AN INTRODUCTION TO BOUNDED RATE SYSTEMS

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SUMMARY

In this work a new class of nonlinear systems is introduced, for which the denomination "bounded rate systems" is proposed. This class appears to be quite relevant for its capability of modeling important physical phenomena in different field such as biology, ecology, engineering.

Bounded rate systems situate between bilinear and linear-in-control systems, so that a bounded rate system theory may be usefully investigated and developed exploiting already available results.

1. A NEW CLASS OF NONLINEAR SYSTEMS

The need of introducing nonlinear systems in modeling the physical world is presently an unquestioned fact especially in a number of important nontechnical fields like biology, ecology, socio-economics.

It is equally well know that the main obstacle in this direction is the difficulties presented by the analytical study of a general nonlinear system.

This situation motivates the actual trend, to spot out specific classes of nonlinear systems which possibly couple the advantages of not too difficult analytical study to the ability of modeling relevant classes of phenomena.

A noteworth step in this direction was the introduction of the class of bilinear systems, which in fact appears quite valid from an applicative point of view and in the mean time allowed to achieve a number of important theoretical results [1,2].

Another significant example of this trend was the study of a more general class of nonlinear systems, that is the systems linear in control for which relevant results are available expecially on the controllability and optimal control [1,3,4,5,6].

This paper is intended to give a futher contribution along the same line by introducing a new class of nonlinear systems. Denoting as usual by x the n-dimension state vector (1) and by u the p-dimension input vector, this class is defined as follows:

$$\dot{x}(t) = \phi(x) + N x u + Bu \tag{1.1}$$

where the operator $\phi : \Omega \to R^n$, Ω open subset of R^n, is assumed to be locally Lipschitzian (2) with at most a linear growth with x, i.e.

$$\|\phi(x)\| \leq c_1 \|x\| + c_2 \ , \qquad x \in \Omega \tag{1.2}$$

and $N: R^n \times R^p \to R^n$ and $B: R^p \to R^n$ are respectively bilinear and linear operators.

It is important to note that as it will be proved in Th.3.1, the assumptions on ϕ (local Lipschitziness and at most linear growth in Ω)

(1) The state vector is taken finite dimensional only for sake of simplicity. The extension to the infinite dimensional case is immediate.

(2) i.e. Lipschitzian within any closed bounded set in Ω.

guarantee existence and uniqueness of solutions of eq. (1.1) in Ω.They furthermore appear to be large enough to include all cases of practical interest.

This class situates between the ones of linear in control and bilinear systems. Indeed (1.1) is clearly linear in control with the restrictions that the free response (u = 0) has a bounded growth rate and the u dependent term is at most linear in x. On the other hand, it is shown in the Appendix that the assumptions on ϕ are equivalent to assume:

$$\phi(x) = F(x)x + f(x) \tag{1.3}$$

where the operators $F:\Omega \to R^{n\times n}$ and $f:\Omega \to R^n$ are locally Lipschitzian with uniformly bounded range. This enlightens the structure of ϕ and at the same time shows that (1.1) may be also considered as a bilinear system with an instantaneous, locally Lipschitzian and uniformly bounded state feedback h (Fig.1), as soon as one defines

$$F(x) = A + N'h(x) \tag{1.4}$$

$$f(x) = B\,h(x) \tag{1.5}$$

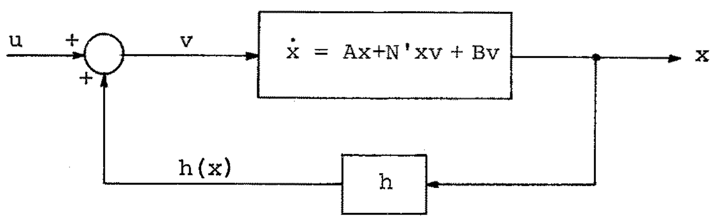

Fig.1

Due to the result (1.3), the considered class of systems may also be represented by the general form:

$$\dot{x}(t) = F(x)x + f(x) + Nxu + Bu \tag{1.6}$$

It is interesting to point out that the u-dependent part of the dynamics in (1.6) has essentially the same behaviour as in linear in control and bilinear systems, while the free response dynamic is indeed characteristic of the new class of systems. In fact it is neither linear as in bilinear systems nor a general nonlinear function as in linear in control systems but it is allowed to have at most a linear growth with x. For this reason in the following we shall denote the system (1.6) *bounded rate system.*

The motivation for the study of bounded rate systems is twofold. First of all it is important to underline the relevance of such a class of systems from an *applicative point of view*. Indeed, as is also shown by the few examples of sect.2, the bounded rate model (1.6) play a fundamental role in a number of important fields like in biology, ecology, engineering, socioeconomics.

Second, because of the structure of bounded rate systems at middle way between linear-in-control and bilinear systems, it is expected that important *theoretical results* may be achieved for such a class of systems both exploiting known results for linear-in-control systems and suitably extending bilinear systems properties.

