IV. THE CONFLUENCE PROPERTY AND THE CHURCH-ROSSER PROPERTY

1. Motivation

The strategy of interpreting expressions by reducing them to normal forms is of questionable value if some expression has more than one normal form. An even stronger condition than uniqueness of normal forms might be wanted because of the following informal argument.

Think of reduction as a means of revealing information which is hidden in an expression. Recall (Lemma 2, p.12) that the reflexive, symmetric, transitive closure of a reduction relation $\rightarrow$ is a relation $\equiv$ such that $A \equiv B$ precisely if $A=B$ is provable. Now if $A=B$ is provable, and the only way to reveal information about A and B is to reduce them, then $A=B$ should be provable by reducing A and B; that is, there should be a C such that A and B both reduce to C.

The property expressed informally above is the Church-Rosser property, stated for the lambda calculus by Church [Ch41] (Th. 7 XXVI, p.24), and Curry and Feys [CF58] (Property (χ), p. 109). Following Rosen, we will use an equivalent property, the confluence property.

Definition 24 [CF 58] (Property (B), p. 110),
[Ro 73] (Def. 3.2, p. 163).

A SRS $<\Sigma, F, \rightarrow, A>$ has the confluence property iff,
for all $A,B,C \in F$ such that $A \rightarrow^* B$ and $A \rightarrow^* C$,
There is a $D \in F$ such that $B \rightarrow^* D$ and $C \rightarrow^* D$.
Figure 12 sketches the confluence property.

$$\begin{tikzcd} & A \arrow[dl, "*"'] \arrow[dr, "*"] & \\ B \arrow[dr, "*"'] & & C \arrow[dl, "*"] \\ & D & \end{tikzcd}$$

Figure 12

The confluence property of Definition 24 is equivalent to the Church-Rosser property as long as the relation involved is transitive. Because of this equivalence, Rosen identifies the confluence property as the Church-Rosser property. The following three theorems are well known.

Theorem 3

Consider a SRS $<\Sigma, F, \rightarrow, A>$ and let $\equiv$ be the reflexive, symmetric, transitive closure of $\rightarrow$. The confluence property is equivalent to the following (Church-Rosser) property:

$B \equiv C$ iff there is a $D \in F$ with $B \to^* D$ and $C \to^* D$.
That is, $\equiv \; = \; \to^* \leftarrow^*$.

Proof See Curry and Feys [CF58], Theorem 3, p. 110. □

Theorem 4

Every SRS with the confluence property has unique normal forms.

Proof Let $A \to^* B$ and $A \to^* C$, where B and C are in normal form. By the confluence property, there is a D with $B \to^* D$ and $C \to^* D$. But, since B and C are in normal form, B=D and C=D, so B=C. □

Theorem 5

Let $\langle \Sigma, F, \to, A \rangle$ be a SRS with the confluence property.
Let F be in normal form.
For all $E \in F$, the following are equivalent:

(1) $A \models E=F$
(2) $A \vdash E=F$
(3) $E \to^* F$
(4) there is a finite reduction sequence
$E=E_0 \xrightarrow[M_1]{} E_1 \xrightarrow[M_2]{} \cdots \xrightarrow[M_n]{} E_n = F$.

Proof Since F is in normal form,
(*) $E \to^* \leftarrow^* F$ iff $E \to^* F$.
(1) <=> (2) by Theorem 1.
(2) <=> (3) by (*) and Theorem 3 and Lemma 2.
(3) <=> (4) by Definition 21, p. □

Theorems 4 and 5 provide the justification for generating reduction sequences in order to solve the computing problem for equational logic.

Rosen gives a natural sufficient condition called "closure" for a SRS to have the confluence property. In order to make a useful generalization and to abstract Lemma 12, which will be applied repeatedly in the next chapter, we repeat the proof in detail.

