HOMOMORPHISMS OF GRAPHS AND THEIR GLOBAL MAPS

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1. Introduction

For directed graphs G_1 and G_2, a homomorphism h of G_1 into G_2 is, roughly speaking, a mapping of the set of arcs of G_1 into the set of arcs of G_2 that preserves the adjacency of arcs. It is naturally extended to a mapping h^* of the set of paths in G_1 into the set of paths in G_2, which is called the extension of h. Also, h naturally induces a mapping h_∞ of the set of bisequences over G_1 into the set of bisequences over G_2, which is called the global map of h. In [9], Hedlund describes the properties of endomorphisms of the shift dynamical system. In [10], using results in [9] and graph-theoretical approaches, the author further investigated the properties of global maps of one-dimensional tessellation automata ("global maps of one-dimensional tessellation automata" and "endomorphisms of the shift dynamical system" are names for the same notion in different fields). Many notions and results in [9] and [10] can be naturally generalized to extensions and global maps of homomorphisms between strongly connected graphs so that we have a new area of graph theory [13][14].

In this paper, we survey a part of the results obtained in [13]and [14] which is mainly concerned with uniformly finite-to-one and onto global maps of homomorphisms between strongly connected graphs.

Most of our results except for Theorem 1 can be considered as generalizations of results described somewhere in [9] and [10]. (We do not say why they can be considered so and which result in [9] or [10] each of them corresponds to. These are found in [13] and [14]. See also [15].) As a generalization of the shift dynamical systems, a class of symbolic flows known as irreducible subshifts of finite type has been studied. Global maps of homomorphisms between strongly connected graphs are closely related to homomorphisms of symbolic flows between irreducible subshifts of finite type, and our results can be directly applied to them. These applications are contained in [13] and [14]. Related results on symbolic flows are found in the works of Coven and Paul [3] [4][5].

2. Basic definitions

A graph (directed graph with labeled points and labeled arcs) G is defined to be a triple $\langle P, A, \zeta \rangle$ where P is a finite set of elements called points, A is a finite set of elements called arcs and ζ is a mapping of A into $P \times P$. If $\zeta(a) = (u, v)$ for $a \in A$ and $u, v \in P$, then u and v are the initial endpoint of a and the terminal endpoint of a, respectively, and are denoted by $i(a)$ and $t(a)$, respectively.

A sequence $x = a_1 \cdots a_p$ $(p \geq 1)$ with $a_i \in A$, $i = 1, \cdots, p$, is a path of length p in G if $t(a_i) = i(a_{i+1})$ for $i = 1, \cdots, p-1$. We call $i(a_1)$ and $t(a_p)$ the initial endpoint of x and the terminal endpoint of x, respectively. Every point u of G is a path of length 0 in G whose initial [terminal] endpoint is u. For any path x in G, we denote by $i(x)$ and $t(x)$ the initial endpoint of x and the terminal endpoint of x, respectively, and if $i(x) = u$ and $t(x) = v$, then we often say that x goes from u to v. The set of all paths in G is denoted by $\Pi(G)$. The set of all paths of length p (≥ 0) in G is denoted by $\Pi^{(p)}(G)$.

Let Z be the set of integers. For a graph $G = \langle P, A, \zeta \rangle$, a mapping $\alpha : Z \to A$ is a bisequence over G if $t(\alpha(i)) = i(\alpha(i+1))$ for all $i \in Z$. Let $\Omega(G)$ denote the set of all bisequences over G. If $\alpha \in \Omega(G)$ and $i \in Z$, then $\alpha(i)$ will often be denoted by α_i.

Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two graphs. A homomorphism h of G_1 into G_2 is a pair (h, ϕ) of a mapping $h : A \to B$ and a mapping $\phi : P \to Q$ such that for any $a \in A$, if $\zeta_1(a) = (u, v)$ with $u, v \in P$, then

$$\zeta_2(h(a)) = (\phi(u), \phi(v)).$$

If G_1 has no isolated point, that is, for each point u of G_1, there exists at least one arc going from or to u, then the homomorphism $h = (h, \phi)$ of G_1 into G_2 is uniquely determined by h. Therefore, when G_1 has no isolated point, we say that h is a homomorphism of G_1 into G_2 and we denote by ϕ_h the unique mapping ϕ such that (h, ϕ) is a homomorphism of G_1 into G_2. In what follows we assume, without loss of generality, that graphs have no isolated point.