As far as the case of bilinear systems in a feedback loop is concerned, it is important to note that if the feedback operator h in Fig. 1 is taken to be linear a *quadratic system* is arrived at.

Quadratic systems may be considered bounded rate within any bounded set in the state space, so that the local properties of the two classes must agree. In particular, if one proves that all the possible solutions for a given quadratic system must be confined within a bounded set in the state space,that system may be regarded as a bounded rate system. The global properties may well be different in that a quadratic system in large (i.e. a bilinear system with a feedback which is linear without saturations on the whole state space) is not obviously bounded rate. However such a feedback is not much realistic from an applicative point of view and at the same time creates some difficulty in proving global existence and uniqueness properties of the solution.

2. SOME EXAMPLES OF BOUNDED RATE SYSTEMS

In this section we intend to show how several natural and technical processes may be conveniently described by means of bounded rate models. We shall consider some important examples in the fields of bio chemistry, population dynamics, and engineering.

2.1. Immune response

The immune response is the sequence of those phenomena which in a mammal are triggered by the injection of a foreign substance (antigen) and lead to the production of specific proteins (antibodies) able to bind the antigen and to neutralize it.

As shown in [8,9], under suitable assumptions, the immune response may be described by the following set of equations:

$$\frac{\partial C(K,t)}{\partial t} = \alpha_c \frac{1-KH}{1+KH} p_s(KH)C(K,t) - \frac{1}{\tau_c} C(K,t) + \beta \bar{p}_c(K) \tag{2.1}$$

$$\frac{\partial C_p(K,t)}{\partial t} = 2\alpha_c \frac{KH}{1+KH} p_s(KH)C(K,t) - \frac{1}{\tau_p} C_p(K,t) \tag{2.2}$$

$$\frac{\partial S(K,t)}{\partial t} = \alpha_s C_p(K,t) + \alpha_s' C(K,t) + 2\alpha_c \int_{-\infty}^{t} i(t-\theta) \frac{p_s(KH)}{1+KH} C(K,\theta)d\theta +$$

$$- Kc(K)S(K,t)\ H(t) + c(K)B(K,t) - \frac{1}{\tau_s} S(K,t) \tag{2.3}$$

$$\frac{\partial B(K,t)}{\partial t} = Kc(K)S(K,t)H(t) - c(K)B(K,t) - \frac{1}{\tau_B} B(K,t) \tag{2.4}$$

$$\frac{dH(t)}{dt} = -H(t) \int_{K_1}^{K_2} Kc(K)S(K,t)dK + \int_{K_1}^{K_2} c(K)B(K,t)dK - \frac{1}{\tau_H} H(t) +$$

$$+ \dot{H}_i(t) \tag{2.5}$$

where:

$C(K,t)$, $C_p(K,t)$, $S(K,t)$, $B(K,t)$ are respectively the concentration densities of immunocompetent and memory cells, plasma cells, antibody sites, immune complex, at time t, for a given affinity constant K (K ranges from K_1 to K_2);

$H(t)$, $H_i(t)$ are respectively the antigen concentration at time t and the amount of antigen introduced per unit volume of circulating fluids up to time t;

$p_s(KH)$ is the probability that a cell with affinity K be stimulated;

$\bar{p}_c(K)$ is the original distribution of immunocompetent cells with respect to K;

α_c is the proliferation rate constant of stimulated cells;

β is the production rate of immunocompetent cells from stem cells;

α_s, α_s' are respectively the rate constant of antibody production by plasma cells and the basal rate constant of antibody production by memory cells;

$i(t)$ is the additional antibody production intensity by a memory cell generated t instants ahead;

$c(K)$ is the dissociation reaction rate constant of the immune complex;

$\tau_c, \tau_p, \tau_s, \tau_B, \tau_H$ are the time constants for death or removal of the various species.

The set of equations (2.1)-(2.5) defines a nonlinear dynamical model for the immune response which is distributed with respect to K. In it $\bar{p}_c(K)$ and $\dot{H}_i(t)$ are two independent variables, which may be considered as inputs:

$$u(t) = \begin{bmatrix} \bar{p}_c(\cdot) \\ \dot{H}_i(t) \end{bmatrix} \tag{2.6}$$

Let us define the state vector x(t):

$$x(t) = [C(\cdot,t)\, C_p(\cdot,t)\, S(\cdot,t)\, B(\cdot,t)\, \tilde{x}^T(\cdot,t)\, H(t)]^T \tag{2.7}$$

where $\tilde{x}(\cdot,t)$ is the state vector for the linear time invariant system with impulse response i(t). From (2.1)-(2.5) we see that the u-dependent part is linear in u and independent of x, while the free response part includes products of state variables by suitable functions of x. Thus the set of equations (2.1)-(2.7) may be given the form:

$$\dot{x}(t) = F(x)\, x + Bu \tag{2.8}$$

where B is a linear time invariant operator. As far as the operator F is concerned, it may be proved [9] that all above mentioned functions of x are locally Lipschitzian and with uniformly bounded range, as soon as we restrict them to a suitably defined open subset Ω of the state space X. Furthermore, it was also proved that, choosing as initial state $x(t_o)$ any point in a given $S \subset \Omega$, possible solutions of (2.8) must evolve within S itself.

