ON THE CONVERGENCE
OF BALAKRISHNAN'S METHOD

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We show that the epsilon method of Balakrishnan gives, in a sense, a constructive proof of an existence theorem of Cesari for the following general problem of optimal control: minimize

$$\int_a^b f(t,x,u)dt$$

over the set of all pairs (u,x), u measurable, x absolutely continuous on $[a,b]$, such that

$$\dot{x} = g(t,x,u) \text{ a.e.},$$

(1) $\quad (a,x(a),b,x(b)) \in T$,

(2) $\quad (t,x(t)) \in G$

(3) $\quad u(t) \in V(t,x(t))$ q.e.

Let

$$I_n(u,x) = \int_a^b \left[f(t,x,u) + n \left| \dot{x}-g(t,x,u) \right| \right] dt \ , \ n = 1,2, \ldots$$

We consider the following problem P_n: minimize

$$I_n(u,x)$$

over the set of all pairs (u,x) as above such that (1),(2),(3) hold.

Under linearity and convexity assumptions (on g,f respectively) it is known that optimal solutions of P_n approximate optimal solutions of P_o.
See Balakrishnan, SIAM J. Control 6 (1968), also in "Control Theory and the calculus of variations", UCLA 1968. In this work these results are extended to the above optimal control problem. In the simplest case, the (Q)-property of Cesari (as weakened by Berkowitz) has an important role in the proof. The type of convergence is the following: given any sequences $\varepsilon_n \longrightarrow 0$ and $\{u_n,x_n\}$ such that

$$I_n(u_n,x_n) \leq \inf I_n + \varepsilon_n$$

for some subsequence we have

$$x_n \longrightarrow x_o \quad \text{uniformly} \ ,$$
$$\dot{x}_n \longrightarrow \dot{x}_o \quad \text{in } L^1 \ ,$$

$$\int_{a_n}^{b_n} f(f,x_n,u_n)dt \longrightarrow \int_{a_o}^{b_o} f(t,x_o,u_o)dt \ ,$$

(u_o,x_o) an optimal pair for the original problem,

$$\inf I_n \longrightarrow \min I_o .$$

This approach avoids any use of generalized controls. Only an approximate minimization is required for every P_n.
When the state equations are linear at least in the control variables, and the cost meets suitable convexity conditions, then we get strong convergence

$$u_n \longrightarrow u_o \text{ in } L^p .$$

If the original problem has a piecewise smooth optimal solution, then by pointwise minimization of
$u \longrightarrow f(t,x(t),u) + n|x(t) - g(t,x(t),u)|$ over the control region $V(t,x(t))$, and then an approximate minimization of I_n with respect of x, we get a sequence (u_n,x_n) such that

$$u_n \longrightarrow u_o \text{ piecewise uniformly.}$$

Complete statements and the details will appear elsewhere.