A homomorphism $h : A \to B$ of a graph $G_1 = \langle P, A, \zeta_1 \rangle$ into a graph $G_2 = \langle Q, B, \zeta_2 \rangle$ is naturally extended to a mapping $h^* : \Pi(G_1) \to \Pi(G_2)$. That is, we define $h^* : \Pi(G_1) \to \Pi(G_2)$ as follows : For each $x \in \Pi(G_1)$, if the length of x is 0, i.e., x is a point of G_1, then $h^*(x) = \phi_h(x)$, and if $x = a_1 \cdots a_p$ $(p \geq 1)$ with $a_i \in A$, $i = 1, \cdots, p$, then $h^*(x) = h(a_1) \cdots h(a_p)$. Mapping h^* is called the extension of h. Another mapping is naturally induced by h. We define $h_\infty : \Omega(G_1) \to \Omega(G_2)$ as follows : For $\alpha \in \Omega(G_1)$, $h_\infty(\alpha) = \beta$ where $\beta_i = h(\alpha_i)$ for all $i \in Z$. We call h_∞ the global map of the homomorphism h.

A graph $G = \langle P, A, \zeta \rangle$ is strongly connected if for any $u, v \in P$, there exists a path going from u to v. (Note that by our assumption, $A \neq \phi$.)

For a positive integer k, a mapping $f : X \to Y$ is k-to-one if $|f^{-1}(y)| = k$ for all $y \in f(X)$. A mapping $f : X \to Y$ is constant-to-one if there exists a positive integer k such that f is k-to-one, uniformly finite-to-one if there exists a positive integer k such that $|f^{-1}(y)| \leq k$ for all $y \in Y$, and finite-to-one if $|f^{-1}(y)| < \infty$ for all $y \in Y$.

3. Uniformly finite-to-one and onto extensions and uniformly finite-to-one and onto global maps

In this section, we state some properties of uniformly finite-to-one and onto extensions and uniformly finite-to-one and onto global maps of homomorphisms of graphs.

For a graph G, let M(G) be the adjacency matrix of G (i.e., if G has n points $u_1, \cdots, u_n$, then M(G) is the square matrix (m_{ij}) of order n such that m_{ij} is the number of arcs going from u_i to u_j.) Since M(G) is a non-negative matrix, by Perron-Frobenius Theorem, M(G) has the non-negative characteristic value that the moduli of all the other characteristic values do not exceed (cf. Gantmacher [7]). We denote by r(G) that "maximal" characteristic value of M(G), which is often called the spectral radius of G.

Theorem 1†. Let h be a homomorphism of a graph G_1 into a graph G_2. If h^* is uniformly finite-to-one and onto, then $r(G_1) = r(G_2)$ and the characteristic polynomial of $M(G_1)$ is divided by the characteristic polynomial of $M(G_2)$.

Let h be a homomorphism of a graph G_1 into a graph G_2. Two paths x and y in G_1 are indistinguishable by h if $i(x) = i(y)$, $t(x) = t(y)$, and $h^*(x) = h^*(y)$.

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† A stronger result than Theorem 1 is found in [13].

Proposition 1. Let G_1 and G_2 be two strongly connected graphs, and let h be a homomorphism of G_1 into G_2. Then the following statements are equivalent. (1) h^* is uniformly finite-to-one. (2) There exist no two distinct paths in G_1 which are indistinguishable by h. (3) h_∞ is uniformly finite-to-one. (4) h_∞ is finite-to-one.

Proposition 2. Let G_1 and G_2 be two graphs such that for each point u of them, there exist at least one arc going to u and at least one arc going from u. Then for any homomorphism h of G_1 into G_2, h^* is onto if and only if h_∞ is onto.

Theorem 2. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$. Then for any homomorphism h of G_1 into G_2, h^* is uniformly finite-to-one if and only if h^* is onto.

Let G_1 and G_2 be two strongly connected graphs and let h be a homomorphism of G_1 into G_2. Then, by the above results, we have many statements which are equivalent to the statement that $r(G_1) = r(G_2)$ and h^* is onto. The following are some of them. (1) $r(G_1) = r(G_2)$ and there exist no two distinct paths in G_1 which are indistinguishable by h. (2) h^* is onto and there exist no two distinct paths in G_1 which are indistinguishable by h. (3) h^* is uniformly finite-to-one and onto. (4) h_∞ is uniformly finite-to-one and onto.