Therefore, as soon as $x(t_o)$ is chosen in S (which is the only possible choice in agreement with the physical meaning of x components), eq. (2.8) may be regarded as a particular case of (1.6).

2.2. *Enzyme Kinetics*

Let us denote by S,E,P,C respectively the substrate,the enzyme, the final reaction product and the substrate-enzyme complex. Once we keep the total amount of enzyme e_o and substrate s_o positive and constant, the general stoichiometric equation for the enzyme kinetics is [10]:

$$S + E \underset{K_{-1}}{\overset{K_1}{\rightleftarrows}} C \overset{K_2}{\rightarrow} P + E \tag{2.9}$$

where K_1, K_{-1}, K_2 are positive reaction rate constants. Denoting the concentrations by low case letters eq. (2.9) leads to the dynamical equation of Michaelis-Menten:

$$\begin{aligned} \dot{s} &= K_{-1}\, c - K_1\, s\, e \\ \dot{p} &= K_2\, c \\ e + c &= e_o \\ s + c + p &= s_o \end{aligned} \tag{2.10}$$

Defined:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ p \end{bmatrix} \tag{2.11}$$

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} s_o \\ e_o \end{bmatrix} \tag{2.12}$$

eq. (2.10) take the form:

$$\dot{x} = F(x)x + N\,x\,u + Bu \tag{2.13}$$

where:

$$F(x) = \begin{bmatrix} -K_{-1} - K_1(x_1+x_2) & -K_{-1} \\ -K_2 & -K_2 \end{bmatrix} \tag{2.14}$$

$$N_1 = \begin{bmatrix} K_1 & 0 \\ 0 & 0 \end{bmatrix} \quad ; \quad N_2 = \begin{bmatrix} -K_1 & 0 \\ 0 & 0 \end{bmatrix} \tag{2.15}$$

$$B = \begin{bmatrix} K_{-1} & 0 \\ K_2 & 0 \end{bmatrix} \tag{2.16}$$

The system (2.13) is a quadratic one and may be looked at as bounded rate as soon as we show that for any initial condition $x(t_o)$ in a bounded subset S of R^2, any possible solution stays in S. Indeed, defined:

$$S = \{x \in R^2 : x_1 \in [0, s_o],\ x_2 \in [0, s_o]\} \tag{2.17}$$

for $x(t_o) \in S$ let $\bar{t} \geq t_o$ the first time in which any of the two components x_1, x_2 changes its sign. Due to the continuity of solutions, we have the following cases:

a) $$x_1(\bar{t}) = 0 \quad , \quad \dot{x}_1(\bar{t}) < 0 \tag{2.18}$$

Recalling (2.13), this means:

$$K_{-1}(u_1(\bar{t}) - x_2(\bar{t})) < 0 \tag{2.19}$$

and therefore $x_2(\bar{t}) > s_o$. This implies the existence of a $0 < t' < \bar{t}$ such that:

$$x_2(t') = s_o \quad , \quad \dot{x}_2(t') \geq 0 \tag{2.20}$$

From (2.13) we then deduce:

$$-K_1 x_1 (t') \geq 0 \tag{2.21}$$

which is in contrast with (2.18).

By a similar proof, we can also reject the possibility that x_1 changes its sign in $\bar{t}$ through a horizontal tangent flex point.

b) $$x_2(\bar{t}) = 0 \quad , \quad \dot{x}_2(\bar{t}) \leq 0 \tag{2.22}$$

Recalling (2.13) this means:

$$x_1(\bar{t}) \geq u_1(\bar{t}) = s_o \tag{2.23}$$

This implies the existence of a $0 < t' < \bar{t}$ such that:

$$x_1(t') = s_o \quad , \quad \dot{x}_1(t') \geq 0 \tag{2.24}$$

From (2.13) we then deduce:

$$-x_2(t')(K_1 + K_{-1}) - K_1 s_o e_o \geq 0 \tag{2.25}$$

which is in contrast with (2.22).

c) $$x_1(\bar{t}) = x_2(\bar{t}) = 0, \ \dot{x}_1(\bar{t}) \leq 0 \ , \ \dot{x}_2(\bar{t}) \leq 0 \tag{2.26}$$

Recalling (2.13), this means:

$$K_2 u_1(\bar{t}) = K_2 s_o \leq 0 \tag{2.27}$$

which is in contrast with the positiveness of s_o.

