ON OPTIMAL PARAMETRIC CONTROL OF PARABOLIC SYSTEM

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Introduction

In the paper we consider a parametric optimization problem for an abstract parabolic equation. Problems of such a type are investigated in the case of elliptic equations in [2] , [10] and parabolic equations in [2] , [3] , [5] , [11] . In this paper two types of observations : in spaces $C(0,T;H)$ and $C(0,T;V)$ are considered.

For both cases sufficient conditions of existence of an optimal control are given. Using the so called generalized adjoint state equation necessary conditions of optimality are formulated. Proofs of presented results are given in [7] , [8] , [9] .

Let there be given Hilbert spaces V, H with

/1.1/ $V \subset H$, V dense in H

By U_{ad} we denote the set of admissible controls which is assumed to be a convex subset of a Hilbert space U.

Let there be given a family of bilinear forms on V :

/1.2/ $a_u(t;y,z)$

where $u \in U_{ad}$

$t \in [0,T]$

$y,z \in V$

We assume :

(i) family /1.2/ is continuous on V, that is

/1.3/ $$|a_u(t;y,z)| \leqslant M \|y\|_V \|z\|_V , \forall t \in [0,T] ,$$
$$\forall u \in U_{ad} , \quad \forall y,z \in V$$

(ii) mapping

$$t \longmapsto a_u(\cdot\ ; y,z)$$

is measurable for all $y,z \in V$ and all $u \in U_{ad}$ with respect to Lebesque measure on the interval $[0,T]$

(iii)

/1.4/ $$a_u(t;y,y) \geqslant \alpha \|y\|_V^2 , \quad \alpha > 0 ,$$

$$\forall u \in U_{ad} , \quad \forall t \in [0,T] , \quad \forall y \in V$$

Following [3] we denote by $W(0,T) \subset L^2(0,T;V)$ a Hilbert space with the scalar product :

/1.5/ $$(y,z)_{W(0,T)} = \int_0^T \left\{ \left(\frac{dy}{dt} , \frac{dz}{dt}\right)_{V'} + (y,z)_V \right\} dt$$

where $(\cdot\ ,\ \cdot)_V$ denotes scalar product in V.

Let there be given elements $y_0 \in H$ and $f \in L^2(0,T;V')$.

For given control $u \in U_{ad}$ we define state trajectory $y_u \in W(0,T)$ as the solution of an abstract parabolic equation of the form :

/1.6/ $$\left(\frac{dy_u}{dt}(t), z\right)_{V'V} + a_u(t;y_u(t),z)$$

$$= (f(t),z)_{V'V} , \quad \forall z \in V, \quad \text{a.e. in }]0,T[$$

/1.7/ $y_u(0) = y_0$

where $(\cdot\,,\,\cdot)_{V'V}$ denotes scalar product between V' and V.
Under the above assumptions problem /1.6/, /1.7/ has the unique solution [4] which continuously depends on the data $y_0 \in H$ and $f \in L^2(0,T;V)$.

2. Observation in $C(0,T;H)$

Let us define on the set $U_{ad} \subset U$ the cost functional

/2.1/ $$J(u) = \frac{1}{2} \| y_u(T) - z_d \|_H^2 + \frac{\varepsilon}{2} \| u \|_U^2$$

where

$$y_u(T) = y_u \Big|_{t=T}$$

and $z_d \in H$ is a given element.
We shall consider the following minimization problem :

/2.2/ $$\text{find} \quad \inf_{u \in U_{ad}} J(u)$$

Lemma :

If we assume that :

(i) bilinear form /1.2/ is Lipschitzian with respect to u, that is

/2.3/ $$| a_{u1}(t;y,z) - a_{u2}(t;y,z)| \leqslant C \| u_1 - u_2 \|_U \|y\|_V \|z\|_V$$

$$\forall t \in [0,T], \quad \forall u_1, u_2 \in U_{ad}, \quad \forall y,z \in V$$

(ii) set $U_{ad} \subset U$ is compact

then there exists a solution $u \in U_{ad}$ to the problem /2.2/.
Proof is given in [8].
Let us assume that at the point $u \in U_{ad}$ there exists Frechet de-

rivative $\tilde{A}_u(\hat{u})$ of the operator $\tilde{A}(u) \in \mathcal{L}(L^2(0,T;V) ; L^2(0,T;V'))$ which is defined by the equality

$$/2.4/ \qquad ((\tilde{A}(u))(t)\, y,z)_{V'V} = a_u(t;y,z)$$
$$\forall u \in U_{ad} \quad , \quad \forall t \in [0,T] \quad , \quad \forall y,z \in V$$

