Gauss-Manin Operators

Introduction

This chapter is devoted to the case of integrands which contain a transcendental parameter apart from the variable of integration, so that we can consider our problem to be the integration of a function in $\{K(x,y) \mid F(u,x,y) = 0\}$, where K is an algebraic extension of $k(u)$ for some field k and u transcendental over it. We shall use this notation, with u being the independent transcendental, and we shall use the prefix operator D to denote differentiation# with respect to u* , and the suffix ' to denote differentiation with respect to x. This case is often more tractable than the case when there is no such transcendental, for integration with respect to x and differentiation with respect to u commute, so that if $G(u,x,y)$ is integrable, then so is $DG(u,x,y)$, $D^2G(u,x,y)$ and so on.

In this case we can sometimes determine that a divisor is not of finite order, so that we then know that the function is unintegrable. If we cannot do that, then we can

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We are going to differentiate with respect to u even if u does represent a constant transcendental parameter (such as e or π). We can do this because, since the parameter is transcendental, it cannot satisfy any algebraic equations relating to other constants in the integrand, and therefore its precise value does not matter.

This brings out the point made under "Theoretical Limitations" in Chapter 1 that we must know all the dependencies among our constants. For example, since it is not known whether e and π are algebraically independent, we cannot consider an integral involving both of them, since we do not know how to express $\mathrm{d}F/\mathrm{d}e$ in terms of $\partial F/\partial e$ and $\partial f/\partial\pi$. We could, of course, assume that they were independent and produce a result of the form "if e and π are algebraically independent, then F has no elementary integral".

* By this we will mean a total differentiation, taking into account the dependence of y on u caused by the functional relationship $F(u,x,y)=0$ (assuming that u appears effectively in this function). We have that $\mathrm{d}y/\mathrm{d}u = -\partial F/\partial u/\partial F/\partial y$, so that $DG = \partial G/\partial u + (\mathrm{d}y/\mathrm{d}u)\partial G/\partial y$.

determine a value u_0 in k for the parameter u such that the divisor $P(u)$ on $F(u,x,y) = 0$ is of finite order iff the divisor $P(u_0)$ is of finite order on $F(u_0,x,y) = 0$. In other words, we can reduce the problem to one not involving u. Recursively, we can reduce the problem to one with no transcendental parameters. We shall not discuss in this chapter how to solve such problems, rather they are left to the next two chapters. If $P(u_0)$ is of infinite order, then we know that $P(u)$ is too, and the problem is solved. Otherwise, let n be the order of $P(u_0)$ and consider $nP(u)$, $2nP(u)$ in turn until we discover what the order of $P(u)$ is, and we know that this process must terminate, though we have no idea when. In the case of curves of genus 1, we can use Cayley's determinant test in order to decrease the amount of work, as explained towards the end of the last chapter.

This work is based on two papers (1958, 1963) of the Russian mathematician Ju.I. Manin, and the reader is referred to them for the full generality of the exposition and most of the proofs, which we will just state. These papers do not make easy reading however, and both the English and Russian versions contain many misprints. Furthermore Manin's exposition is complicated by his desire to work with n parameters (and hence n differentiations) rather than just 1. We may, of course, have several parameters $u_1,\ldots,u_n$, but we shall eliminate then one-by-one, using the methods of this chapter recursively, rather than try to consider them all simultaneously. I have been unable to conduct any experiments with curves with two parameters, so I have no firm ideas as to which or the two approaches (parallel or recursive) is better, but intuitively it seems that we want to make the problem as small as possible, by eliminating parameters, as soon as possible, rather than work with then all until we reach the end.

Example

Before giving the general theory, we will consider a worked example of this theory, taken essentially from Manin (1963, pp. 190-192). Consider the "general" elliptic curve $y^2 = x(x-1)(x-u)$ and take the ground field K to be a finite extension of $k(u)$ for some field k. Let $w = y^{-1}dx$ be a differential of the first kind (and since the curve is elliptic, all others are constant (in the sense of not depending on x or y) multiples of this form). Then if C is any closed curve on the surface $\{K(x,y) \mid F(u,x,y) = 0\}$, then $e = \int_C w$ is infinitely many valued (and an analytic function of u), and in fact the space of these is generated by an arbitrary pair e_1, e_2 of such periods*. These functions are the solutions of the Gauss linear differential equation $4u(u-1)D^2e - 4(1-2u)De + e = 0$.

