color the remaining $q/2 - m_2$ edges of the second 2-factor white.

Before developing Step 3, we note that the edges already colored form three monochromatic linear forests. It remains to color the edges $e(2,i)$, $i = 1$ to m_2. It is convenient to denote the path formed from $C(k,i)$ upon deletion of edge $e(k,i)$ by $P(k,i)$.

<u>Step 3.</u> We color the edges $e(2,1), e(2,2), \ldots$ blue so long as the blue subgraph remains a linear forest. Suppose $e(2,j)$ is the first edge, if any, which cannot be colored blue because its addition to the blue subgraph forms a cycle, as we now see.

Since the edges $e(1,i)$ are independent, as are the edges $e(2,i)$, coloring $e(2,j)$ blue cannot create a vertex of degree 3 in the blue subgraph. Thus so coloring $e(2,j)$ must complete a blue cycle. This means that two blue edges $e(1,j_1)$ and $e(1,j_2)$ must be adjacent to $e(2,j)$. So we color $e(2,j)$ red, thereby making one red path out of the paths $P(1,j_1)$, $P(1,j_2)$ and the edge $e(2,j)$.

We follow this pattern in coloring the remainder of the edges $e(2,i)$. That is, we color them blue so long as this leaves the blue subgraph a linear forest. And when any $e(2,i)$ cannot be colored blue, we color it red.

We now show that the red subgraph is a linear forest. If coloring any $e(2,k)$ blue creates a blue cycle, then there must be edges $e(1,k_1)$ and $e(1,k_2)$ adjacent to $e(2,k)$. Further, each of $e(1,k_1)$ and $e(1,k_2)$ is adjacent to a blue edge of the form $e(2,i)$ since coloring $e(2,k)$ blue would have completed a blue cycle, and $e(1,k_1)$ and $e(1,k_2)$ are independent. Thus the other endvertices of the paths $P(1,k_1)$ and $P(1,k_2)$ are incident with blue edges $e(2,i)$. This observation means that each path $P(1,i)$ has at most one endvertices incident with a red $e(2,k)$, and of course as observed above, no interior vertex of such a path is incident with any red $e(2,k)$. So coloring edges $e(2,k)$ red as needed leaves the red subgraph a linear forest. ∎

4. The linear arboricity of 5-regular and 6-regular graphs

We heard very recently that B. Peroche [12] proved that the linear arboricity for 5-regular graphs (or 6-regular graphs) is 3 (or 4) respectively. We state these results without proofs, since it is rather long.

Theorem 6. The linear arboricity of 5-regular graph is 3. ∎

Theorem 7. The linear arboricity of 6-regular graph is 4. ∎

5. Bounds on the linear arboricity of a graph

In [3], the bounds of the linear arboricity for a graph G with maximum degree Δ is given as follows:

$$\{\Delta/2\} \leq \Xi(G) \leq \{3\{\Delta/2\}/2\}.$$

However, Peroche [12] obtained the better bounds of the linear arboricity for a graph G with maximum degree Δ by applying Theorem 7 recursively, which is stated as follows.

Theorem 8. If G is a graph with maximum degree Δ, then

$$\{\Delta/2\} \leq \Xi(G) \leq \{2\Delta/3\} \text{ if } \Delta \text{ is even,}$$

$$\{\Delta/2\} \leq \Xi(G) \leq \{(2\Delta + 1)/3\} \text{ if } \Delta \text{ is odd.} \blacksquare$$

6. Unsolved problem

We proved in [4] that the arboricity $T(G) = \{(r + 1)/2\}$ for any r-regular graph G. It was conjectured in [2] that for r-regular graphs $\Xi(G) = T(G) = \{(r + 1)/2\}$ and this equation was proved for $0 \leq r \leq 6$ as seen in the previous sections.

We do not know any graph G which is r-regular for which $\Xi(G) > \{(r + 1)/2\} = T(G)$. Thus the conjecture of equality is still open.

Appendix. The linear arboricity for multigraphs has been studied in [5].

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