STRUCTURAL EQUIVALENCE OF CONTEXT-FREE GRAMMAR FORMS IS DECIDABLE

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ABSTRACT:

Two grammars G_1 and G_2 are structural equivalent if the corresponding parenthesized grammars generate the same language. This definition transfers to grammar forms in a natural way. It is shown that structural equivalence of context-free grammar forms is decidable.

KEY WORDS AND PHRASES:

Contextfree grammars, parenthesized grammars, grammar forms, grammar homomorphisms, reduction, grammatical language families.

CR Categories: 4.12, 5.23, 5.25

0. INTRODUCTION.

In [2] Cremers and Ginsburg define grammar families by the concept of grammar forms. Intuitively a grammar form F defines a grammar family such that any member of this family is "similar" to a fixed prototype grammar G_F. Two natural equivalence problems arise by these definitions, first the problem wether two grammars G_{F_1} and G_{F_2} generate the same grammar family (strong equivalence) and second the problem wether two grammars G_{F_1} and G_{F_2} generate the same language family (weak equivalence).

In [2] it is shown that the first problem is decidable. The second one is still open. We want to show that another equivalence problem which is more general than strong equivalence and less general than weak equivalence is decidable.

Following [5] we introduce the structural equivalence of two grammars G_1 and G_2. G_1 and G_2 are structural equivalent if the associated parenthesized grammars generate the same language. This definition transfers in a natural way to grammar forms. It is known ([5],[6]) that structural equivalence of context-free grammars is decidable. We generalize this result to structural equivalence of context-free grammar forms.

1. BASIC NOTATIONS.

We adopt the usual notations of phrase structure grammars $G=(V,\Sigma,P,\sigma)$, context-free grammars, etc. ([3],[6]) with the (trivial) change $\sigma \subseteq V - \Sigma$. To avoid trivialities we assume (without loss of generality), that all grammars in consideration are reduced ([3],[6]) and contain no unnecessary symbols.

By $\overset{*}{\vdash}$ ($\vdash$) we denote the notion of a (direct) derivation.

The set $\mathcal{L}(G) = \{w \in \Sigma^* \mid \exists \sigma_1 \in \sigma : \sigma \overset{*}{\vdash} w\}$

is the generated language and

$$D(G) = \{\sigma_1 \overset{*}{\vdash} w \mid w \in \Sigma^* \ \& \ \sigma_1 \in \sigma\}$$

is the set of derivations.

A <u>grammar morphism</u> $\phi : G_1 \to G_2$ is a monoid homomorphism $\phi : V_1^* \to V_2^*$ with

(i) $\phi(\sigma_1) \subseteq \sigma_2$

(ii) $\phi(V_1 - \Sigma_1) \subseteq (V_2 - \Sigma_2)$

(iii) $\phi(\Sigma_1) \subseteq \Sigma_2^*$

(iv) $\phi(P_1) = \{(\phi(p),\phi(q)) \mid (p,q) \in P_1\} \subseteq P_2$.

We single out various classes of grammar morphisms.
If ϕ is lengthpreserving, we call ϕ a <u>fine</u> morphism, if $\phi(\xi) = \xi\ (\xi \in V_1 - \Sigma_1)$ ϕ is <u>external</u> and if $\phi(t) = t\ (t \in \Sigma_1)$ ϕ is <u>internal</u>.
In a natural way ϕ induces $\hat{\phi} : D(G_1) \to D(G_2)$. ([4])

We call ϕ <u>closed</u> if $\hat{\phi}$ is surjective. For closed morphisms ϕ we get: $\phi(\mathcal{L}(G_1)) = \mathcal{L}(G_2)$. A closed, internal morphism ϕ is called a <u>reduction</u>. ϕ is an isomorphism if ϕ and ϕ^{-1} are morphisms.