On the other hand, functions of the form $\int_O^P w$, where O is the point at infinity, are extremely important for the investigation of the geometry of the curve. Such functions are only defined up to some period, because we could choose any path to get from O to P, and satisfy the equation $I(P) + I(Q) = I(P + Q)$ modulo such a period, where $P+Q$ is the sum according to the group law (see the previous chapter). We can remove this ambiguity by acting on both sides of the equation with the Gauss operator, which destroys such periods, thus obtaining the function $J(P) = (4u(u-1)D^2 - 4u(1-2u)D + 1)I(P)$, which can be identified with an element of $\{K(x,y) \mid F(u,x,y) = 0\}$. Because of the linearity of the Gauss operator and the relationship between $I(P)$, $I(Q)$ and $I(P+Q)$, J defines a homomorphism from the group of points on the curve to the additive group of this function field.

This can be made explicit in the following way: first we observe that

$$[4u(u-1)D^2 - 4(1-2u)D + 1]y^{-1}\mathrm{d}x = -2\mathrm{d}(y(x-u)^{-2}).$$

If we then integrate round a closed curve, the right hand side is the integral of an exact differential, so vanishes, and on the left hand side the integration and the differentiation with respect to u commute, so that we have

$$[4u(u-1)D^2 - 4(1-2u)D + 1]\int_c y^{-1}\mathrm{d}x = [4u(u-1)D^2 - 4(1-2u)D + 1]e(t)$$

thus giving the Gauss differential equation. Then

$$J(P) = [4u(u-1)D^2 - 4(1-2u)D + 1]\int_O^P \frac{\mathrm{d}x}{y},$$

* A *period* is defined to be the integral of a differential of the first kind round a closed curve on the Riemann surface of the curve.

but this time we cannot commute the integration and D, because P depends on u. Write P as the point $(X(u),Y(u))$. Then if G is an algebraic function of x and u, rational in x and y, we can state that

$$D\int_O^P G(x,u)\mathrm{d}x = (DX(u))G(X(u),u) + \int_O^P DG(x,u).$$

We can then differentiate again to get that

$$D^2\int_O^P G(x,u)\mathrm{d}x =$$

$$(D^2X(u))G(X,u) + (DX(u))[(DG(x(u),u)) + \partial G(X(u),u)/\partial u] + \int_O^P D^2G(x,u).$$

In the special case under consideration, this gives us that#

$$J(P) = -2Y(X-u)^{-2} + D\frac{2u(u-1)(DX)}{y} + 2u(u-1)D(XDY).$$

Picard-Fuchs Equations

Let C be a curve of genus g defined by $F(x,y)=0$, where the coefficients of F lie in K, an algebraic extension of $k(u)$, and involve u effectively. Then we can find $2g$ closed curves on $C(u)$ such that any closed curve on $C(u)$ is continuously deformable to a sum of these $2g$ curves*. Let these curves be $c_1,\ldots,c_{2g}$ and let $w_1,\ldots,w_g$ be g linearly independent

Observe the difference between the two halves of the second term in this summation. The first contains a total derivative of $G(X(u),u)$ with respect to u, while the other contains only a partial derivative, and is in fact the result of substituting $X(u)$ for x in $DG(x,u)$. The difference between these two is too subtle for many algebra systems, and this adds to the difficulty of implementing this work in a straight-forward fashion (see item 2 in Appendix 1).

* The precise wording is that we can find $2g$ curves $c_1,\ldots,c_{2g}$ which form a basis for the 1-dimensional homology of the Riemann surface.

differentials of the first kind on $C(u)$. Define $L_{a,b}$ to be $\int_{c_b} w_a$, so that this is a period of the curve.

Lemma 1 For any a, the periods $L_{a,b}$ satisfy a linear differential equation of order $2g$ (or possibly less in degenerate cases):

$$p_{a,2g}(u)D^{2g}L_{a,b} + \dots + p_{a,1}DL_{a,b} + p_{a,0}L_{a,b} = 0$$

(i.e. the equation does not depend on b).