To obtain a simple form of necessary conditions of optimality we introduce adjoint state $p_u \in W(0,T)$ which is defined as the solution of adjoint state equation :

$$/2.5/ \qquad -\left(\frac{dp_u}{dt}(t),z\right)_{V'V} + a_u(t,z,p_u(t)) = 0$$
$$\forall z \in V \quad , \quad \text{a.e. in }]0,T[$$

$$/2.6/ \qquad p_u(T) = -\, y_u(T) + z_d$$

Optimal control $\hat{u} \in U_{ad}$ is characterized [2] , [8] by inequality :

$$/2.7/ \qquad \int_0^T (\langle \tilde{A}_u(\hat{u}), u-u\rangle\, y_{\hat{u}}(t),\, p_{\hat{u}}(t))_{V'V}\, dt \geqslant 0 \quad ,$$
$$\forall u \subset U_{ad}$$

3. Observation in $C(0,T;V)$

In order to consider the case where cost dunctional is defined on $C(0,T;V)$ we have to use some representation of bilinear form /1.2/ . To do that we need some additional definictions.

Let there be given a Hilbert space S and linear, bounded operator $\gamma \in \mathcal{L}(V,S)$. We assume

/3.1/ operator γ maps V onto S

/3.2/ kernel $\ker \gamma = V_0$ is dense in H

It is easy to show that for given element $\varphi \in L^2(0,T;S')$ linear functional

/3.3/ $$V \ni z \longmapsto (\varphi(t), \ \gamma z)_{S'S} \in R^1$$

is continuous a.e. in $]0,T[$.

With the form /1.2/ we associate the so called formal operator :

/3.4/ $$A(u) \in \mathcal{L}(L^2(0,T;V) \ ; \ L^2(0,T;V_0'))$$

whichis defined by the formula

/3.5/ $$\int_0^T a_u(t,y(t),z(t))\,dt = \int_0^T ((A(u))(t)\,y(t), \ z(t))_{V_0' V_0} \ dt$$
$$\forall z \in L^2(0,T;V_0), \qquad \forall y \in L^2(0,T;V)$$

Furthermore we assume that the domain x/ of operator $A(u)$ considered as an unbounded operator in $L^2(0,T;H)$ is a space $L^2(0,T;D)$ where $D \subset V$ is a given Hilbert space.

It can be shown [1] that there exists the unique operator, called Neumann operator :

/3.6/ $$\sigma(u) \in \mathcal{L}(L^2(0,T \ ;D) \ ; \ L^2(0,T;S'))$$

such that the following representation takes place :

/3.7/ $$\int_0^T a_u(t \ ; \ y(t), \ z(t))\,dt$$
$$= (A(u)\,y,z)_{\mathcal{H}} + ((\sigma(u)\,y \ , \ \tilde{\gamma}\,z))$$
$$\forall y \in L^2(0,T \ ;D) \ , \qquad \forall z \in L^2(0,T;V)$$

where

$$\mathcal{H} = L^2(0,T;H)$$

$$(\tilde{\gamma}\,z)(t) = \gamma z(t) \qquad \text{a.e. in }]0,T[$$

and $\tilde{\gamma} \in \mathcal{L}(L^2(0,T;V); \ L^2(0,T;S'))$

$((\cdot \ , \ \cdot))$ denotes scalar product between $L^2(0,T;S)$

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x/ It is the set of elements $y \in L^2(0,T;V)$ such that $A(u)y \in L^2(0,T;H)$.

and $L^2(0,T;S)$.