To any grammar G we associate a grammar family Γ_G to be the collection of all grammars G' such that there exists a diagram

$$G \xleftarrow[\phi_1]{} G'' \xrightarrow[\phi_2]{} G'$$

where ϕ_1 is fine and ϕ_2 is external and closed. By $\lambda(G)$ we denote the collection of all languages $\mathcal{L}(G')$ with $G' \in \Gamma_G$.

The existence of such a diagram is by [10] equivalent to the notion of grammar forms [2] and therefore $\lambda(G)$ is exactly the grammatical language family ([2]) associated to this grammar form. We call G_1 and G_2 <u>strong equivalent</u> ($G_1 \approx G_2$) if $\Gamma_{G_1} = \Gamma_{G_2}$, and <u>weak equivalent</u> ($G_1 \sim G_2$) if $\lambda(G_1) = \lambda(G_2)$.

Now, consider a universal bracket pair $\{(,)\}$.
If G is a grammar, then the associated <u>parenthesized grammar</u> $G^{()}$ is obtained by

$$G^{()} = (V \cup \{(,)\}, \Sigma \cup \{(,)\}, \{(p,(q)) \mid (p,q) \in P\}, \sigma) \quad ([5],[6]).$$

Denote by $\lambda^{()}(G)$ the collection of all languages $\mathcal{L}(G'^{()})$ with $G' \in \Gamma_G$.

G_1 and G_2 are called <u>structural equivalent</u> ($G_1 \equiv G_2$) if $\lambda^{()}(G_1) = \lambda^{()}(G_2)$.

It is easy to check:

$$G_1 \approx G_2 \Longrightarrow G_1 \equiv G_2 \Longrightarrow G_1 \sim G_2.$$

<u>Example:</u> Consider G_1 and G_2 specified by

$$\begin{array}{lllllll} P_1 : \sigma & \to & \sigma_1 & + & a\sigma & + & \varepsilon \\ \ \sigma_1 & \to & \sigma_1\sigma_1 & + & \sigma_1 & + & \varepsilon \\ P_2 : \sigma & \to & \sigma_1 & + & a\sigma & + & \varepsilon \end{array}$$

It is easy to check that $\lambda(G_1) = \mathcal{L}_{reg} = \lambda(G_2)$, $\lambda^{()}(G_2) \sqsubseteq \mathcal{L}_{lin}$ and $D_1 \in \lambda^{()}(G_1)$, where D_1 is the Dyck language over $\{(,)\}$, hence $G_1 \sim G_2$ but $G_1 \not\equiv G_2$.

From now on, we assume without further mentioning all grammars to be context-free.

2. PRELIMINARY RESULTS.

In this section we derive some results on grammar morphisms and parenthesized grammars, which are necessary to prove our main result.
We state without proof.

<u>Lemma 1:</u> Let $\phi : G_1 \to G_2$ be a morphism

There exists a factorization $\phi = \phi_1 \circ \phi_2$ resp. $\phi_2 \circ \phi_1$ with ϕ_2 internal and ϕ_1 external. If ϕ is closed then ϕ_2 is closed.

<u>Lemma 2:</u> If $\phi_1 : G_0 \to G_1$ is closed and external and $\phi_2 : G_0 \to G_2$ is internal, then there exist $\psi_1 : G_2 \to G_3$ closed and external and $\psi_2 : G_1 \to G_3$ internal with $\psi_2\phi_1 = \psi_1\phi_2$. Moreover, if ϕ_1 is fine then ψ_1 is fine and if ϕ_2 is a reduction then ψ_2 is a reduction.

We now study the effect of parenthesizing. Obviously there is a canonical external and closed morphism $e : G^{()} \to G$ defined via bracket erasing.