By a similar analysis, it can also be rejected the possibility that $x_1(t) > s_o$, or $x_2(t) > s_o$, for $t \geq t_o$.

2.3. Bacterial growth

The continuous bacterial colture problem is a typical problem in biochemical engineering [11]. A dynamical model for that process which was proposed in [12] in connection with an optimal control problem is the following:

$$\begin{aligned} \dot{b} &= \left(\frac{s}{K_2 s^2 + s + K_1} - q \right) b \\ \dot{s} &= (1 - s) q - \frac{sb}{K_2 s^2 + s + K_1} \end{aligned} \tag{2.28}$$

where b, s are respectively the bacterial and nutritive substrate concentration in the growth vessel, q is the input and output flow rate and K_1, K_2 are suitable positive constants which determine the bacterial growth rate as a function of s. Defining:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b \\ s \end{bmatrix} \tag{2.29}$$

$$u = q \tag{2.30}$$

eq. (2.28) take the form:

$$\dot{x} = F(x)x + N x u + Bu \tag{2.31}$$

where:

$$F(x) = \begin{bmatrix} h(x_2) & 0 \\ -h(x_2) & 0 \end{bmatrix} \qquad h(x_2) = \frac{x_2}{K_2 x_2^2 + x_2 + K_1} \tag{2.32}$$

$$N = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.33}$$

As in previous cases, defined

$$S = \{x \in R^2 \ : \ x_1 \geq 0 \ , \ x_2 \geq 0\} \tag{2.34}$$

we first show that for $x(t_o) \in S$ and for non negative u, any possible solution of (2.31) stay in S. Indeed by integrating the first equation in (2.31) we obtain:

$$x_1(t) = x_1(t_o)\, e^{-\int_{t_o}^{t} [u(\tau) - h(x_2(\tau))]d\tau} \tag{2.35}$$

which for $x_1(t_o) \geq 0$ is always nonnegative. As far as x_2 is concerned, assume there exists a $\bar{t} \geq t_o$ such that:

$$x_2(\bar{t}) = 0 \quad , \quad \dot{x}_2(\bar{t}) < 0 \tag{2.36}$$

From (2.31) it then follows $\dot{x}_2(\bar{t}) = u(\bar{t}) < 0$ which is in contrast with the assumed nonnegativeness of u. With a similar reasoning, we can also reject the possibility of x_2 changing its sign through a horizontal tangent flex point.

Finally, h and therefore F is easily seen to be uniformly Lipschitzian and with uniformly bounded range on an open set $\Omega \subset S$. In fact, the derivative of $h(x_2)$ vanishes for $x_2 = \pm \sqrt{K_1/K_2} = \pm \hat{x}_2$.

In the case $4K_1K_2 > 1$, h is uniformly Lipschitzian on $\Omega = R^2$

$$\sup_{x \in \Omega} |h(x_2)| = |h(-\hat{x}_2)| < \infty \tag{2.37}$$

On the contrary, in the case $4K_1K_2 \leq 1$, we may take

$$\Omega = \{x \in R^2 \; : \; x_2 > \frac{\bar{x}_2}{2}\} \tag{2.38}$$

where $\bar{x}_2$ is the greatest singular point of $h(x_2)$; again $\Omega \subset S$, h is uniformly Lipschitzian on Ω and

$$\sup_{x \in \Omega} |h(x_2)| = \max(|h(\frac{\bar{x}_2}{2})|, \; |h(+\hat{x}_2)|) < \infty \tag{2.39}$$

In conclusion, eq. (2.31) may be looked at as a bounded rate system, for any $x(t_o) \in S$ and nonnegative u.

2.4. Interacting Populations

A well-known and fairly general model to describe the dynamics of n interacting species is given by the Volterra-Lotka equations [13,14]

$$\dot{x}_i = K_i x_i + \sum_{1j}^{n} \alpha_{ij} x_j x_i \; , \qquad i = 1,2,\ldots,n \tag{2.40}$$

where the x_i's are the population numbers of the i-th species, the K_i's are real numbers, not all positive, representing the intrinsic increasing rates and the α_{ij} are antisimmetric real numbers ($\alpha_{ij} = -\alpha_{ji}$; $\alpha_{ii} = 0$) representing the "predation efficiency" of the i-th species on the j-th one.

The model (2.40) is clearly a quadratic one and may be rewritten as:

$$\dot{x} = F(x)x \tag{2.41}$$

where:

$$F(x) = \begin{bmatrix} K_1 + \alpha_1^T x & 0 & \cdots & 0 \\ 0 & K_2 + \alpha_2^T x & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & K_n + \alpha_n^T x \end{bmatrix} \tag{2.42}$$

$$\alpha_i^T = [\alpha_{i1} \ \alpha_{i2} \ \cdots \ \alpha_{in}] \tag{2.43}$$

The model (2.41) is therefore bounded rate as soon as we prove that all possible solutions with initial state within a given bounded set S, stay in S itself.