2. Closure

In applications of SRS theory, the axioms or reduction rules A of a SRS will often be presented by some schematic descriptions, and we will want to establish the confluence property and other properties by examining the characteristics of A as manifested in its description. The closure property is a useful sufficient condition for the confluence property because it may be easily verified in many such presentations.

If $\langle A=B\rangle \in \mathcal{A}$, and similar reductions of A and B produce A' and B', then it is reasonable to hope that $\langle A'=B'\rangle \in \mathcal{A}$. The use of a residual map allows a formal definition of "similar" reductions: $A \xrightarrow[x]{} A'$ and $B \xrightarrow[r\langle A=B\rangle x]{} B'$.

Figure 13 sketches the informal property above.

$$\begin{tikzcd} & A \arrow[dl, "x"'] \arrow[dr, "()"] & \\ A' \arrow[dr, "()"'] & & B \arrow[dl, "r\langle A=B\rangle x"] \\ & B' & \end{tikzcd}$$

Figure 13

Example 13

In $\mathcal{E}$ of Example 6, (p. 18) with r of Example 11, (p. 23) the above property holds, see Figure 14 for instance.

$$\begin{tikzcd} & F(p(s(0)),0) \arrow[dl, "(1)"'] \arrow[dr, "()"] & \\ F(0,0) \arrow[dr, "()"'] & & Cond(p(s(0)),0,F(p(p(s(0)),F(p(s(0)),0)) \arrow[dl, "r\langle A=B\rangle(1)"] \\ & Cond(0,0,F(p(0),F(0,0)) & \end{tikzcd}$$

Figure 14

See Figure 15 on the next page for a picture of the trees above, with (1) and its residuals marked by *.

The intuitive similarity between the above pictures and the diagram of the confluence property gives hope that we have found a sufficient condition for the confluence property. Definition 25 generalizes Rosen's definition of closure [Ro73] (Def. 5.4, p. 170) to use the more general notion of residual map.

Definition 25

A SRS $\langle \Sigma, F, \rightarrow, \mathcal{A}\rangle$ is closed with respect to a residual map r iff the following holds. Let $\langle A=B\rangle \in \mathcal{A}$, $x \neq ()$ and $A \xrightarrow[x]{} A'$. Then there is a B' such that

(1) $\langle A'=B'\rangle \in \mathcal{A}$ and $B \xrightarrow[r\langle A=B\rangle x]{} B'$