Example 1. Let $G = \langle P, A, \zeta \rangle$ be a graph. For any non-negative integer p, we define a graph $L^{(p)}(G)$ as follows. $L^{(0)}(G) = G$. For $p \geq 1$, $L^{(p)}(G) = \langle \Pi^{(p)}(G), \Pi^{(p+1)}(G), \zeta^{(p)} \rangle$ where $\zeta^{(p)}(a_1 \cdots a_{p+1}) = (a_1 \cdots a_p, a_2 \cdots a_{p+1})$ for $a_1 \cdots a_{p+1} \in \Pi^{(p+1)}(G)$ with $a_i \in A$, $i = 1, \cdots, p+1$. We call $L^{(p)}(G)$ the <u>path graph of length p</u> of G. ($L^{(1)}(G)$ is usually known as the <u>line digraph</u> of G (cf. Harary [8]) or the <u>adjoint</u> of G(cf. Berge [2])). Clearly, if G is strongly connected, then $L^{(p)}(G)$ is strongly connected for all $p \geq 0$. For any positive integers p and q with $p \geq q$, we define a mapping $h_{G,p,q} : \Pi^{(p)}(G) \to A$ as follows. For any $a_1 \cdots a_p \in \Pi^{(p)}(G)$ with $a_i \in A$, $h(a_1 \cdots a_p) = a_q$. Then clearly $h_{G,p,q}$ is a homomorphism of $L^{(p-1)}(G)$ into G. If G is a graph such that for each point u of it, there exist at least one arc going to u and at least one arc going from u, then $(h_{G,p,q})^*$ is uniformly finite-to-one and onto and $(h_{G,p,q})_\infty$ is one-to-one and onto. Hence, by Theorem 1, we know that under the same condition for G, $r(L^{(p)}(G)) = r(G)$ and $\psi_{G'}(X)$ is divided by $\psi_G(X)$, where $G' = L^{(p)}(G)$ and we denote by $\psi_H(X)$ the characteristic polynomial of M(H) for any graph H. In fact, Adler and Marcus [1] pointed out a stronger result : For any graph G (without any restriction imposed on it), $\psi_{G'}(X) = X^{m-n}\psi_G(X)$ where $G' = L^{(p)}(G)$, $m = |\Pi^{(p)}(G)|$, n is the number of points of G, and we assume that $\psi_\phi(X) = 1$ for the graph ϕ with no point.

4. Compatible sets and Complete sets

Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be graphs, and let h be a homomorphism of G_1 into G_2. Let $U \subset P$ and let $y \in \Pi(G_2)$. Define

$$C_h(U, y) = \{t(x) \mid x \in \Pi(G_1),\ i(x) \in U,\ h^*(x) = y\}$$

and

$$\bar{C}_h(y, U) = \{i(x) \mid x \in \Pi(G_1),\ t(x) \in U,\ h^*(x) = y\}.$$

For $u \in P$ and $y \in \Pi(G_2)$, we denote $C_h(\{u\}, y)$ [$\bar{C}_h(y, \{u\})$] by $C_h(u, y)$ [$\bar{C}_h(y, u)$]. A subset U of P is called a <u>compatible set</u> [a <u>backward-compatible</u> (abbreviated <u>b-compatible</u>) <u>set</u>] <u>for h</u> if $U = C_h(u, y)$ [$U = \bar{C}_h(y, u)$] for some $u \in P$ and $y \in \Pi(G_2)$.

A subset U of P is a <u>complete set</u> [a <u>backward-complete</u> (abbreviated <u>b-complete</u>) <u>set</u>] <u>for h</u>, if there exists $v \in Q$ such that $U \subset \phi_h^{-1}(v)$, and $C_h(U, y) \neq \phi$ [$\bar{C}_h(y, U) \neq \phi$] for all $y \in \Pi(G_2)$ with $i(y) = v$ [$t(y) = v$].

Theorem 3. Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two strongly connected graphs with $r(G_1) = r(G_2)$. Let h be a homomorphism of G_1 into G_2 with h^* onto. Then every maximal compatible [b-compatible] set for h is a minimal complete [b-complete] set for h.