Consider a morphism $\phi : G_1 \to G_2$, then ϕ induces a unique morphism $\phi^{()} : G_1^{()} \to G_2^{()}$ such that

$$\begin{tikzcd} G_1^{()} \arrow[r, dashed, "\phi^{()}"] \arrow[d, "e"'] & G_2^{()} \arrow[d, "e"] \\ G_1 \arrow[r, "\phi"'] & G_2 \end{tikzcd}$$

is commutative. Observe that in a certain sense parenthesizing is a functor. Obviously properties of ϕ transfer to $\phi^{()}$, especially if ϕ is a reduction then $\phi^{()}$ is a reduction.
This yields

<u>Fact:</u> If $\phi : G_1 \to G_2$ is a reduction, then $\mathcal{L}(G_1^{()}) = \mathcal{L}(G_2^{()})$.

Conversely if $\phi : G_1^{()} \to G_2^{()}$ is a "bracket" morphism $(\phi(()=(\&\phi())=))$, then ϕ induces $\hat{\phi} : G_1 \to G_2$ with $\phi = \hat{\phi}^{()}$. Again properties of ϕ transfer to $\hat{\phi}$.
Call a grammar G <u>invertible</u> if (p,q) & $(p,q') \in P \Longrightarrow q = q'$.

<u>Lemma 3:</u> To any grammar G there exists a diagram

$$G \xleftarrow{\phi_1} G_o \xrightarrow{\phi_2} G^{inv}$$

where ϕ_1 and ϕ_2 are reductions and G^{inv} is invertible.

<u>Proof:</u> We analyze the construction of G^{inv} given in [5], [6].

<u>Step 1:</u> "G^{inv}":
Consider $Z' = 2^{V-\Sigma} - \{\emptyset\}$. Define a substitution $\mu: (Z' \cup \Sigma)^* \to 2^{V^*}$ by

$$\mu(z_1) = \begin{cases} z_1 & , z_1 \in Z' \\ \{z_1\} & , z_1 \in \Sigma . \end{cases}$$

to any $q \in (Z' \cup \Sigma)^*$ consider

$$S_1 = \{\xi \mid q' \text{ with } \xi \to q' \in P \ \& \ q' \in \mu(q)\}.$$

Now define G^{inv} as follows

$$G^{inv} = (V', \Sigma, P', \sigma')$$

(i) $\quad V' - \Sigma = \{S_q \mid S_q \neq \emptyset,\ q = t_o S_{q_1} t_1, \ldots, S_{q_i} t_i$
with $S_{q_\lambda} \neq \emptyset,\ t_\lambda \in \Sigma^*,\ q, q_\lambda \in (Z' \cup \Sigma)^*; 0 \leq \lambda \leq i\}$

(ii) $\quad P' = \{(S_q, q) \mid q \in (Z' \cup \Sigma)^* \ \& \ S_q \in V' - \Sigma\}$

(iii) $\quad \sigma' = \{S_q \mid S_q \cap \sigma \neq \emptyset\}$

It is easy to check, that G^{inv} is invertible.

<u>Step 2:</u> "G_O, ϕ_1,ϕ_2"

Define $G_O = (V_O,\Sigma,P_O,\sigma_O)$ by

(i) $V_O - \Sigma \quad = \{(S_q,\xi) \mid \xi \in S_q,\ q \in V' - \Sigma\}$

(ii) $\sigma_O \quad = \{(S_q,\sigma_1) \mid \sigma_1 \in \sigma \cap S_q\}$

(iii) $P_O \quad = \{(S_q,\xi) \to t_O(S_{q_1},\eta_1),\dots,(S_{q_i},\eta_i)t_i \mid$

$\alpha)$ $i \geq O,\ t_\lambda \in \Sigma^*(O \leq \lambda \leq i)$

$\beta)$ $\xi \in S_q, \eta_\lambda \in S_{q_\lambda} \quad (1 \leq \lambda \leq i)$

$\gamma)$ $q = t_O S_{q_1},\dots,S_{q_i} t_i$

$\delta)$ $\xi \to t_O\eta_1,\dots,\eta_i t_i \in P\}$.

Induce ϕ_1 and ϕ_2 by $\phi_1((S_q,\xi)) = \xi$ and $\phi_2((S_q,\xi) = S_q$.