(2) $y \perp x \Rightarrow r\langle A'=B'\rangle y = r\langle A=B\rangle y$

Clause (1) formalizes the diagram above; and clause (2), which is

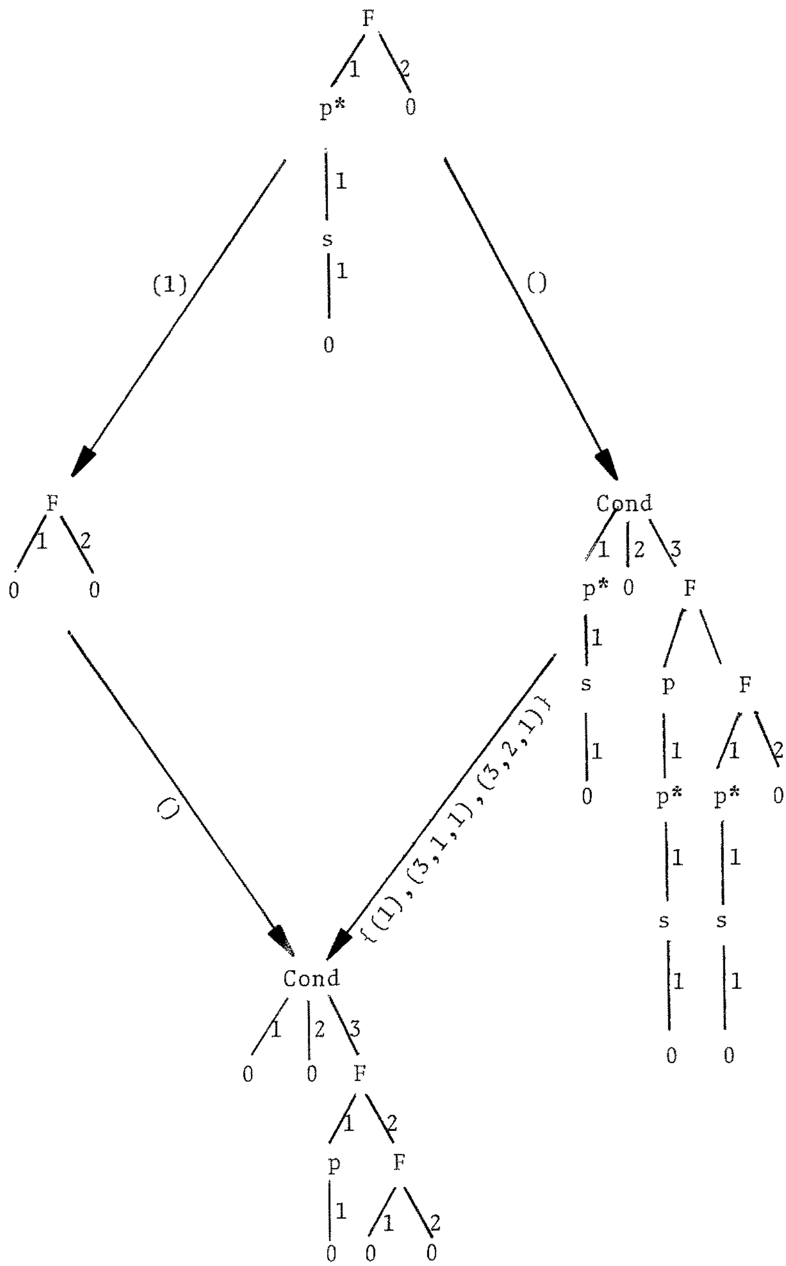


Figure 15

intuitively acceptable when residuals are thought of as rearrangements, is necessary for the inductive argument of Lemma 11. Clause (1) is clearly a special case of the confluence property.

Closure is a natural and common property for SRSs. Nonclosed SRSs often represent undesirable axiom systems.

Example 14

Extend $\mathcal{E}$ of Example 6 (p. 18) to $\mathcal{L} = \langle \Sigma_{\mathcal{L}}, (\Sigma_{\mathcal{L}})_{\#}, \rightarrow, \mathcal{R}_{\mathcal{L}} \rangle$, where $\Sigma_{\mathcal{L}} = \Sigma \cup \{\text{First}, \text{Tail}\} = \{0, s, p, \text{Cond}, F, \text{First}, \text{Tail}\}$, ρ First $= \rho$ Tail $= 1$. First and Tail represent the standard list operators defined by

$$\text{First}(a(x)) = a$$
$$\text{Tail}(a(x)) = x$$

To represent these two equations, let

$$\mathcal{A}_{\mathcal{L}} = \mathcal{A} \cup \{\langle \text{First}(a(A)) = a \rangle, \langle \text{Tail}(a(A)) = A \rangle \mid a \in \{0, s, p, \text{First}, \text{Tail}\}, A \in (\Sigma_{\mathcal{L}})_{\#}\}$$

Extend the residual map r of Example 11 (p. 23) in the natural way:

$$r\langle \text{First}(a(A)) = a \rangle x = \phi$$
$$r\langle \text{Tail}(a(A)) = A \rangle x = \begin{cases} \{z\} & \text{if } x = (1,1) \cdot z \\ \phi & \text{otherwise} \end{cases}$$

$\mathcal{L}$ is <u>not</u> closed with respect to r, as Figure 16 shows.