Proof: If w is a differential with no residues (i.e. of the second kind), then Dw is a differential with no residues (Manin, 1963, Corollary 2, p. 198). Then $w_a, Dw_a, \dots, D^{2g}w_a$ are $2g+1$ differentials of the second kind, so there is a linear relationship between them:

$$p_{a,2g}D^{2g}w_a + . + \mathrm{p}_{a,1}Dw_a + p_{a,0}w_a = \mathrm{d}f$$

for some function f. Then we merely integrate round the periods, and the integral of $\mathrm{d}f$ round any closed curve is 0.

After Manin, we will call such equations *Picard-Fuchs equations.* The differential operator $L = \Sigma\, p_{a,i}D^i$ is termed a *Gauss-Manin operator*. We can endow the space of such equations of the form $L\int_c w = o$ with the structure of a module by allowing the more general equation* $\Sigma\, L_i \int_c w_i = 0$. Now if D is the divisor $\Sigma\, n_i P_i$, where the n_i are integers, define $\int^D f(x)\mathrm{d}x = \Sigma\, n_i \int_O^{P_i} f(x)\mathrm{d}x$, where the lower limit of integration is some fixed point O. This is clearly independent of the precise choice of O for divisors of degree 0.

If J is such a Picard-Fuchs equation, then we have that $\Sigma\, L_i w_i$ is an exact differential, say $\mathrm{d}z$. Then, if P and Q are any two points on an Abelian variety A, we can define#

* More formally (Manin, 1963, p. 199) a *Picard-Fuchs equation* is any relation of the form J: $\Sigma\, L_i \int_c w_i = 0$ where the w_i are differentials of the first kind and L_i are linear differential operators. Such equations can be defined on any Abelian variety A, but we will only be concerned with the case $A = \mathrm{Jac}(C)$.

We are using J in two senses, both for the Picard-Fuchs equation and for the operator which takes (P,Q) onto $z(Q) - z(P)$ (the so-called *Picard-Fuchs* operator), but this should cause no confusion.

$J(P,Q)$ to be $z(Q)-z(P)$, and this is well-defined (Manin, 1963, p. 202, Theorem 1) and is in fact $\Sigma L_i \int_Q^P w_i$, and it does not matter along which contour we integrate since the integral of an exact differential round any closed curve (i.e. the difference of two contours) is 0.

Furthermore $J(P,Q) = J(P,R) + J(R,Q)$, since we can choose a contour from P to Q which passes through R, and then split the integral at R (This is part (b) of Manin, 1963, p. 202, Theorem 1).

Picard-Fuchs operators as homomorphisms

We now make a fundamental remark about the relationship of the Picard-Fuchs operator J to addition of points on our Abelian variety:

$$J(P + R,Q + R) = J(P,Q).$$

The proof of this follows from the fact that differentials of the first kind are invariant under translations of our variety: this is made formal by Manin (1963, p.207, Lemma 12).

We now define $J(P)$ to be $J(O,P)$ where O is the zero of the group law on our Abelian variety. This brings us in line with the notation used in our example earlier.

Lemma 2 J is a homomorphism from the points of an Abelian variety (as an additive group) into the ground field.

Proof: In order to prove this, it is sufficient to prove that $J(P + Q) = J(P) + J(Q)$, since $J(O) = 0$ and O is the zero of the additive group on the Abelian variety.

$$\begin{aligned} J(P + Q) &= J(0,P + Q) \\ &= J(O,P) + J(P,P + Q) = J(O,P) + J(O,Q) \text{ by the remark above} \\ &= J(P) + J(Q). \end{aligned}$$

Corollary 3 For any Picard-Fuchs equation J, the set of points of A of finite order lie in the kernel of the corresponding Picard-Fuchs operator J.

Proof: Because the ground field is torsion free and homomorphisms map torsion parts into torsion parts.

This means that the points of A of finite order lie in the intersection of the kernels of all the Picard-Fuchs operators. It would be wonderful if the converse were true, but that cannot be the case. To see this, let C be a curve over $\mathbf{Q}$ with a point P of infinite order. Now consider C and P over $\mathbf{Q}(u)$. Then $P(u)$ is certainly still of infinite order, but we cannot say that it does not lie in the kernel of J, for J must take it and O to the same value, for both depend equally (not at all) on u. However, the next best thing is true, i.e. that this is the only way in which things can go wrong.