Corollary 1. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Let U be an arbitrary maximal compatible [b-compatible] set for h. Then for any path y in G_2 with $i(y) \in \phi_h(U)$ [$t(y) \in \phi_h(U)$], $C_h(U, y)$ [$\bar{C}_h(y, U)$] is a maximal compatible [b-compatible] set for h.

Proof. Let y be a path in G_2 with $i(y) \in \phi_h(U)$. From Theorem 3, U is a complete set. Hence $C_h(U, y)$ is a complete set. Since U is a compatible set, $C_h(U, y)$ is a compatible set. Let V be a maximal compatible set such that $V \supset C_h(U, y)$. Then from Theorem 3, V is a minimal complete set. Therefore, since $C_h(U, y)$ is a complete set, we have $V = C_h(U, y)$. Therefore $C_h(U, y)$ is a maximal compatible set.

The proof of the second reading is similar.

5. Induced regular homomorphisms

A homomorphism h of a graph G_1 into a graph G_2 is regular [backward-regular (abbreviated b-regular)] if for each point u of G_1 and for each arc b going from [to] $\phi_h(u)$, there exists exactly one arc a going from [to] u with $h(a) = b$.

By virtue of the Corollary 1 in the preceding section, we can introduce the notion of "induced regular [b-regular] homomorphism" which is associated with every homomorphism h between two strongly connected graphs such that h^* is uniformly finite-to-one and onto.

Throughout this section, we assume that $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ are two strongly connected graphs with $r(G_1) = r(G_2)$ and h is a homomorphism of G_1 into G_2 such that h^* is onto.

Denote by $\mathcal{C}_h$ $[\bar{\mathcal{C}}_h]$ the set of all maximal compatible [b-compatible] sets for h. For any $U \subset P$ and $y \in \Pi(G_2)$, we define

$$B_h(U, y) = \{x \in \Pi(G_1) \mid i(x) \in U,\ h^*(x) = y\}$$

and

$$\bar{B}_h(y, U) = \{x \in \Pi(G_1) \mid t(x) \in U,\ h^*(x) = y\}.$$

We define the bundle-graph induced by h as the graph $\mathcal{G}_h = \langle \mathcal{C}_h, \mathcal{E}_h, \zeta_h \rangle$ where $\mathcal{E}_h$ is the set of all pairs of the form $(U, B_h(U, b))$ where $U \in \mathcal{C}_h$ and $b \in B$ with $i(b) \in \phi_h(U)$, and $\zeta_h : \mathcal{E}_h \to \mathcal{C}_h \times \mathcal{C}_h$ is defined as follows :

$$\zeta_h((U, B_h(U, b))) = (U, C_h(U, b))$$

for all $U \in \mathcal{C}_h$ and $b \in B$ with $i(b) \in \phi_h(U)$. By Corollary 1, $C_h(U, b) \in \mathcal{C}_h$ for any $U \in \mathcal{C}_h$ and $b \in B$ with $i(b) \in \phi_h(U)$. Hence ζ_h is well-defined. Furthermore, we define a mapping $\tilde{h} : \mathcal{E}_h \to B$ as follows :

$$\tilde{h}((U, B_h(U, b))) = b$$

for all $U \in \mathcal{C}_h$ and $b \in B$ with $i(b) \in \phi_h(U)$.

Similarly, the backward bundle-graph (abbreviated b-bundle-graph) induced by h is defined to be the graph $\bar{\mathcal{G}}_h = \langle \bar{\mathcal{C}}_h, \bar{\mathcal{E}}_h, \bar{\zeta}_h \rangle$ where $\bar{\mathcal{E}}_h$ is the set of all pairs of the form $(\bar{B}_h(b, U), U)$ where $U \in \bar{\mathcal{C}}_h$ and $b \in B$ with $t(b) \in \phi_h(U)$ and $\bar{\zeta}_h : \bar{\mathcal{E}}_h \to \bar{\mathcal{C}}_h \times \bar{\mathcal{C}}_h$ is defined as follows.

$$\bar{\zeta}_h((\bar{B}_h(b, U), U)) = (\bar{C}_h(b, U), U)$$

for all $U \in \bar{\mathcal{C}}_h$ and $b \in B$ with $t(b) \in \phi_h(U)$. We define a mapping $\bar{\tilde{h}} : \bar{\mathcal{E}}_h \to$

B as follows :

$$\bar{\bar{h}}((\bar{B}_h(b,\ U),\ U)) = b$$

for all $U \in \bar{C}_h$ and $b \in B$ with $t(b) \in \phi_h(U)$.