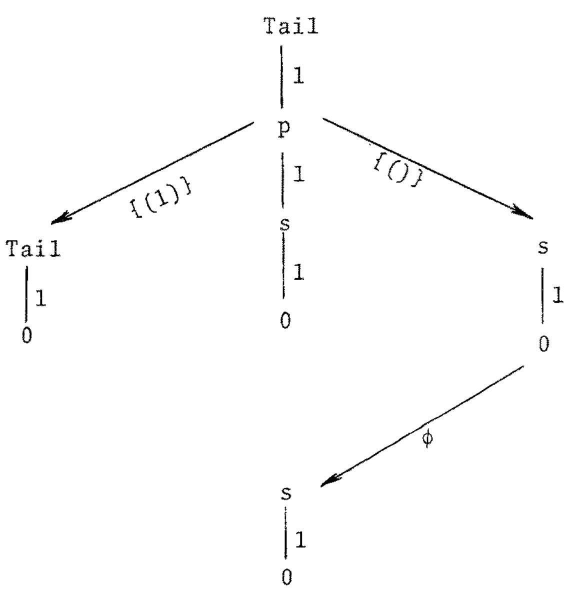


Figure 16

$r\langle Tail(p(s(0)))=s(0)\rangle(1) = \phi$, but $\langle Tail(0)=s(0)\rangle \notin A_L$.

In fact, there is no residual map for which L is closed. L represents an undesirable formal system, since the meaning of Tail(p(s(0)) is intuitively ambiguous. Notice that, since Tail(0) and s(0) are both in normal form, L does not have the confluence property.

In a closed SRS, many important properties of the residual map r generalize to the extended residual map $\hat{r}$. Lemma 9 generalizes clause (1) of Definition 22 of a residual map (p. 23).

Lemma 9

Let $\langle\Sigma,F,\to,A\rangle$ be closed with respect to residual map r, and let $\hat{r}$ be the extended residual map (Def. 23, p. 25). Also let each $x \in N$ be a redex in A, and

$$A \xrightarrow[M_1]{} \cdots \xrightarrow[M_n]{} B.$$

Then each $y \in \hat{r}[A,(M_1,\ldots,M_n)]N$ is a redex in B.

Proof An easy induction on $|M_1|+\ldots+|M_n|$, $|N|$. □

Lemma 10 generalizes Definition 25 (p. 30) of closure.

Lemma 10

Let $\langle\Sigma,F,\to,A\rangle$ be closed with respect to r.
Let $A \xrightarrow[z]{} B$ and $A \xrightarrow[M]{} A'$, where $\forall_{y \in M} \neg(y \ \underline{anc}\ z)$.
Then there is a B' such that

(1) $A' \xrightarrow[z]{} B'$ and $B \xrightarrow[\hat{r}[A,(\{z\})]M]{} B'$

(2) $N \perp M \Rightarrow \hat{r}[A',(\{z\})]N = \hat{r}[A,(\{z\})]N$.

Proof Induction on $|M|$, $|N|$ using Definition 25 (p. 30). □

3. Proof of the Confluence Property

Lemmas 11 and 12 give more and more general cases of the confluence property, each proved by a single induction or by a noninductive argument. The full confluence property (Theorem 6) follows from Lemma 12 by two final inductions.

Lemma 11

Let $\langle\Sigma,F,\to,A\rangle$ be closed with respect to r, and let M and N be sets of independent redexes such that $\forall_{x \in M, y \in N} \neg(y \ \underline{anc}\ x)$. Also let $A \xrightarrow[M]{} B$ and $A \xrightarrow[N]{} C$.
Then there is a D such that $B \xrightarrow[\hat{r}[A,(M)]N]{} D$ and $C \xrightarrow[M]{} D$.

Figure 17 sketches the Lemma.

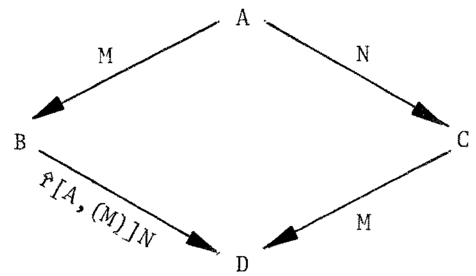


Figure 17

Proof By induction on $|M|$.
Basis step: If $|M| = 0$, then let D = C.
Induction step: Assume the lemma holds for sets of size $|M| - 1$.
Let $z \varepsilon M$. Figure 18 outlines the induction step.