Let us assume that there exist Hilbert spaces:

/3.8/ (i) $W^1(0,T) \subset C(0,T;V)$,

injection $W^1(0,T) \to C(0,T;V)$ is continuous

/3.9/ (ii) $Y \subset L^2(0,T;S')$

injection $Y \longrightarrow L^2(0,T;S')$ is continuous

such that for any given

/3.10/ $y_0 \in V$

/3.11/ $f \in L^2(0,T;H)$

/3.12/ $\varphi \in Y$

the following state equation has the unique solution $y_u \in W^1(0,T)$:

/3.13/
$$\left(\frac{dy_u}{dt}(t), z \right)_{V'V} + a_u(t; y_u(t), z)$$
$$= (f(t), z)_H + (\varphi(t), \gamma z)_{S'S} , \qquad \forall z \in V$$
a.e. in $]0,T[$

/3.14/ $y_u(0) = y_0$

Remark

Problem of existence of the spaces /3.8/, /3.9/ is discussed in [9] . Using /3.7/ we obtain another representation [9] of the system /3.13/, /3.14/ namely

/3.15/
$$\frac{dy_u}{dt} + A(u)y = f$$
$$\sigma(u)\, y_u = \varphi$$
$$y_u(0) = y_0$$

We introduce an optimization problem similar to that in previous section. We define cost functional

/3.16/ $$J(u)= \frac{1}{2} \left\| y_u(T)- z_d \right\|_V^2 + \frac{\varepsilon}{2} \left\| u \right\|_U^2 , \quad \varepsilon \geqslant 0$$

where $z_d \in V$ is a given element.

We consider the problem of minimization of the cost functional on a given convex set $U_{ad} \subset U$.

If mappings

/3.17/ $$U \supset U_{ad} \ni u \longmapsto A(u) \in \mathcal{L}(W^1(0,T);L^2(0,T;H))$$

/3.18/ $$U \supset U_{ad} \ni u \longmapsto \sigma(u) \in \mathcal{L}(W^1(0,T);Y)$$

are locally Lipschitzian and the set $U_{ad} \subset U$ is compact then there exists an optimal control $\hat{u} \in U_{ad}$ such that

/3.19/ $$J(\hat{u}) \leqslant J(u), \qquad \forall u \in U_{ad}$$

Proof is given in [9] .

To obtain a simple form of necessary conditions of optimality we introduce [8] the so called generalized adjoint state $(p,r) \in L^2(0,T;H) \times Y$ defined at the given point $u \in U_{ad}$ as a solution of generalized adjoint state equation of the form :

/3.20/ $$\left(p, \frac{dw}{dt} + A(u)w \right)_{\mathcal{H}} + \left(r, \sigma(u)w \right)_Y = - \left(y_u(T)- z_d, w(T) \right)_V , \qquad \forall w \in W_0^1(0,T)$$

where

$$W_0^1(0,T) = \left\{ w \in W^1(0,T) \mid w|_{t=0} = 0 \right\}$$

It can be shown [8] , that there exists the unique solution of /3.20/ for each $u \in U_{ad}$.

If $\hat{u} \in U_{ad}$ is an optimal control, mappings /3.17/, /3.18/ are

Frechet differentiable and $(\hat{p},\hat{r})$ is a generalized adjoint state at $\hat{u} \in U_{ad}$ then necessary conditions of optimality takes on the form :

/3.21/
$$\left(\langle A_u(\hat{u}); u-u\rangle \; y_{\hat{u}}, \hat{p}\right)_{\mathcal{H}}$$
$$+\left(\langle \sigma_u(\hat{u}); u-\hat{u}\rangle \; y_{\hat{u}}, \hat{r}\right)_Y + \varepsilon(\hat{u}, u-\hat{u})_U \geqslant 0 \quad , \quad \forall u \in U_{ad}$$

where $A_u(\hat{u})$, $\sigma_u(\hat{u})$ denotes Frechet derivatives of mappings /3.17/ /3.18/ taken at optimal point $\hat{u} \in U_{ad}$.

Example :

Let Ω be an open region in R^n with smooth boundary $\Gamma = \partial\Omega$

We introduce the following functional spaces :

(i) $V = H^1(\Omega)$, $H = L^2(\Omega)$

(ii) $W^1(0,T) = H^{2,1}(Q)$

where $Q = \Omega \times]0,T[$, $T > 0$

(iii) $Y = H^{1/2,1/4}(\Sigma)$

where $\Sigma = \Gamma \times]0,T[$

(iv) $D = H^1(\Omega;\Delta)$

where
$$H^1(\Omega;\Delta) = \{ y \in H^1(\Omega) | \; y \in L^2(\Omega) \}$$
$$S = H^{1/2}(\Gamma)$$

We define the set of admissible controls $U_{ad} \subset U = H^2(0,T)$ as the set of solutions of ordinary differential equation

$$\begin{cases} \dfrac{du}{dt} = -a_1 u + v_1 \\ \dfrac{dv_1}{dt} = -a_2 v_1 + v \end{cases}$$
$$u(0) = v_1(0) = 0$$

for all $v \in L^2(0,T)$ such that

$$0 \leqslant v(t) \leqslant 1 \qquad \text{a.e. in }]0,T[$$

where $a_1, a_2 > 0$ are given constants.