Indeed, it was proved that if $\det[\alpha_{ij}] \neq 0$ (which may happen only for n even) then, the system (2.40) admits a unique nontrivial equilibrium point $\bar{x}$, which is in $\bar{R}_n^+ = \{x : x_i > 0, \quad i = 1,2,\ldots,n\}$ and is stable.

Furthermore, a scalar function $\phi(x)$ may be defined on $\bar{R}_n^+$:

$$\phi(x) = \sum_1^n {}_i \ \bar{x}_i \left(\frac{x_i}{\bar{x}_i} - \ln \frac{x_i}{\bar{x}_i} \right) \tag{2.44}$$

which on each trajectory takes a constant value, greater than or equal to $\sum_1^n {}_i \ \bar{x}_i$. As a consequence, any trajectory starting in $\bar{R}_n^+$ stays in $\bar{R}_n^+$.

For each constant $c \geq \sum_1^n {}_i \ \bar{x}_i$, we now define the set:

$$S_c = \{x \in \bar{R}_n^+ : \phi(x) \leq c\} \tag{2.45}$$

Clearly, being ϕ constant along the trajectory, S_c contains all the trajectories starting from points belonging to S_c itself. Finally, S_c is bounded; indeed, if $x \in S_c$, it easily follows that:

$$\frac{x_i}{\bar{x}_i} - \ln \frac{x_i}{\bar{x}_i} \leq \frac{c}{\bar{x}_i} , \quad i = 1,2,\ldots,n \tag{2.46}$$

which implies a finite upper bound for each x_i.

2.5. Chemical reactor

The system under consideration is a continuous-flow stirred tank reactor, in which a single irreversible chemical reaction takes place. The dynamics of this system are easily described by material and heat balance equations [15]:

$$\dot{c} = \frac{F}{V}\left(c_o - c\right) - K_o \exp\left(-\frac{E}{RT}\right) c$$
$$\dot{T} = \frac{F}{V}\left(T_o - T\right) - \frac{U}{V\rho c_p}\left(T - T_k\right) + \frac{(-\Delta H)}{\rho c_p} K_o \exp\left(-\frac{E}{RT}\right) c \qquad (2.47)$$

where: c,T are respectively reactant concentration and reactor internal absolute temperature; c_o, T_o are the reactant concentration and temperature in the input flow; T_k is the coolant temperature. We also used the following notation:

F, input and output flow rate, assumed constant

V, reactor volume

$K = K_o \exp\left(-\frac{E}{RT}\right)$, specific reaction velocity constant at temperature T

U, product of area of the coolant surface in the coil and its heat transfer coefficient

$(-\Delta H) > 0$, enthalpy variation in the reaction, assumed exothermic

ρ, c_p, density and specific heat of the input flow.

We now define the constants:

$$K_1 = \frac{F}{V} \; ; \; K_2 = \frac{U}{V\rho c_p} \; ; \; K_3 = \frac{(-\Delta H)}{\rho c_p} \; ; \; K_4 = \frac{E}{R} \qquad (2.48)$$

and the state and input variables;

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c \\ T \end{bmatrix} \qquad (2.49)$$

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} c_o \\ T_o \\ T_k \end{bmatrix} \qquad (2.50)$$

Then eq. (2.47) take the form:

$$\dot{x} = F(x)x + Bu \qquad (2.51)$$

where:

$$F(x) = \begin{bmatrix} -(K_1 + K_o e^{-\frac{K_4}{x_2}}) & 0 \\ K_3 K_o e^{-\frac{K_4}{x_2}} & -(K_1 + K_2) \end{bmatrix} \qquad (2.52)$$

$$B = \begin{bmatrix} K_1 & 0 & 0 \\ & & \\ 0 & K_1 & K_2 \end{bmatrix} \tag{2.53}$$

The function F is locally Lipschitzian with uniformly bounded range in the open set $\Omega = \{x \in R^2 : x_2 > 0\}$. To prove that (2.54) is actually a bounded rate system, we now show that, defined:

$$S = \{x \in R^2 ;\ x_1 \geq 0,\ x_2 > 0\} \tag{2.55}$$

$$U = \{u : u_1(t) \geq 0,\ u_2(t) > 0,\ u_3(t) > 0\} \tag{2.56}$$

for any choice of $x(t_o)$ in S and of u in U (which are the only physically meaningful choices), any possible solution of (2.51) stays in S itself. Indeed, the first equation in (2.51) may be integrated out:

$$x_1(t) = x_1(t_o)\, e^{-\int_{t_o}^{t} (K_1 + K_o e^{-\frac{K_4}{x_2(\tau)}}) d\tau} +$$

$$+ \int_{t_o}^{t} e^{-\int_{\theta}^{t} (K_1 + K_o e^{-\frac{K_4}{x_2(\tau)}}) d\tau} K_1 u_1(\theta) d\theta \tag{2.57}$$

and therefore $x_1(t_o) \geq 0$ and $u_1(\theta) \geq 0$ imply $x_1(t) \geq 0$.