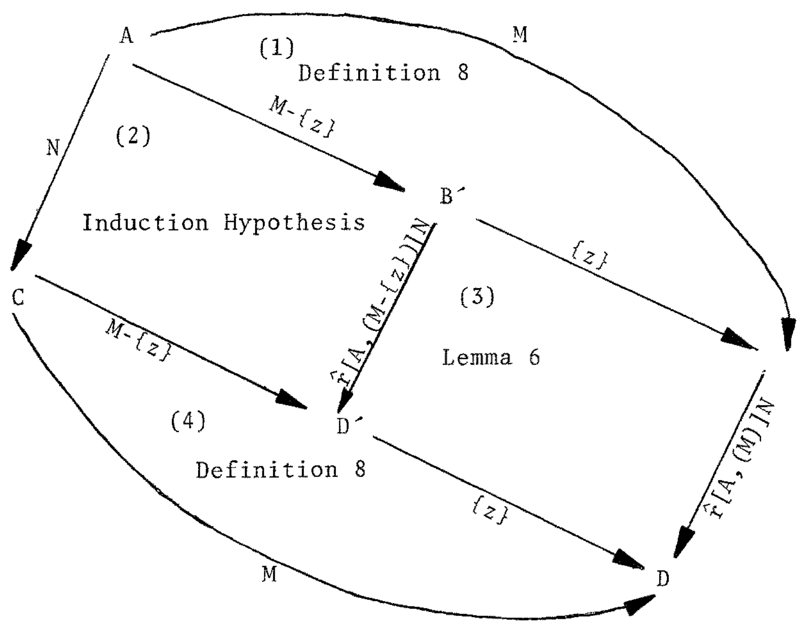


Figure 18

(1) By Definition 19, there is a B' such that
$A \xrightarrow[M-\{z\}]{} B' \xrightarrow[\{z\}]{} B$

(2) By the induction hypothesis applied to the reductions $A \xrightarrow[M-\{z\}]{} B'$

and $A \xrightarrow[N]{} C$, there is a D' such that $C \xrightarrow[M-\{z\}]{} D'$ and $B' \xrightarrow[\hat{r}[A,(M-\{z\})]N]{} D'$.

(3) By Lemma 6 (p. 25), since $(M-\{z\}) \cup N$ contains no ancestor of z, $\hat{r}[A,(M-\{z\})]N$ contains no ancestor of z. So, by Lemma 10, there is a D such that $D' \xrightarrow[\{z\}]{} D$ and $B \xrightarrow[\hat{r}[B',(\{z\})](\hat{r}[A,(M-\{z\})]N)]{} D$. And, by Definition 23 (p. 25), $\hat{r}[B',(\{z\})](\hat{r}[A,(M-\{z\})]N) = \hat{r}[A,(M)]N$. So, $B \xrightarrow[\hat{r}[A,(M)]N]{} D$.

(4) We have $C \xrightarrow[M-\{z\}]{} D' \xrightarrow[\{z\}]{} D$, so, by Definition 19, $C \xrightarrow[M]{} D$. □

Lemma 12 (General Residual Lemma)

If $<\Sigma,F,\rightarrow,A>$ is closed with respect to r, and $A \xrightarrow[M]{} B$, $A \xrightarrow[N]{} C$, then there is a D such that

$$B \xrightarrow[\hat{r}[A,(M)]N]{} D, \quad C \xrightarrow[\hat{r}[A,(N)]M]{} D.$$

Figure 19 sketches the Lemma.