In order to explain this more precisely, we need a little more notation. This is taken from Lang(1959, p. 213), though we gain some simplicity by only considering the characteristic 0 case. Let K be any overfield of $\bar{k}$ for the purpose of this paragraph (though the application will be to K an algebraic extension of $\bar{k}(u)$). Let A be a variety over K. A pair (A',τ) is said to be a *K/k-trace* of A if A' is an abelian variety over k and τ is a homomorphism from A' to A and has finite kernel, such that, for any abelian variety B defined over k, and homomorphism $\beta:B -> A$ defined over K, there is a homomorphism $\beta':B -> A'$ defined over k such that $\tau\beta' = \beta$. This may appear a somewhat abstract definition, as indeed it is, but it defines the trace as that portion of A which is essentially independent of K/k (u in our case). The presence of τ is, in some sense, technical — the problem is that A' may have a few more points defined in it than we would like, which correspond to points defined over algebraic extensions of K.

Theorem 4 * If P is a point of A, and $J(P)=0$ for all Picard-Fuchs operators J on A, then there is an integer n such that nP (in the sense of the addition on A), lies on the image of the $k(u)/k$-trace of A. Furthermore if such an n exists, then $J(P)=0$ for all Picard-Fuchs operators J.

What this theorem means for our purposes is that, if $J(P) = 0$ for all J, then P is essentially independent of u. Now, there are infinitely many such Picard-Fuchs operators

* (Manin, 1963, p. 208, Theorem 2) I would like to thank Professor M.F. Singer for his invaluable assistance with this theorem.

J, but we need only consider a basis for the space of Picard-Fuchs operators, rather than all Picard-Fuchs operators. The dimension of such a basis is equal to the genus of the algebraic curve C in the case $A = \mathrm{Jac}(C)$.

Divisors of Finite Order

Now let D be a divisor on an algebraic curve C. D corresponds to a point D' on Jac(C), and D is rationally equivalent to 0 iff D' is of finite order on Jac(C). If D' depends essentially on u, then it is certainly of infinite order, and if it does not, then the problem has been reduced to a simpler one. The way the reduction is performed is by substituting a value in k for u, so that the problem is reduced to one over k, rather than $k(u)$. Not every value of u will do - consider substituting $u = 0$ in $y^2 = ux^3-1$. The question of which values of u will work is called "Good Reduction", and is discussed in greater detail in chapter 8 (see the section "Criteria for Good Reduction, especially Theorem 8), where it plays a much more important part in the argument. It suffices here to say that there are only finitely many values of u which are not of good reduction (i.e which do not work) and there is a simple *a priori* test* to determine whether or not a value of u is of good reduction.

FIND_ORDER_MANIN

Input:

F(X,Y): the equation of an algebraic curve

> There is no actual need for this to be in primitive, rather than multivariate, form. The only explicit use made of F is in the calculation of the differentials of the first kind.

* The value u_0 in k of u is of good reduction if $F(u_0,x,y)$ is absolutely irreducible over k, and has the same genus as $F(u,x,y)$.

D: a divisor on the curve, written as $\sum_{j=1}^{M} n_j P_j$.

We will sometimes write P_j as (X_j, Y_j).

U: a parameter over which the curve is defined.

Output:

INFINITE or an integer N, depending on whether D is of infinite or finite order. The integer N signifies that the image was of order N, so that we should consider ND, $2ND$, ... in our search to find the order of D.

[1] DIFF__1 :=

A linearly independent basis for the differentials of first kind on curve F(X,Y)=0. Let G be the length of DIFF__1, viz the genus of the curve $F(X,Y)=0$.

This can be done by a simple variant of Coates' Algorithm: see Chapter 3 for details.

[2] For each W in DIFF__1 do:

[2.1] Let $A_{2G-1},\ldots,A_0$ be indeterminates, and let A_{2G} be $1-\sum_{i=0}^{2G-1} A_i$, so that the sum of all the A_i is 1 (because Picard-Fuchs operators are indeterminate up to constant multiples).

[2.2] Solve the equation $\sum A_i \mathrm{d}^i W/\mathrm{d}U^i = \mathrm{d}R(X,Y)/\mathrm{d}X$ for the unknowns A_i and the rational function $R(X,Y)$.