As far as x_2 is concerned, if we assume that there exists a time $\bar{t} \geq t_o$ such that:

$$x_2(\bar{t}) = 0 \quad , \quad \dot{x}_2(\bar{t}) \leq 0 \tag{2.58}$$

from the second equation in (2.51) we get:

$$K_1 u_2(\bar{t}) + K_2 u_3(\bar{t}) \leq 0 \tag{2.59}$$

which is in contrast with the assumed positiveness of $u_2(t)$, $u_3(t)$.

2.6. Nuclear reactor Kinetics

Other important examples of quadratic systems which are also bounded rate systems are the usually adopted models for the nuclear reactor kinetics.

As is extensively reported [16,17] a model for the free response of a point reactor with one group of delayed neutrons and one feedback region with Newton cooling is:

$$\dot{\nu} = - \frac{\alpha(T-T_o) + \beta}{l} \nu + \lambda c$$

$$\dot{c} = \frac{\beta}{l} \nu - \lambda c \qquad (2.60)$$

$$\dot{T} = K(\nu - \nu_o) - \gamma(T - T_o)$$

where ν is the neutron density, c is the precursor density, T is an average reactor temperature, with equilibrium values respectively ν_o, $c_o = \frac{\beta}{l \cdot \lambda} \nu_o$, T_o. Furthermore:

α = temperature coefficient of reactivity

β = delayed neutron fraction

λ = precursor decay constant

l = neutron generation time

$1/K$ = reactor heat capacity

$1/\gamma$ = mean time for heat transfer to the coolant

Defining:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \nu - \nu \\ c - c_o \\ T - T_o \end{bmatrix} \qquad (2.61)$$

equations (2.60) take the form:

$$\dot{x} = F(x)x \qquad (2.62)$$

where:

$$F(x) = \begin{bmatrix} -\frac{\beta}{l} & \lambda & -\frac{\alpha}{l}(\nu_o + x_1) \\ \frac{\beta}{l} & -\lambda & 0 \\ K & 0 & -\gamma \end{bmatrix} \qquad (2.63)$$

The system (2.62) is clearly a quadratic one so that all we have to prove is the uniform boundedness of its possible trajectories. For that purpose let us define the set S:

$$S = \{x \in R^3 : x_1 > -\nu_o,\ x_2 > -c_o\} \qquad (2.64)$$

and the scalar function V on S:

$$V(x) = \nu_o \left[\frac{x_1}{\nu_o} - \ln\left(1 + \frac{x_1}{\nu_o}\right)\right] + c_o\left[\frac{x_2}{c_o} - \ln\left(1+\frac{x_2}{c_o}\right)\right] + \frac{\alpha}{2K1}x_3^2 \qquad (2.65)$$

V is a Lyapunov function on S since V is positive definite on S and $\dot{V}$ is negative semidefinite along the trajectories of (2.62) which belong to $S^{(*)}$. Let us now define the subset $S_L \subset S$:

$$S_L = \{x \in R^3 : V(x) \leq L\} \qquad (2.66)$$

Recalling (2.65) we see that each of the three terms at the RHS is non negative in S, and therefore each of them is bounded above by L in S_L. This means that in S_L each of the three components of x belongs to a closed bounded interval.

Thus for any L, S_L is closed and bounded. Moreover, for any initial state $x(t_o) \in S_L$, all the possible trajectories stay in S_L. Indeed, would this not be the case, there should exist a $\bar{t}$ such that $x(t) \in S$, $t \leq \bar{t}$, and $x(\bar{t}) \in S - S_L$. But this is impossible since $x(t) \in S$ for $t \leq \bar{t}$ implies that $V(x(\bar{t}))$ cannot be increased with respect to the initial value; so that $V(x(\bar{t})) \leq L$ and therefore $x(\bar{t}) \in S_L$.

As a conclusion, the model (2.62) is bounded rate in S_L, for any $L < \infty$.

2.7. *Other examples*

Other important examples of bounded rate systems may be found in virtually all the applicative fields when a multiplicative control is indeed a bounded function of the state itself. This for instance may happen in macro-economics. In [18,19] a model is proposed for the growth of a national economy, which in [10] is interpreted as a bilinear system: the state is the vector of total national output and the multiplicative control is the matrix of coefficients of material inputs. As a matter of fact, this control may be taken as a bounded function of the state, thus generating a bounded rate system.

Another interesting case in ecology is the insect-pest control model [20] where the state vector is the insect population at different ages. The multiplicative control is build up by two terms. The first

(*) Despite of the semidefiniteness of $\dot{V}$, it has been shown [17] that the origin, which is the only equilibrium point of (2.62) in S, is actually globally asymptotically stable.

one is an intrinsic control which describes the effects of overcrowding and is a bounded function of the state; the second one is the external control implemented by releasing a sterilized male population.