Theorem 4. G_h [$\bar{G}_h$] is a strongly connected graph, $\tilde{h}$ [$\bar{\tilde{h}}$] is a regular [b-regular] homomorphism of G_h [$\bar{G}_h$] into G_2, and hence $r(G_h) = r(\bar{G}_h) = r(G_2)$ (= $r(G_1)$).

We call $\tilde{h}$ [$\bar{\tilde{h}}$] the induced regular [b-regular] homomorphism of h.

To each path Z of length p (≥ 0) in G_h [$\bar{G}_h$], the subset of paths $B_h(U,\ y)$ [$\bar{B}_h(y,\ U)$] of length p in G_1 where $i(Z) = U$ [$t(Z) = U$] and $y = \tilde{h}(Z)$ [$y = \bar{\tilde{h}}(Z)$], corresponds and is called the bundle of Z. Clearly each subset of paths of length p in G_1 of the form $B_h(U,\ y)$ [$\bar{B}_h(y,\ U)$] where $U \in C_h$ [$U \in \bar{C}_h$] and $y \in \Pi(G_2)$ with $i(y) \in \phi_h(U)$ [$t(y) \in \phi_h(U)$], is the bundle of some path of length p in G_h [$\bar{G}_h$], and is also called a bundle [backward bundle, abbreviated b-bundle] of length p for h.

6. Mergible homomorphisms

In this section, we introduce the notion of "mergible" for homomorphisms between strongly connected graphs with uniformly finite-to-one and onto extensions, and we give an outline of the proof that for each mergible homomorphism h, h_∞ is constant-to-one.

Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Let p be a non-negative integer. We say that h is p bundle-mergible [p b-bundle mergible] if for each bundle [b-bundle] X of length p for h, all paths in X have the same initial [terminal] endpoint. We say that h is mergible if for some non-negative integers p and q, h is both p bundle-mergible and q b-bundle-mergible. We note that h is 0 bundle-mergible [0 b-bundle-mergible] if and only if h is regular [b-regular].

It is easily verified that for a homomorphism h between two strongly connected graphs G_1 and G_2 with $r(G_1) = r(G_2)$, h^* is onto and h is p bundle-mergible [p b-bundle-mergible] if and only if for any two paths x_1 and x_2 of length $\ell \geq p$ in G_1, if $i(x_1) = i(x_2)$ [$t(x_1) = t(x_2)$] and $h^*(x_1) = h^*(x_2)$, then x_1 and x_2 have the same initial [terminal] subpath of length $\ell - p$. (For paths x and y in a graph G, y is an initial subpath [a terminal subpath] of x if there exists a path w in G such that $x = yw$ [$x = wy$]. Here we assume that $i(x)x = xt(x) = x$ for each path x in a graph G.)

Another restatment of the property of being p bundle-mergible [p b-bundle-mergible] is given as the following proposition.

Proposition 3. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$ and let h be a homomorphism of G_1 into G_2 with h^* onto. Let p be a non-negative integer. Then h is p bundle-mergible [p b-bundle-mergible] if and only if for any point u of G_1 and $y \in \Pi^{(p)}(G_2)$ with $i(y) = \phi_h(u)$ [$t(y) = \phi_h(u)$], $C(u, y)$ [$\bar{C}(y, u)$] is either empty or a maximal compatible [b-compatible] set.

Now we shall state six lemmas used in the proof of the main theorem of this section.

Lemma 1. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. If h is mergible, then the induced regular homomorphism $\tilde{h}$ of h is mergible.

Lemma 2. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. If h is p bundle-mergible [p b-bundle-mergible] for a non-negative integer p, then there exists a one-to-one and onto mapping $\rho: \Omega(G_h) \to \Omega(G_1)$ [$\rho : \Omega(\bar{G}_h) \to \Omega(G_1)$] such that $\tilde{h}_\infty = h_\infty \rho$ [$\bar{\tilde{h}}_\infty = h_\infty \rho$].