The denominator of $R(X,Y)$ can be chosen to be the least common multiple of the denominators on the left hand side, and after clearing denominators the equation breaks up into a series of linear equations in the A_i and the coefficients of $R(X,Y)$, all of which can depend on U, but not on X or Y. Furthermore the degree of R is at most one more than the highest degree on the left hand side.

Our Gauss-Manin operator J corresponding to the differential of the first kind W is now $\Sigma A_i d^i/dU^i$ and $J\ W = dR(X,Y)/dX$.

We now have to compute $\frac{d^i}{dU^i}\int_O^P W dX$ for $1 \leq i \leq 2G$. This can be written as $\int_O^P d^i W/dU^i + B_i(X,U)$, where B is the contribution of all the other terms from the repeated differentiation.

[2.3] SUM := 0.

This will be used to accumulate $J(D)$ in.

[2.4] For j = 1 ... M do:

[2.4.1] B_0 := 0.

[2.4.2] For i = 1 ... 2G do:
$B_i = \frac{dB_{i-1}}{dU} + \frac{dX_j}{dU}\frac{d^{i-1}W}{dU^{i-1}})\ |_P$ (where $|P$ means evaluation at the X,Y values of the point P_j).

The first term in this expression comes from the differentiation of the B term for the previous i, and the second term comes from the fact that $\frac{d}{dU}\int_O^P f(x)dX = \int_O^P \frac{df(x)}{dU}dX + \frac{dP}{dU}f(p)$ (since $dO/dU = 0$).

$J(P)$ is now $\Sigma A_i \frac{d^i}{dU^i}\int_O^P W dX$, which can be re-ordered as

$$\sum B_i(P,U)A_i + \int_O^P \sum A_i \frac{d^i W}{dU^i} dx.$$

[2.4.3] SUM:=SUM + $n_j \Sigma B_i A_i + R(P_j)$.

The above expression is the previous formula multiplied by n_j and is the contribution of $n_j P_j$ to $J(D)$.

[2.5] If SUM is non-zero, then return INFINITE.

[3] For U0 = 0,1,2,...

If GOOD_REDUCTION(U0,F,K)

Then do:

This is not necessarily the best way of performing this choice of the value for U. For example, suppose the equation F depends on U and $\sqrt{U^2+1}$, and that 0 is not of good reduction. Then choosing U0 = 1 will give us a curve defined over $\boldsymbol{Q}(\sqrt{2})$, whereas choosing U0 = 3 will give us one defined over $\boldsymbol{Q}$. However appealing such intuitive choices may be, it is hard to devise a program for finding 'good' values of U in that sense.

[3.1] D:= Substitute U0 for U in D.

[3.2] F:= Substitute U0 for U in F.

[3.3] Return FIND_ORDER(F,D).

FIND_ORDER is implemented as one of FIND_ORDER_MANIN (in the event that, even after substituting U0 for U, there is still a transcendental parameter), FINITE_ORDER_ELLIPTIC (see Chapter 7) or BOUND_TORSION (see Chapter 8).

Implementation

The implementation of this algorithm is technically fairly difficult, though few mathematical problems are raised by it. As was mentioned earlier when discussing the example, one major source of difficulty is the differentiation with respect to U and X occurring in the same expression, and the need to distinguish between partial derivatives with respect to U (as in step 2.2) and total derivatives (i.e. with the point P_j substituted for X and Y, which may well depend on U). In order to do this I have found it easier to write my own special-purpose top-level differentiation routines, rather than try to use REDUCE's and manipulate REDUCE's data structure for derivatives, which has no provision for distinguishing between total and partial derivatives. I do use a modified version of several of REDUCE's differentiation routines for simpler parts of the task: details of the modifications are given in Appendix 1.

Another major source of difficulty is the need for an efficient implementation, with as few calculations as possible being repeated. Since all the expressions involved are multivariate (with both the *Gauss-Manin parameter* and the variable of integration involved, "simple" operations such as the computation of greatest common divisors can be very time-consuming. This leads to the requirement for a variety of "look-aside" tables containing, for example, $\mathrm{d}^i W/\mathrm{d}U^i$ or the (partial) derivatives $\partial^i Y/\partial U^i$, which must be created and purged as appropriate. There are several other efficiency points: for example the linear equations in step 2.2 can be partially sparse#, and it is necessary to take advantage of this in order to obtain an implementation with reasonable efficiency. Despite these and other tricks, this algorithm can still be very expensive because of the size of expressions that have to be manipulated, especially in step 2.2. When attempting to discover whether the divisor consisting of the point $(d(d-1), d(d-1)^3)$ with multiplicity 1 and the point at infinity with multiplicity -1 on Tate's curve (Appendix 2, Example 4) was of finite order or not (it is in fact of order 7) a carefully coded draft implementation of this algorithm consumed approximately 15 minutes CPU on the IBM 370/168 at IBM Thomas J. Watson Research Centre at Yorktown Heights*.