<u>Step 3:</u> "ϕ_1 is a reduction"

We observe first for all $\sigma_1 \in \sigma$

$$\phi_1^{-1}(\sigma_1) = \{(S_q,\sigma_1) \mid \sigma_1 \in S_q\} \subseteq \sigma_O.$$

Now consider $(S_q,\xi) \to t_O(S_q,\eta_1)\ \dots\ (S_{q_i},\eta_i)t_i \in P_O$ and

$q' = t_O'(S_{q_1'},\eta_1')\ \dots\ (S_{q_j'},\eta_j')\ t_j$ such that

$$t_O\eta_1\ \dots\ \eta_i t_i = \phi_1(q') = t_O'\eta_1'\ \dots\ \eta_j' t_j'.$$

We conclude $j = i$, $t_O = t_O',\dots,t_i = t_i'$, $\eta_1 = \eta_1',\dots,\eta_i = \eta_i'$.

On the other hand, by $\xi \to t_O\eta_1\ \dots\ \eta_i t_i \in P$ we get $\xi \in S_{q'}$, hence

$$(S_{q'},\xi) \to t_O(S_{q_1},\eta_1)\ \dots\ (S_{q_i},\eta_i)t_i \in P_O.$$

This proves, that ϕ_1 is coperfect in the sense of [11]. Hence ϕ_1 is a reduction ([11]).

<u>Step 4:</u> "ϕ_2 is a reduction"

Again, we observe the following fact.

If $S_q \in \sigma'$, then there exists $\sigma_1 \in \sigma \cap S_q$, hence $(S_q,\sigma_1) \in \sigma_O$.

This proves $\phi_2(\sigma_O) = \sigma'$.

Consider

$$(S_q,\xi) \rightarrow t_0(S_{q_i},\eta_1) \ldots (S_{q_i},\eta_i)t_i \quad P_0$$

and

$$(S_{q'},\xi') \text{ with } \phi_2(S_{q'},\xi') = \phi_2((S_q,\xi)).$$

We get $S_q = S_{q'}$. Since $\xi' \in S_q$ and $q = t_0 S_{q_1} \ldots S_{q_i} t_i$ there exist $\eta_1' \ldots \eta_i'$ with $\xi' \rightarrow t_0\eta_1' \ldots \eta_i' t_i \in P$ and $\eta_\lambda' \in S_{q_\lambda}$ $(1 \leq \lambda \leq i)$.
We conclude

$$(S_q,\xi') \rightarrow t_0(S_{q_1},\eta_1') \ldots (S_{q_i},\eta_i')\, t_i \in P_0$$

and

$$\phi_2(t_0\ (S_{q_1},\eta_1') \ldots (S_{q_i},\eta_i')t_i) = \phi_2(t_0(S_{q_1},\eta_1) \ldots (S_{q_i},\ \eta_i)t_i).$$

This proves that ϕ_2 is perfect in the sense of [11]. Hence ϕ_2 is a reduction ([11]).

We are now able to prove one of the key theorems.

<u>Theorem 1:</u> The following statements are equivalent for two grammars G_1 and G_2

(1) $\mathcal{L}(G_1^{()}) = \mathcal{L}(G_2^{()})$

(2) There exists a diagram

$$G_1 \xleftarrow[\phi_1]{} G_0 \xrightarrow[\phi_2]{} G_2$$

where ϕ_1 and ϕ_2 are reductions.

<u>Proof:</u> (2) $\Longrightarrow$ (1) with the above fact.
(1) $\Longrightarrow$ (2). By lemma 3 there exist two diagrams

$$G_i \xleftarrow[\psi_1^i]{} G_i' \xrightarrow[\psi_2^i]{} G_i^{inv} \quad (i = 1,2)$$

with ψ_1^i and ψ_2^i reductions (i=1,2).