Let there be given real functions $F(.)$, $g(.)$ such that

(i) $F(.)\ ,\ g(.) \in C^2[0,1]$

(ii) $F(r) \geqslant \alpha > 0 \ , \quad \forall r \in [0,1]$

$g(r) \geqslant 0 \ , \quad \forall r \in [0,1]$

We introduce the state equation of the form :

$$\int_{\Omega} \frac{\partial y_u}{\partial t} z \, d\Omega + F(u(t)) \sum_{i=1}^{n} \int_{\Omega} \frac{\partial y_u}{\partial x_i} \frac{\partial z}{\partial x_i} \, d\Omega$$

$$+ g(u(t)) \int_{\Gamma} y_u z \, d\Gamma = \int_{\Omega} f z \, d\Omega + \int_{\Gamma} \varphi z \, d\Gamma$$

$$\forall z \in H^1(\Omega), \qquad \text{a.e in }]0,T[$$

$$y_u(x,0) = y_0(x) \ , \qquad x \in \Omega$$

It can be shown that for given

$$\varphi \in H^{1/2,1/4}(\Sigma)$$

$$f \in L^2(Q)$$

$$y_0 \in H^1(\Omega)$$

$$u \in U_{ad}$$

There exists the unique solution $y_u \in H^{2,1}(Q)$ of the above problem.

In this case operators /3.17/, /3.18/ have the form

$$A(u)y = F(u)\,\Delta y \ , \qquad y \in D = H^1(\Omega;\Delta)$$

$$\sigma(u)y = \frac{\partial y}{\partial n} + g(u)\,y$$

where $\frac{\partial}{\partial n}$ denotes normal derivative to the boundary $\Gamma = \partial\Omega$

We introduce the cos t functional of the form

$$J(u) = \frac{1}{2} \| y_u(T) - z_d \|^2_{H^1(\Omega)} + \frac{\varepsilon}{2} \| u \|^2_{H^2(0,T)}$$

where $\varepsilon \geqslant 0$ and $z_d \in H^1(\Omega)$ is a given element.

For any given $\varepsilon \geqslant 0$ there exists [9] an optimal control $\hat{u} \in U_{ad}$ which is characterized by the following inequality :

$$\int_Q \left[\frac{dF}{du}(\hat{u})(u-u)\right] \Delta y_{\hat{u}} \hat{p} \, dQ$$

$$+ \int_\Sigma \left[\frac{dF}{du}(\hat{u}) \frac{\partial y_{\hat{u}}}{\partial n} + \frac{dg}{du}(\hat{u}) \, y_{\hat{u}}\right](u-\hat{u})\hat{r} \, d\Sigma$$

$$+ \varepsilon(\hat{u}, u-\hat{u})_{H^2(0,T)} \geqslant 0 \quad , \qquad \forall u \in U_{ad}$$

where $(\hat{p},\hat{r}) \in L^2(Q) \times (H^{1/2,1/4}(\Sigma))'$ is the unique solution [9] of the problem :

$$\int_Q \left(\frac{dw}{dt} + F(\hat{u}) \Delta w\right) \hat{p} \, dQ$$

$$+ \int_\Sigma \left(F(\hat{u}) \frac{\partial w}{\partial n} + g(\hat{u}) \, w\right) \hat{r} \, d\Sigma$$

$$= - \left(y_u(\cdot \, ,T) - z_d \, , \, w(\cdot \, , \, T)\right)_{H^1(\Omega)} \qquad \forall w \in H^{2,1}(Q)$$

such that $w(\cdot, 0) = 0$

where we use the same notation for the scalar product in $L^2(\Sigma)$ and the scalar product between $H^{1/2,1/4}(\Sigma)$ and $(H^{1/2,1/4}(\Sigma))'$.

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