A DECOMPOSITION TECHNIQUE IN INTEGER LINEAR PROGRAMMING

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1. INTRODUCTION

The size of linear integer programming problems that can be successfully solved is generally not very large and only a relatively small number of integer variables can be considered.

In fact the normally used packages are conceived for about 150-300 variables.

The techniques normally used to solve larger integer or mixed integer problems using a branch and bound search method, are based on the "penalty" approach and the choice of suitable lower and upper bounds for the optimal value of objective function. In order to obtain such bounds the Gomory's group theoretic methods together with Lagrange multipliers have been used in many works |1,2,3,4,5,6,7,8,13,15,16|.

The solution procedure proposed in the present work makes use of a decomposition technique that generates a number of subproblems of the original one.

Let the problem be written as:

$$\begin{aligned} & \text{minimize } z, \\ & z = c^T x \\ & \text{s.t.: } Ax = b \\ & x \geq 0, \text{ integer} \end{aligned} \tag{1}$$

where A is a matrix of rank m of order m×n, (m<n), x and c are n-vectors and b is an m-vector. Further, let A be partitioned as B and N, B being the optimal linear programming basis. Vectors x and c are similarly partitioned into x_B, x_N, c_B and c_N, respectively. Without loss of generality, assume that all the coefficient of A and b are integer. (This is equivalent to assuming that A and b consist of rational numbers).

Expression (1) may be written as follows:

$$
\begin{aligned}
&\text{minimize } z, \\
&z = c_B^T x_B + c_N^T x_N \\
&\text{s.t.: } Bx_B + Nx_N = b \\
&x_B,\ x_N \geq 0, \text{ integers}
\end{aligned}
\qquad (2)
$$

where B is of order m×m and nonsingular N is of order m×(n-m), c_B and x_B are of order m×1, and c_N and x_N are of order (n-m)×1.

Consider the linear programming problem (2) in the updated form (3):

$$
\begin{aligned}
&\text{minimize } z, \\
&z = (c_N^T - c_B^T B^{-1} N) x_N + c_B^T B^{-1} b \\
&\text{s.t.: } x_B + B^{-1} N x_N = B^{-1} b \\
&x_B,\ x_N \geq 0
\end{aligned}
\qquad (3)
$$

The optimal conditions of a linear programming problem, $c_N^T - c_B^T B^{-1} N \geq 0$, must be satisfied, and the non integer optimum is $x_B = B^{-1}b$ and $x_N = 0$. In all but trivial cases, $B^{-1}b$ will not be all-integer.

Therefore the strategy for finding an integer optimum will be to examine certain solutions of the set $x_N \geq 0$ and integer.

2. THE GROUP THEORETIC APPROACH

There are three problems in examining the solutions of the set $\{x_N \geq 0 \ , \ \text{integer}\}$ in general:

1) $x_N \geq 0$ and integer are not sufficient to assure that x_B will be integer;
2) $x_N \geq 0$ and integer are not sufficient to assure that the inequalities $x_B \geq 0$ will be satisfied;
3) $x_N \geq 0$ and integer are not sufficient to assure the optimality of an integral solution to (3).

When a solution $x_N \geq 0$ and integer overcomes these three problems simultaneously such x_N determines an optimal integer solution of (3).

The first problem can be resolved by adding the constraints |11, 17|

$$\sum_{j\in N} (a_{ij} - |a_{ij}|)x_j \equiv (b_i - |b_i|) \pmod{1} \qquad \forall i=1,\dots,m \qquad (4)$$

where $a \equiv b \pmod{c}$ means that a and b are congruent modulo c, or that a and b differ by an integer multiple of c (i.e., $a-b=r\,c$, r integer). In addition, a_{ij} are the updated matrix coefficients of (3) and $|a_{ij}|$ is the largest integer not larger than a_{ij}. (Note that no component of x_B appears in (4)).