Parenthezising leads to diagrams

$$G_i^{()} \xleftarrow[\hat{\psi}_1^i]{} G_i'^{()} \xrightarrow[\hat{\psi}_2^i]{} G_i''^{()} \quad (i = 1,2)$$

where ψ_1^i and ψ_2^i are reductions and $G_i''^{()}$ is invertible (i = 1,2).
Since

$$\mathcal{L}(G_1''^{()}) = \mathcal{L}(G_1^{()}) = \mathcal{L}(G_2^{()}) = \mathcal{L}(G_2''^{()})$$

we obtain from [5], that there exists an internal isomorphism between

$G_1''^{()}$ and $G_2''^{()}$.

Hence we get a diagram

$$G_1^{()} \xleftarrow[\hat{\psi}_1^1]{} G_1'^{()} \xrightarrow[\hat{\psi}_2^1]{} G_1''^{()} \xleftarrow[\hat{\psi}_2^2]{} G_2'^{()} \xrightarrow[\hat{\psi}_1^2]{} G_2^{()},$$

where all arrows are reductions.

Removing the brackets, we get a diagram

$$\begin{tikzcd}
 & G_1' \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & & G_2' \arrow[dl, "\psi_3"'] \arrow[dr, "\psi_4"] & \\
G_1 & & G_0 & & G_2
\end{tikzcd}$$

where all $\psi_i (i = 1,2,3,4)$ are reductions.

By a theorem of [8], [9] we can fill in this diagram with reductions ψ_5 and ψ_6.

$$\begin{tikzcd}
 & & G_0' \arrow[dl, "\psi_5"'] \arrow[dr, "\psi_6"] & & \\
 & G_1' \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & & G_2' \arrow[dl, "\psi_3"'] \arrow[dr, "\psi_4"] & \\
G_1 & & G_0 & & G_2
\end{tikzcd}$$

Now, letting $\phi_1 = \psi_1\psi_5$ and $\phi_2 = \psi_4\psi_6$, we get the theorem.

3. THE MAIN THEOREM

We are now in the position to prove a characterization of the structural equivalence.

Theorem 2: The following statements are equivalent for two grammars G_1 and G_2:

(1) $\lambda^{()}G_2 \subseteq \lambda^{()}(G_1)$

(2) There exists a diagram

$$G_1 \xleftarrow{\phi_1} G_1' \xrightarrow{\phi_2} G_0 \xleftarrow{\phi_3} G_2' \xrightarrow{\phi_4} G_2$$

with

(α) ϕ_1 is internal

(β) ϕ_2 is a reduction

(γ) ϕ_3 is fine, external and closed

(δ) ϕ_4 is external and closed.

Proof: (1) $\Longrightarrow$ (2)

Consider G_2. Since $\mathcal{X}(G_2^{()}) \in \lambda^{()}(G_1)$, there is a $\hat{G}_2 \in \Gamma_{G_1}$ with $\mathcal{X}(\hat{G}_2^{()}) = \mathcal{X}(G_2^{()})$.

By theorem 1 and the definition of Γ_{G_1} we get a diagram

$$G_1 \xleftarrow{\text{fine}} \hat{G}_1 \xrightarrow[\text{external}]{\text{closed}} \hat{G}_2 \xleftarrow{\text{reduction}} \hat{\hat{G}}_2 \xrightarrow{\text{reduction}} G_2$$

consider the diagram

$$\hat{G}_1 \xrightarrow[\text{external}]{\text{closed}} \hat{G}_2 \xrightarrow{\text{reduction}} \hat{\hat{G}}_2.$$

By a theorem of [10] we can fill in this diagram in the following way

$$\begin{tikzcd}
\hat{G}_1 \arrow[d, "\text{closed}"', "\text{ext.}"] & G_1^* \arrow[l, dashed, "\text{red.}"'] \arrow[d, dashed, "\text{closed ext.}"] \\
\hat{G}_2 & \hat{\hat{G}}_2 \arrow[l, "\text{red.}"]
\end{tikzcd}$$

Consider the diagram

$$G_1^* \xrightarrow[\text{ext.}]{\text{closed}} \hat{\hat{G}}_2 \xrightarrow{\text{red.}} G_2.$$