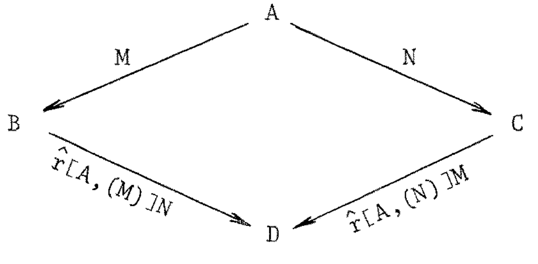


Figure 19

Proof Partition M into three pieces:

$M \cap N$,

$M_{out} = \{x \mid x \in M \wedge \forall_{y \in N} \neg\, y \underline{\text{ anc }} x\}$,

$M_{in} = \{x \mid x \in M \wedge \exists_{y \in N}\, y \underline{\text{ anc}\neq} x\}$.

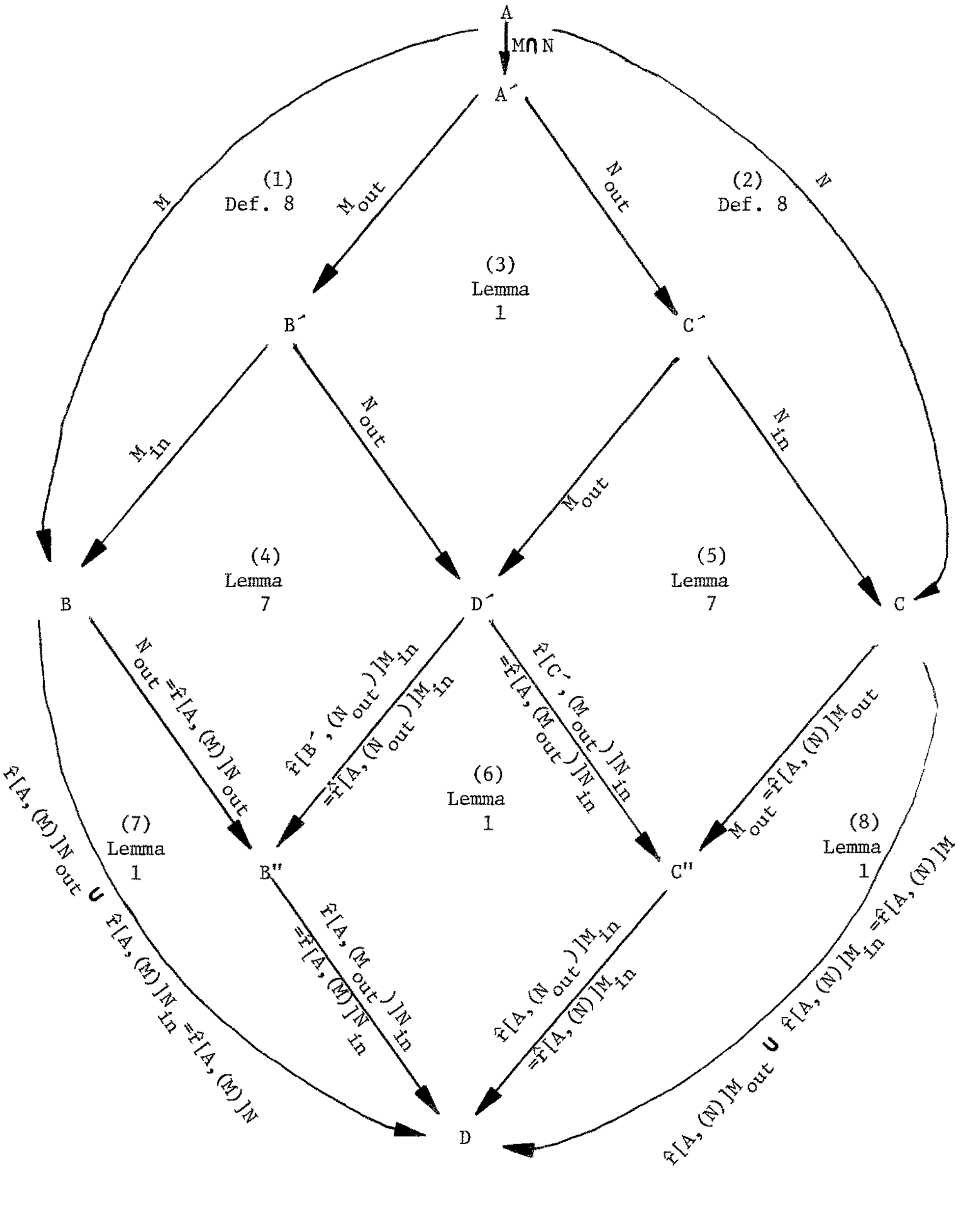
Similarly partition N into

$N \cap M = M \cap N$,

$N_{out} = \{y \mid y \in N \wedge \forall_{x \in M} \neg\, x \underline{\text{ anc }} y\}$

$N_{in} = \{y \mid y \in N \wedge \exists_{x \in M}\, x \underline{\text{ anc}\neq} y\}$

Figure 20 on the next page summarizes the proof.


A
M∩N
A´
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(1)
Def. 8
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Def. 8
M out
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Lemma
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Figure 20

By Definition 19, there are A', B' and C' such that:

(1) $A \xrightarrow[M \cap N]{} A' \xrightarrow[M_{out}]{} B' \xrightarrow[M_{in}]{} B$

(2) $A \xrightarrow[M \cap N]{} A' \xrightarrow[N_{out}]{} C' \xrightarrow[N_{in}]{} C$

(3) $M_{out} \perp N_{out}$, so, by Lemma 5 (p. 21) applied to $A' \xrightarrow[M_{out}]{} B'$ and $A' \xrightarrow[N_{out}]{} C'$, there is a D' such that $B' \xrightarrow[N_{out}]{} D'$ and $C' \xrightarrow[M_{out}]{} D'$.