Although the Gauss-Manin operator is generally of order $2G$ (where G is the genus of the curve), there are many special cases in which it is degenerate and has lower degree. For example, while the Gauss-Manin operator of a general elliptic curve has degree 2, it has degree 1 in the following special cases (Manin, 1958, p. 77): $Y^2 = X^3 + aX$ $Y^2 = X^3 + a$ $Y^2 = X^3 + a^2X + ba^3$ where b is any constant (i.e. not depending on X, Y or a) not equal to $27/4$ (or else the curve ceases to be elliptic). Furthermore, in these cases

More information on sparsity, and an example of how it can make equations much simpler to solve is given in Appendix 2 Example 4. There is also a general discussion of linear equations in Chapter 9.

* Unfortunately a more precise figure is not available because of a bug in the CMS-Batch simulation of OS installed at Yorktown Heights, which meant that the LISP timing features were inoperative and the CPU time had to be calculated by multiplying elapsed time by the "service ratio" (i.e. the ratio of CPU to elapsed time) produced by the operating system. However the figure is probably accurate to within 25%. The 370/168 is approximately 5% faster than the 370/165 installed at Cambridge, and on which the remainder of the timings quoted in this monograph were measured.

the Picard-Fuchs operators are extremely simple (being $\frac{XDa-2aDX}{2aY}$, $\frac{XDa-3aDX}{3aY}$, and $\frac{XDa-aDX}{aY}$ respectively). It is therefore important, in the interests of efficiency, to recognise these degenerate cases as early as possible, and this is not easy.

A further special case is the one where X_j is independent of U. In this case all the B_i are 0 and the computation simplifies considerably. This case frequently arises in practice, and hence has to be tested for.

Special Values of Parameters

In this section we shall suppose that our integrand $f(x,u)\,\mathrm{d}x$ depends algebraically on u. This is not really a restriction, since if it depends transcendentally on u, we can replace a transcendental function of u by a new transcendental parameter u' without altering the problem, since we know that, if $f(x,u)$ is integrable, then its integral is defined over the *algebraic closure* of the original ground field, i.e. no new transcendentals can be introduced.

Proposition 5 If $f(x,u)$ depends algebraically on u, then the residues of $f(x,u)\,\mathrm{d}x$ are algebraic functions of u (because they lie in the algebraic closure of the constant field).

If the algorithm FIND__ORDER__MANIN returns the answer INFINITE because one of the values of SUM was non-zero, one can conclude that the integral is not expressible in elementary terms. It might seem reasonable to consider those special values of u for which SUM happens to be zero, and ask whether the integral is elementary in this special case. This leads on the the more general question:

"*For what values of u is $\int f(x,u)\mathrm{d}x$ an elementary function?*".

Unfortunately, those values of u which make SUM zero are not the only values which can make the integral elementary, for there are several other ways in which the problem can reduce when a special value is substituted for u.

Let us now consider the various ways in which the substitution of a special value for the parameter u can alter the nature of the integration problem:

1) The curve can change genus. This can only happen for finitely many values of u, and we can decide what these values are by considering the canonical divisor, and the possibilities for it to degenerate.
2) The places at which residues of the integrand occur can change. There are only finitely many of these, and they can be detected by looking for all the values of u for which factors of the numerator and of the denominator coincide, or factors of the denominator coincide with each other (or for which the numerator or denominator change degree, to allow for the case of the factor x – infinity).
3) The dimension of the space of residues can collapse. This is an exceptionally tricky case, and we will postpone a full discussion of it.
4) A divisor may be a torsion divisor for a particular value of u, even though not generally. This is where we started on this discussion, and these cases (of which there are only finitely many) can be detected by looking at all the roots (in u) of the functions SUM in the algorithm FIND_ORDER_MANIN.
5) The algebraic part may be integrable for a particular value of u, though not in general. These cases can be detected by looking at the equations generated in the algorithm FIND_ALGEBRAIC_PART to see when the contradicting equation, which proves that the function is unintegrable, becomes degenerate.