By lemma 1 we can fill in this diagram in the following way

$$\begin{tikzcd}
G_1^* \arrow[r, "\text{closed}", "\text{ext.}"'] \arrow[d, dashed, "\text{red.}"'] \arrow[dr, dashed, "\text{closed}"] & \hat{\hat{G}}_2 \arrow[d, "\text{red.}"] \\
G_2' \arrow[r, dashed, "\text{closed}", "\text{ext.}"'] & G_2
\end{tikzcd}$$

Now, we reached the following situation:

$$G_1 \xleftarrow[\text{fine}]{} G_1^* \xrightarrow[\text{red.}]{} G_2' \xrightarrow[\text{closed}]{\text{ext.}} G_2$$

By lemma 1 again we find a diagram

$$\begin{tikzcd} G_1 & G_1 \arrow[l, "\text{internal}"'] & G_1^* . \arrow[l, "\text{ext.}", "\text{closed, fine}"'] \arrow[ll, bend right, "\text{fine}"'] \end{tikzcd}$$

Consider the diagram

$$G_1' \xleftarrow[\text{fine}]{\text{ext.}} G_1^* \xrightarrow[\text{red.}]{} G_2' .$$

By lemma 2 we can fill in this diagram:

$$\begin{tikzcd} G_1' \arrow[d, dashed, "\text{red.}"'] & & G_1^* \arrow[ll, "\text{ext.}"', "\text{closed, fine}"] \arrow[d, "\text{red.}"] \\ G_0 & & G_2' \arrow[ll, dashed, "\text{ext.}", "\text{closed, fine}"'] \end{tikzcd}$$

But this proves the if-part of theorem 2.

(2) $\Longrightarrow$ (1) Consider an arbitrary diagram

$$G_2 \xleftarrow{\text{fine}} G_3' \xrightarrow[\text{closed}]{\text{ext.}} G_3$$

By a theorem of [10] we get a diagram

$$G_0 \xleftarrow[\text{fine}]{} G_2' \xleftarrow[\text{fine}]{} G_3^* \xrightarrow[\text{closed}]{\text{ext.}} G_3 .$$

On the other hand by a theorem of [8], [9] we get a diagram

$$G_1' \xleftarrow[\text{fine}]{} G_3^{**} \xrightarrow[\text{red.}]{} G_3^* .$$

In summary we obtain

$$G_1 \xleftarrow[\text{fine}]{} G_3^{**} \xrightarrow[\text{red.}]{} G_3^* \xrightarrow[\text{closed}]{\text{ext.}} G_3$$

By definition: $G_3^{**} \in \Gamma_{G_1}$. By section 2 we get

$$\mathcal{L}(G_3^{**()}) = \mathcal{L}(G_3^{*()}) \text{ and } \mathcal{L}(G_3^{()}) = h(\mathcal{L}(G_3^{*()}))$$

where h is a certain, bracket preserving homomorphism. But this proves $\mathcal{L}(G_3^{()}) \in \lambda^{()}(G_1)$.

It remains to show the decidability of statement (2) of theorem 2. Denote by $||G|| = \operatorname*{Max}\limits_{(p,q)\in P} |q|$. To any G and Z with $Z \cap \Sigma = \emptyset$ associate

$$\mathcal{G}(G,Z) = \{G' \mid G' = (Z \cup \Sigma, \Sigma, P', \sigma'),\ ||G'|| \leq ||G|| \ \&\ \text{there exists a diagram } G \xleftarrow{\text{int.}} G'' \xrightarrow{\text{red.}} G' \}$$

<u>Lemma 4:</u> $|\mathcal{G}(G,Z)| < \infty$ and there is an algorithm which computes $\mathcal{G}(G,Z)$.