$\forall_{x \in M_{in}, y \in N_{out}} \neg\, x\ \underline{\text{anc}}\ y$, and
$\forall_{y \in N_{in}, x \in M_{out}} \neg\, y\ \underline{\text{anc}}\ x$.

(4) So, by Lemma 11 (p. 33) applied to $B' \xrightarrow[M_{in}]{} B$ and $B' \xrightarrow[N_{out}]{} D'$, there is a B" such that $B \xrightarrow[N_{out}]{} B''$ and $D' \xrightarrow[\hat{r}[B',(N_{out})]M_{in}]{} B''$.

(5) Applying Lemma 11 to $C' \xrightarrow[N_{in}]{} C$ and $C' \xrightarrow[M_{out}]{} D'$, there is a C" such that $C \xrightarrow[M_{out}]{} C''$ and $D' \xrightarrow[\hat{r}[C',(M_{out})]N_{in}]{} C''$.

(6) By clause (2) of Lemma 10 (p.),
$\hat{r}[B',(N_{out})]M_{in} = \hat{r}[A,(N_{out})]M_{in}$
$\hat{r}[C',(M_{out})]N_{in} = \hat{r}[A,(M_{out})]N_{in}$.

By Lemma 7 (p. 27),
$\hat{r}[A,(N_{out})]M_{in} \perp \hat{r}[A,(M_{out})]N_{in}$.

Now, by Lemma 5 (p. 21) applied to
$D' \xrightarrow[\hat{r}[A,(N_{out})]M_{in}]{} B''$ and $D' \xrightarrow[\hat{r}[A,(M_{out})]N_{in}]{} C''$,
there is a D such that
$B'' \xrightarrow[\hat{r}[A,(M_{out})]N_{in}]{} D$ and $C'' \xrightarrow[\hat{r}[A,(N_{out})]M_{in}]{} D$.

By Definition 23(p. 25),
$\hat{r}[A,(M_{out})]N_{in} = \hat{r}[A,(M)]N_{in}$, and
$\hat{r}[A,(N_{out})]M_{in} = \hat{r}[A,(N)]M_{in}$.