Thus we have shown that the number of "exceptional" values of the parameter u is finite (and these values are effectively computable) for cases 1,2,4 and 5.

Case 3 is substantially more difficult. As an example, we can have infinitely many values of u for which the $\boldsymbol{Z}$-module of residues decreases in dimension:

Consider an integrand whose 4 residues are 1, -1, u, $-u$; for example

$$\frac{1}{x\sqrt{x^2+1}} + \frac{1}{x\sqrt{x^2+u^2}}\mathrm{d}x.$$

Then for every rational value of u the residue space becomes 1-dimensional, and hence potentially has to be considered as a special case.

Lemma 6 Let the $\boldsymbol{Z}$-module of residues $(r_1,...,r_k)$ of $f(x,u)\ \mathrm{d}x$ have dimension k. Suppose that there are values $u_1,...,u_k$ of u such that $f(x,u_i)\mathrm{d}x$ has an elementary logarithmic part (without lying in cases 1,2 or 4 above) for $1 \leq i \leq k$ and such that the set of vectors $\{(r_i(u_a)\ 1 \leq i \leq k)\ 1 \leq a \leq k\}$ is of dimension k. Then $f(x,u)$ dx has an elementary logarithmic part.

Proof: There is an integer n such that the vector $(n,0,\ldots,0)$ can be expressed as a linear combination, with integer coefficients, of the residue vectors $(r_i(u_a))$. Then the divisor d_1 corresponding to the residue r_1 must be a torsion divisor, because it has been expressed as the n-th root of a sum of torsion divisors. Similarly all the other divisors are torsion divisors, and hence the logarithmic part of the general integral must exist.

Theorem 7 If $f(x,u)$ dx is not elementarily integrable, then there are only finitely many values u_i of u for which $f(x,u_i)$dx is elementarily integrable.

Proof: The only problem is case 3 above, for we have shown (and our arguments can easily be made completely rigorous) that there are only finitely many values which correspond to cases 1,2,4,5. So suppose that there are infinitely many values corresponding to case 3, but not to case 1,2 or 4. Then by Lemma 6 above, the $\boldsymbol{Z}$-module spanned by the residue vectors $(r_i(u_a))$ is of dimension less than k, and so can be embedded in a space of dimension $k-1$. Then there is a linear relationship between $r_1(u),\ldots,r_k(u)$ which is not true in general, but which is true for infinitely many particular values of u. but since the $r_1(u)$ are algebraic in u, by Proposition 5 above, this means that we have an algebraic expression which is not identically zero, but which has infinitely many roots, and this establishes the required contradiction.

Note that this Theorem is not completely constructive, in that we have shown no way of finding out what the finitely many values in case 3 are. That this problem is not completely trivial can be shown by the following example:

> Let E be an elliptic curve over $\boldsymbol{Q}$ with a point of infinite order and a point of finite order (i.e. the infinite portion of the Mordell-Weil group is to have rank at least 1 and the torsion part is to be non-trivial, and such curves exist, as is shown by the tables in Birch & Swinnerton-Dyer (1963) or Swinnerton-Dyer (1974)) known as P and Q respectively. Let D be a divisor linearly equivalent to $3P$, and D' be a divisor linearly equivalent to $5P$-Q. Let $f(x)$ be a function on the elliptic with divisor of

residues D, and $f'(x)$ one with divisor of residues D'. Then consider the function $f(x)+uf'(x)$, whose residue space is 2-dimensional for irrational u, and 1-dimensional for rational u. When u is irrational, the logarithmic part cannot be found, while if u is rational, say m/n, the divisor is n(3P) + m(5P−Q), which is a torsion divisor only if the coefficient of P is zero, viz $u = -3/5$. This example demonstrates the necessity for the restriction that $f(x,u)$ should depend algebraically on u, because if we had written $f(x)$ + sin u $f'(x)$ then there would have been infinitely many solutions, viz. all the roots of $\sin^{-1} -3/5$.