<u>Proof:</u> We show first, that to any diagram

$$G \xleftarrow[\phi_1]{\text{int.}} G'' \xrightarrow[\phi_2]{\text{red.}} G'$$

exists a diagram

$$G \xleftarrow{\text{int.}} G''' \xrightarrow{\text{red.}} G'$$

with $V''' - \Sigma \subseteq (V - \Sigma) \times (V' - \Sigma)$.

To prove this we consider the following equivalence relation on $V'' - \Sigma$:

$$\xi \equiv \xi' \Longleftrightarrow \phi_1(\xi) = \phi_1(\xi') \ \&\ \phi_2(\xi) = \phi_2(\xi').$$

Then define $G''' = (\{[\xi]_\equiv \mid \xi \in V'' - \Sigma\} \cup \Sigma, \Sigma, P''', \{[\sigma_1]_\equiv \mid \sigma_1 \in \sigma''\})$ where

$$[\xi]_\equiv \to t_0 [\eta_1]_\equiv \ \dots\ [\eta_i]_\equiv t_i \in P'''$$

if and only if

$$\xi \to t_0 \eta_1 \ \dots\ \eta_i t_i \in P''$$

for all $\xi, \eta_1, \dots, \eta_i \in V'' - \Sigma, t_0, \dots, t_i \in \Sigma^*$ and $i \geq 0$.

By definition of $\xi \equiv \xi'$ we get in a natural way internal ϕ_1' and ϕ_2' such that

$$\begin{tikzcd}
G & G'' \arrow[l, "\phi_1"'] \arrow[d, "\phi_2"] \arrow[dl, dashed] \\
G''' \arrow[u, "\phi_1'"] \arrow[r, "\phi_2'"'] & G'
\end{tikzcd}$$

is commutative. This shows: ϕ_2' is a reduction. By construction $[\xi]_\equiv$ is uniquely determined by $\phi_1'([\xi]_\equiv)$ and $\phi_2'([\xi]_\equiv)$. This proves the above assertion.

To compute $\mathcal{G}(G,Z)$ we now have to inspect only (G'',G'), such that

(i) $\Sigma'' = \Sigma' = \Sigma$ (ii) $V'' - \Sigma \subseteq (V - \Sigma) \times (V' - \Sigma)$

iii) $||G''|| \leq ||G||$, (iv) $||G'|| \leq ||G||$

Obviously, there are only a finite number of such pairs and the set of all these (G'',G') can be effectively constructed. To any such pair associate

$$\phi_1 : V'' - \Sigma \to V - \Sigma \quad \text{and} \quad \phi_2 : V'' - \Sigma \to V' - \Sigma$$

by $\phi_1(\xi,\eta) = \xi$ and $\phi_2(\xi,\eta) = \eta$ $((\xi,\eta) \in V'' - \Sigma)$.

We can deaide wether ϕ_1 and ϕ_2 define internal morphism. Now, if both are morphisms we can decide by an algorithm due to C.P. Schnorr [7] wether ϕ_2 is a reduction.

Now, consider a grammar G and Σ' with $(V - \Sigma) \cap \Sigma' = \emptyset$ and $k \in Z_+$. To any such triple associate

$$\mathcal{H}(G,\Sigma',k) = \{G' \mid G' = (\Sigma' \cup (V - \Sigma), \Sigma', P', \sigma),$$
$$||G'|| \leq k \ \& \text{ there exists a diagram}$$
$$G' \xleftarrow[\text{ext.fine}]{\text{closed}} G'' \xrightarrow[\text{ext.}]{\text{closed}} G\}$$

<u>Lemma 5:</u> $|\mathcal{H}(G,\Sigma',k)| < \infty$ and there is an algorithm which computes $\mathcal{H}(G,\Sigma',k)$.