Let G_1 and G_2 be two strongly connected graphs and let h be a homomorphism of G_1 into G_2. Let n be a non-negative integer. We define a mapping $h^{(n)} : \Pi^{(n+1)}(G_1) \to \Pi^{(n+1)}(G_2)$ by

$$h^{(n)}(x) = h^*(x) \qquad (x \in \Pi^{(n+1)}(G_1)).$$

It is easily seen that $h^{(n)}$ is a homomoprhism of $L^{(n)}(G_1)$ into $L^{(n)}(G_2)$, $(h^{(n)})^*$ is onto if and only if h^* is onto, and if $r(G_1) = r(G_2)$, then $r(L^{(n)}(G_1)) = r(L^{(n)}(G_2))$. (Cf. Example 1.)

Lemma 3. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomoprhism of G_1 into G_2 with h^* onto. Let n and p be non-negative integers. If h is p bundle-mergible [p b-bundle-mergible], then $h^{(n)}$ is a p bundle-mergible [p b-bundle-mergible] homomorphism of $L^{(n)}(G_1)$ into $L^{(n)}(G_2)$.

Lemma 4. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Suppose that h is p bundle-mergible [p b-bundle-mergible] for a non-negative integer p. Then any two distinct maximal compatible [b-compatible] sets for $h^{(p)}$ are disjoint.

A homomorphism h between graphs is <u>biregular</u> if h is both regular and b-regular.

Lemma 5. Let h be a regular homomorphism of G_1 into G_2 where G_1 and G_2 are strongly connected graphs. If every two distinct maximal b-compatible sets for h are disjoint, then $\bar{\bar{h}}$ is biregular.

Lemma 6. Let h be a biregular homomorphism of a graph G_1 with $\Omega(G_1) \neq \phi$ into a strongly connected graph G_2. Let G_1 have p points and G_2 have q points. Then h_∞ is p/q-to-one.

Theorem 5. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. If h is mergible, then h_∞ is constant-to-one.

Proof. Assume that h is mergible.

Let $G_3 = \mathcal{G}_h$ and let $h_1 = \tilde{h}$. Then from Theorem 4 and Lemma 1, G_3 is a strongly connected graph with $r(G_3) = r(G_2)$ and h_1 is a regular and mergible homomorphism of G_3 into G_2. From Lemma 2, there exists a one-to-one and onto mapping $\rho : \Omega(G_3) \to \Omega(G_1)$ such that

$$(h_1)_\infty = h_\infty \rho.$$

Since h_1 is regular and mergible, h_1 is 0 bundle-mergible and there exists $p \geq 0$ such that h_1 is p b-bundle mergible. Let $G_5 = L^{(p)}(G_3)$, let $G_4 = L^{(p)}(G_2)$, and let $h_2 = h_1^{(p)}$. Then, $r(G_5) = r(G_4)$, h_2^* is onto, and from Lemma 3, h_2 is 0 bundle-mergible and p b-bundle-mergible. Moreover, from Lemma 4, any two distinct maximal b-compatible sets for h_2 are disjoint. Let $\rho_1 = (h_{G_3,p+1,\ 1})_\infty$ and let $\rho_2 = (h_{G_2,p+1,\ 1})_\infty$ (cf. Example 1). Then ρ_1 is a one-to-one mapping of $\Omega(G_5)$ onto $\Omega(G_3)$ and ρ_2 is a one-to-one mapping of $\Omega(G_4)$ onto $\Omega(G_2)$, and we have

$$(h_2)_\infty = \rho_2^{-1}(h_1)_\infty \rho_1$$

Let $G_7 = \bar{\mathcal{G}}_{h_2}$ and let $h_3 = \bar{\bar{h}}_2$. Then since h_2 is regular (because h_2 is 0 bundle-mergible) and any two distinct maximal b-compatible sets for h_2 are disjoint, it follows from Lemma 5 that h_3 is biregular. Since

h_2 is p b-bundle-mergible, it follows from Lemma 2 that there exists a one-to-one and onto mapping $\rho' : \Omega(G_7) \to \Omega(G_5)$ such that

$$(h_3)_\infty = (h_2)_\infty \rho'.$$

Since h_3 is biregular, it follows from Lemma 6 that $(h_3)_\infty$ is constant-to-one. Therefore, since ρ, ρ_1, ρ_2, and ρ' are one-to-one and onto mappings, it follows that h_∞ is constant-to-one.

7. Characterizations of constant-to-one and onto global maps of homomorphisms between strongly connected graphs.

Let G be a graph with $\Omega(G) \neq \phi$. Two bisequences $\alpha, \beta \in \Omega(G)$ are point-separated if $i(\alpha_i) \neq i(\beta_i)$ for all $i \in Z$.

Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2. It is easy to see that if for each $\beta \in \Omega(G_2)$, any two distinct members of $h_\infty^{-1}(\beta)$ are point-separated, then h^* is onto and h is mergible. Moreover, using topological arguments similar to those in Hedlund [9] or those in Ferguson [6] (see the proof of Lemma 2.3 of [6]), we can prove that if h_∞ is constant-to-one, then for each $\beta \in \Omega(G_2)$, any two distinct members of $h_\infty^{-1}(\beta)$ are point-separated. Therefore, using these and Theorem 5, we have the following result.

Theorem 6. Let G_1 and G_2 be two strongly connected graphs and let h be a homomorphism of G_1 into G_2. Then the following statements are equivalent.

(1) h_∞ is constant-to-one and onto.

(2) $r(G_1) = r(G_2)$ and for each $\beta \in \Omega(G_2)$, any two distinct members of $h_\infty^{-1}(\beta)$ are point-separated.

(3) $r(G_1) = r(G_2)$, h^* is onto, and h is mergible.

By the fact stated after Theorem 2, statement (3) in the above theorem can be restated as follows :

(3') h^* is onto, there exist no two distinct paths in G_1 which are indistinguishable by h, and h is mergible.

It is easy to see that there exists an algorithm to determine, for a homomorphism h between two strongly connected graphs G_1 and G_2, whether (3') holds. This gives an algorithm to determine whether h_∞ is constant-to-one and onto for an arbitrary homomorphism h between strongly connected graphs.

As an application of Theorem 6, we have the following result which can be considered as a generalization of Theorem 2 of [12].

Theorem 7. Let G_1, G_2 and G_3 be strongly connected graphs with $r(G_1) = r(G_2) = r(G_3)$, and let h_1 be a homomorphism of G_1 into G_2 and h_2 be a homomorphism of G_2 into G_3. Then if $(h_1h_2)_\infty$ is constant-to-one, then each of $(h_1)_\infty$ and $(h_2)_\infty$ is constant-to-one.

8. One-to-one and onto global maps of homomorphisms between strongly connected graphs.

Let h be a regular [b-regular] homomorphism of G_1 into G_2 and let p be a non-negative integer. We say that h is p definite if for any x_1, $x_2 \in \Pi^{(p)}(G_1)$, $h^*(x_1) = h^*(x_2)$ implies $t(x_1) = t(x_2)$ $[i(x_1) = i(x_2)]$. We say that h is definite if h is p definite for some non-negative integer p.

A definite regular homomorphism is considered to be a generalization of the state transition diagram of an automaton having a definite table, which was introduced by Perles, Rabin, and Shamir [16]. We remark that properties of definite tables and a practical decision procedure for definiteness of tables presented in [16], are straightforwardly extended to definite regular [b-regular] homomorphisms of graphs. We can characterize homomorphisms between strongly connected graphs with one-to-one and onto global maps in terms of definiteness of their induced regular and b-regular homomorphisms.

Theorem 8. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Let p be a non-negative integer. Then the induced regular [b-regular] homomorphism $\tilde{h}$ $[\bar{\tilde{h}}]$ of h is p definite if and only if h is p b-bundle-mergible [p bundle-mergible] and $U \cap V \neq \phi$ $(|U \cap V| = 1)$ for any $U \in \mathcal{C}_h$ and $V \in \bar{\mathcal{C}}_h$ with $\phi_h(U) = \phi_h(V)$.

Theorem 9. Let G_1 and G_2 be two strongly connected graphs with $r(G_1) = r(G_2)$, and let h be a homomorphism of G_1 into G_2 with h^* onto. Then h_∞ is one-to-one if and only if h is mergible and $U \cap V = \phi$ $[|U \cap V| = 1]$ for any $U \in \mathcal{C}_h$ and $V \in \bar{\mathcal{C}}_h$ with $\phi_h(U) = \phi_h(V)$.

By the above two theorems, we have the following result.

Theorem 10. Let G_1 and G_2 be two strongly connected graphs with

$r(G_1) = r(G_2)$, and let h be a homomorphism with h^* onto. Then h_∞ is one-to-one if and only if both the induced regular homomorphism and induced b-regular homomorphism of h are definite.

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