A final example in engineering is the heat exchanger: the coolant flow rate which acts as a multiplicative control on the heat exchange process [2] may well be thought as a bounded function of the temperature (state) by means of an external feedback loop.

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

A first important theoretical result that may be established for a bounded rate system defined by (1.6) :

$$\dot{x} = F(x)x + f(x) + Nxu + Bu \tag{1.6}$$

is related to the problem of existence and uniqueness of its solution. We have the following:

THEOREM 3.1. *Let us consider a bounded rate system (1.6) with the assumption: there exists a subset* $S \subset \Omega$ *and class* U *of continuous input functions such that for any initial condition* $x_o = x(t_o) \in S$ *and any* $u \in U$ *any possible solution of (1.6) takes value* $x(t) \in S$, $\forall t \geq t_o$.

Then for any $x_o \in S$, $u \in U$, *eq. (1.6) admite a unique solution for all* $t \geq t_o$.

PROOF. First of all, note that, being F, f locally Lipschitzian and uniformly bounded in $\Omega \supset S$, and being u continuous for any $(t_o,x_o) \in R^1 xS$, there exists a $R_o = \{(t,x): |t-t_o| \leq \alpha, \|x-x_o\| \leq \beta\}$ such that:

a) $R_o \subset \Omega$

b) the RHS of (1.6), in R_o, is uniformly bounded, continuous in t for each fixed x and uniformly Lipschitzian in x.

Therefore (1.6) admits a local unique solution through (t_o,x_o).

Furthermore the RHS of (1.6) in $Rx\Omega$ is continuous and recalling (1.2) is such that:

$$\|F(x(t))x(t) + f(x(t)) + Nx(t)u(t) + Bu(t)\| \leq C_1 \|x(t)\| + C_2 +$$

$$+ \|N\| \; \|u(t)\| \; \|x(t)\| + \|B\| \; \|u(t)\| =$$

$$= K_1(t)\|x(t)\| + K_2(t) \tag{3.1}$$

where $K_1(t)$, $K_2(t)$ are suitable scalar positive continuous functions of t.

The scalar equation:

$$\dot{z} = K_1(t)z + K_2(t) \tag{3.2}$$

admits a unique solution for any initial value $z(t_o) \in R$ and for all $t \geq t_o$. Consequently, due to Thm. 5.6.1 in [7] any local solution of (1.6) may be uniquely extended for all $t \geq t_o$.

REMARK 1. All the previously mentioned examples of bounded rate systems satisfy the assumptions of Thm. 3.1. and therefore admit a unique solution in large.

REMARK 2. A similar theorem, for systems evolving on group, is proved in [21]. The set of assumptions seems to be similar to that one of Thm. 3.1. and somehow more restrictive than the latter, in those cases in which both are applicable.

4. CONCLUSIONS

In the previous section we enlightened the relevance of the class of bounded rate systems. We showed how a number of important physical processes may be included in that class.

From the theoretical point of view a first general result for bounded rate systems was given in Sect. 3. We must however point out that the theory for such a class of systems is still to be developed and in our mind this should be an important task to be pursued.

In particular results on stability, controllability and optimal control would be useful. As far as stability is concerned, a first contribution was given in [9] where asymptotic stability for the immune response model was proved exploiting the general structure of a bounded rate system as a bilinear system in a uniformly bounded feedback loop.

This is a good example of how to achieve theoretical results for bounded rate systems by suitably exploiting already available results for bilinear systems.

As far as controllability is concerned, a number of important results are already available for the class of linear in control systems [3,4,5,6,21,24] which hopefully will be a useful starting point to build a controllability theory for bounded rate systems.

We also mention some papers [25,26] in which it is shown how to derive controllability properties by looking only at the u-dependent part of the state equation. Under this aspect bounded rate systems behave as bilinear systems, for which some results on controllability theory are already available [1].

APPENDIX

In order to prove eq. (1.3) we may state the following

THEOREM. *Given an open subset* Ω *of* R^n *a necessary and sufficient condition for a function* $\phi : \Omega \to R^n$ *to be locally Lipschitzian with the growth property:*

$$\|\phi(x)\| \leq C_1 \|x\| + C_2 \quad , \quad \forall x \in \Omega \tag{A.1}$$

where C_1, C_2 *are nonnegative constants, is that there exist two functions* $F : \Omega \to R^{n\times n}$, $f : \Omega \to R^n$ *which are locally Lipschitzian and with uniformly bounded range such that:*

$$\phi(x) = F(x)x + f(x) \quad , \quad \forall x \in \Omega \tag{A.2}$$

PROOF. *Sufficiency.* If (A.2) holds, then for x_1, x_2 in any closed bounded subset $M \subset \Omega$ we have:

$$\begin{aligned}\|\phi(x_1) - \phi(x_2)\| &= \|F(x_1)x_1 + f(x_1) - F(x_2)x_2 - f(x_2)\| \\ &\leq \|F(x_1)\| \ \|x_1 - x_2\| + \|F(x_1) - F(x_2)\| \ \|x_2\| + \|f(x_1) - f(x_2)\| \\ &\leq \max_{x_1 \in M}\|F(x_1)\| \ \|x_1 - x_2\| + \max_{x_2 \in M}\|x_2\| \ L_F \ \|x_1 - x_2\| + L_f\|x_1 - x_2\| \\ &\leq L_\phi \|x_1 - x_2\| \end{aligned} \tag{A.3}$$

where L_F, L_f are the Lipschitz constants of F, f in M. Consequently ϕ is locally Lipschitzian in Ω. As far as (A.1) is concerned, we have:

$$\|\phi(x)\| = \|F(x)x + f(x)\| \leq \|F(x)\| \ \|x\| + \|f(x)\| \leq C_1\|x\| + C_2 \qquad \forall x \in \Omega \tag{A.4}$$

where C_1, C_2 are the finite upper bounds for $\|F(x)\|$, $\|f(x)\|$ in Ω.

Necessity. Denoting by ϕ_i the i-th component of ϕ, it obviously is locally Lipschitzian and inequality (A.1) implies:

$$|\phi_i(x)| \leq C_1 \|x\| + C_2 \quad , \quad i = 1,2,\dots,n \quad , \quad \forall x \in \Omega \tag{A.5}$$

so that necessity of (A.2) is proved as soon as we prove for each i the existence of two functions $F_i : \Omega \to R^n$, $f_i : \Omega \to R^1$, locally Lipschitzian and with uniformly bounded range, such that

$$\phi_i(x) = F_i^T(x)x + f_i(x) \qquad \forall x \in \Omega \tag{A.6}$$

Eq. (A.6) will now be proved by means of a constructive procedure.

For a fixed $\rho > 0$ we construct the function $f_i : \Omega \to R^1$ in the following way:

$$f_i(x) = \begin{cases} \phi_i(x) & , \quad x \in \Omega_i' = \{x \in \Omega : -C_1\rho - C_2 \leq \phi_i(x) \leq C_1\rho + C_2\} \\ C_1\rho + C_2 & , \quad x \in \Omega_i'' = \{x \in \Omega : \phi_i(x) > C_1\rho + C_2\} \\ -C_1\rho - C_2 & , \quad x \in \Omega_i''' = \{x \in \Omega : \phi_i(x) \leq -C_1\rho - C_2\} \end{cases} \tag{A.7}$$

By construction f_i has a uniformly bounded range. Moreover, it is also locally Lipschitzian in Ω since for any choice of $x_1, x_2 \in \Omega$:

$$|f_i(x_1) - f_i(x_2)| \leq |\phi_i(x_1) - \phi_i(x_2)| \tag{A.8}$$

We now observe that the distance of $\Omega_i'' \cup \Omega_i'''$ from the origin cannot be less than ρ. In fact, recalling (A.5), (A.7), if $x \in \Omega_i'' \cup \Omega_i'''$, than

$$C_1\|x\| + C_2 \geq |\phi_i(x)| > C_1\rho + C_2 \to \|x\| > \rho \tag{A.9}$$

Therefore the function $F_i : \Omega \to R^n$:

$$F_i(x) = \begin{cases} 0 & , \quad x \in \Omega_i' \\ \dfrac{\phi_i(x) - f_i(x)}{x^T x}\, x & , \quad x \in \Omega_i'' \cup \Omega_i''' \end{cases} \tag{A.10}$$

is well defined.

The functions f_i, F_i defined in (A.7), (A.10) already satisfy (A.6) in all Ω. Due to the growth property (A.5) F_i turns out to have a uniformly bounded range; indeed in Ω_i' we have $F_i(x) = 0$ and in $\Omega_i'' \cup \Omega_i'''$ we have:

$$\|F_i(x)\| = \frac{|\phi_i(x) - f_i(x)|}{\|x\|^2}\|x\| \le \frac{|\phi_i(x)|+|f_i(x)|}{\|x\|} \le$$

$$\le \frac{C_1\|x\| + C_2 + C_1\rho + C_2}{\|x\|} \le 2C_1 + \frac{2C_2}{\rho} \tag{A.11}$$

Finally because of the definition (A.10), F_i is a continuous function in Ω which in particular is a constant on Ω_i', while in $\Omega_i'' \cup \Omega_i'''$ is the product of locally Lipschitzian functions ($\phi_i(x)-f_i(x)$ and x) by a continuous uniformly bounded function $\left(\frac{1}{x^T x}\right)$. Therefore F_i is easily seen to be locally Lipschitzian on Ω.

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