<u>Proof:</u> We show first, that to any diagram

$$G' \underset{\phi_1}{\xleftarrow[\text{ext.fine}]{\text{closed}}} G'' \underset{\phi_2}{\xrightarrow[\text{ext.}]{\text{closed}}} G$$

there exists a diagram

$$G' \xleftarrow[\phi_1']{} G''' \xrightarrow[\phi_2']{} G$$

where ϕ_1' is closed, external and fine and ϕ_2' is closed and external, such that

$$\Sigma''' \subseteq \Sigma' \times \bigcup_{j=0}^{||G||} \Sigma^j.$$

Observe first that $|\phi_2(t)| \leq ||G||$ for all $t \in \Sigma''$. Now consider the following equivalence relation on Σ'':

$$t \equiv t' \text{ iff } \phi_1(t) = \phi_1(t') \;\&\; \phi_2(t) = \phi_2(t').$$

Define $G''' = (\{[t]_\equiv \mid t \in \Sigma''\} \cup (V'' \setminus \Sigma''), \{[t]_\equiv \mid t \in \Sigma''\}, P''', \sigma'')$
where

$$\xi \to \eta_0 [t_1]_\equiv \ldots\ldots [t_i]\ \eta_i \in P'''$$

iff

$$\xi \to \eta_0 t_1 \ \ldots\ t_i \eta_i \in P''$$

for all $\eta_0, \ldots, \eta_i \in (V'' - \Sigma'')^*$, $t_1, \ldots, t_i \in \Sigma''$ and $i \geq 0$.

Again we obtain by construction a commutative diagram

$$\begin{tikzcd}
G' & G'' \arrow[l, "\phi_1"'] \arrow[d, "\phi_2"] \arrow[dl, dashed] \\
G''' \arrow[u, "\phi_1'"] \arrow[r, "\phi_2'"'] & G
\end{tikzcd}$$

where ϕ_1' is closed, external and fine and ϕ_2' is closed and external. Moreover $[t]_\equiv$ is now uniquely determined by $\phi_1'([t]_\equiv)$ and $\phi_2'([t]_\equiv)$.

To compute $\mathcal{H}(G, \Sigma', k)$ we have to consider all pairs (G'', G') with

(i) $\quad ||G'|| \leq k, \quad ||G''|| \leq k$

(ii) $\quad \Sigma'' \subseteq \Sigma' \times \bigcup_{j=0}^{||G||} \Sigma^j$

(iii) $\quad V'' - \Sigma'' = V' - \Sigma' = V - \Sigma.$

Obviously, there are only a finite number of such pairs and all these pairs can be constructed effectively. Now, associate to any such pair (G'', G') two mappings ϕ_1 and ϕ_2 by $\phi_1(t,w) = t$ and $\phi_2(t,w) = w$ for all $(t,w) \in \Sigma''$. Decide wether both mappings define external morphisms. If they do, check: $P' = \phi_1(P'')$ and $P = \phi_2(P'')$ which is a necessary and sufficient condition for ϕ_1 resp. ϕ_2 to be closed.

Consider now a diagram from theorem 2(2)

$$G_1 \xleftarrow{\text{int.}} G_1' \xrightarrow{\text{red.}} G_0 \xleftarrow[\text{ext., fine}]{\text{closed}} G_2' \xrightarrow[\text{ext.}]{\text{closed}} G_2$$

By definition of $\mathcal{G}(G, Z)$ and $\mathcal{H}(G, \Sigma', k)$ we get

$$G_0 \in \mathcal{G}(G_1, V_2 - \Sigma_2) \;\&\; G_0 \in \mathcal{H}(G_2, \Sigma_1, ||G_1||).$$

This shows, that the existence of such a diagram is equivalent to

$$\mathcal{G}(G_1, V_2 - \Sigma_2) \cap \mathcal{H}(G_2, \Sigma_1, ||G_1||) \neq \emptyset.$$

Using lemma 4 and lemma 5 we can decide this last relation. This completes the proof of our

Theorem 3: For any two contextfree grammars G_1 and G_2 it is decidable wether or not $G_1 \equiv G_2$ holds.

Remark: We have shown, that the relation
$\lambda^{()}(G_2) \sqsubseteq \lambda^{()}(G_1)$ is decidable, too.

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