By Clause (3) of Lemma 8 (p. 27),
$\hat{r}[A,(M)](N \cap M) = \hat{r}[A,(N)](M \cap N) = \phi$.

Finally, by Lemma 5,

(7) $B \xrightarrow[\hat{r}[A,(M)]N_{out} \cup \hat{r}[A,(M)]N_{in} \cup \hat{r}[A,(M)](N \cap M)]{} D$, and

(8) $C \xrightarrow[\hat{r}[A,(N)]M_{out} \cup \hat{r}[A,(N)]M_{in} \cup \hat{r}[A,(M)](M \cap N)]{} D$.

Now, by Definition 23,

$B \xrightarrow[\hat{r}[A,(M)]N]{} D$ and $C \xrightarrow[\hat{r}[A,(N)]M]{} D$ □

Corollary 2

Let $<\Sigma,F,\to,A>$ be closed with respect to r, and let $A \underset{\perp}{\to} B$ $A \underset{\perp}{\to} C$. Then there is a D with $B \underset{\perp}{\to} D$ and $C \underset{\perp}{\to} D$.

Proof Direct from Lemma 12. □

Theorem 6 (Confluence Property)

If $<\Sigma,F,\to,A>$ is closed with respect to r, then it has the confluence property.

Proof An easy induction on m and n where $A \underset{\perp}{\to}^m B$ and $A \underset{\perp}{\to}^n C$ (see Def. 24, p. 28), using Corollary 2 and the fact that $\underset{\perp}{\to}{}^* = \to^*$. □

Theorem 6 justifies the confluence property on the reduction relation $\to$ from the closure condition on the axioms or reduction rules A. In Chapter VII we will see that, with certain natural notations for defining reduction rules, closure will be easy to verify syntactically. In Chapters IV and V, all SRSs will be assumed to be closed with respect to some appropriate residual map r.

*4. Continuous Semantics and the Confluence Property

Given a continuous interpretation $<D,v>$ of Σ, a reasonable formalization of the notion that A is simpler or clearer than B is $\hat{v}A \sqsubseteq \hat{v}B$. Consider a SRS $<\Sigma,F,\to,A>$ where $<A=B>\in A \Rightarrow \hat{v}A \sqsubseteq \hat{v}B$. If this SRS has the confluence property, then we may treat A as a set of definitions on v, and reduction sequences will reveal all possible information about expressions in Σ_*.

Theorem 7

Let $<D,v>$ be a continuous interpretation of Σ.
Let A be a set of equations.
Let $<\Sigma,F,\to,A>$ be a SRS with the confluence property.
If $<A=B>\in A \Rightarrow \hat{v}A \sqsubseteq \hat{v}B$, then, for all A,

$\underline{\text{def}}A = \sqcup\{\hat{v}B \mid A \vdash A=B\}$ exists and

$\underline{\text{def}}A = \sqcup\{\hat{v}C \mid A \to^* C\}$

If A has normal form E, then

$\underline{\text{def}}A = \hat{v}E$.

Proof The confluence property and the assumption $<A=B>\in A \Rightarrow \hat{v}A \sqsubseteq \hat{v}B$ guarantee that, for all A

$\{\hat{v}B \mid A \vdash A=B\}$ is a directed set.

So $\sqcup\{\hat{v}B \mid A \vdash A=B\}$ exists. By Theorem 3 and Lemma 2 $A \vdash A=B$ iff $\exists_C\ A \to^* C \wedge B \to^* C$. By assumption $\hat{v}B \sqsubseteq \hat{v}C$, so

$\sqcup\{\hat{v}B \mid A \vdash A=B\} = \sqcup\{\hat{v}C \mid A \to^* C\}$. If A has normal form E then $\sqcup\{\hat{v}C \mid A \to^* C\} = \hat{v}E$. $\square$