In other words, satisfaction of constraints (4) assures us that for x_N, only integer values of x_B will be considered, and the objective function categorizes the optimal solution, therefore solving problem 3. Thus, if we could solve the following problem, we would overcome problems 1 and 3.

$$\begin{aligned}
&\text{minimize } z\\
&z = (c_N^T - c_B^T B^{-1} N)x_N + c_B^T B^{-1} b\\
&\text{s.t.: } B^{-1} N x_N \equiv B^{-1} b \pmod{1}\\
&x_N \geq 0, \text{ integer}
\end{aligned} \qquad (5)$$

or

$$\text{minimize } z$$

$$z = \tilde{c}^T x_N \qquad (6)$$

$$\text{s.t.: } Dx_N \equiv p \ (\text{mod. } 1)$$

$$x_N \geq 0, \text{ integer}$$

where:

$$\tilde{c}^T = [c_N^T - c_B^T B^{-1} N]$$

$$D = B^{-1} N$$

$$p = B^{-1} b$$

From (6) follows that:

$$x_B = B^{-1} b - B^{-1} N x_N \qquad (7)$$

It is usually possible to eliminate some of the constraints of (4). Any constraints which can be shown to be congruent modulo one to other equations or congruent modulo one to linear combinations are redundant and may be dropped.

The constraints that cannot be deleted are generating constraints for the group, and are sufficient to admit only valid solutions to the group of constraints. Thus, when the group is cyclic, there is only one constraint necessary to solve the group problem |9,10,11,12,17,18|.

Nevertheless in many real cases the number of the constraints and especially the number of integer variables in (6) is too large for an efficient solution.

In the following a procedure is proposed to formulate two or more I.L.P. problems in a fewer number of variables that can be solved indipendently. The optimal solution obtained is obviously the same of problem (6).

3. DECOMPOSITION TECHNIQUE

Consider the I.L.P. problem written in the last form (6). Let

g.c.d.$\{\cdot\} \triangleq$ greater common divisor of the set of integer numbers $\{\cdot\}$

l.c.m.$\{\cdot\} \triangleq$ least common multiple of the set of integer numbers $\{\cdot\}$

THEOREM - If there exist a column partition of matrix D (by reordering rows and columns of D)

$$D \triangleq |D_1 \vdots D_2|$$

and two positive integers $\{k_1,k_2\}$ such that

i) $$k_1 D_2 \equiv 0 \pmod{1} \qquad k_2 D_1 \equiv 0 \pmod{1} \tag{8}$$

ii) For each $i=1,2$ $\exists$ (l_i,m_i) with $(d_{l_i m_i} \varepsilon D_i)$ such that $(k_i d_{l_i m_i} \not\equiv 0 \pmod{1})$ $i=1,2$ (9)

iii) $$\text{g.c.d.}\{k_1,k_2\} = 1 \tag{10}$$

Then the optimal solution of (6) is the same of the optimal solution of the following block diagonal form problem (reordering rows of $\tilde{c}$, x_N, P, according to D):

$$\begin{aligned} &\text{minimize } z \\ &z = \tilde{c}^T x_N \\ &\text{s.t. } k_i D_i \, x_N^{(i)} \equiv k_i p \pmod{1} \qquad i=1,2 \\ &x_N \geq 0 \text{ integer} \end{aligned} \tag{11}$$

where $x_N^T \triangleq [x_N^{(1)^T} \vdots x_N^{(2)^T}]$ is a partition of the x_N vector according to the column partition of matrix D.

Proof - Let us consider the sets

$$\Delta \overset{\Delta}{=} \{x_N | Dx_N \equiv p(\text{mod}.1),\ x_N \geq 0,\ \text{integer}\} \tag{12}$$

$$\Delta_{k_i} \overset{\Delta}{=} \{x_N | k_i Dx_N \equiv k_i\, p\ \ (\text{mod}.1),\ x_N \geq 0,\ \text{integer}\} \tag{13}$$

$i=1,2$

k_i positive integer

Since

$$\begin{aligned} &(x_N \varepsilon \Delta) \Rightarrow x_B \text{ is an integer vector} \\ &(x_N \varepsilon \Delta k_i) \Rightarrow k_i x_B \text{ is an integer vector} \end{aligned} \tag{14}$$

$i=1,2$

we can write

$$\Delta \subseteq \bigcap_{i=1}^{2} \Delta_{k_i} \tag{15}$$

Further, since (8) and (9), hold, we can write

$$\Delta_{k_i} = \{x_N | k_i D_i x_N^{(i)} \equiv k_i\ p\ (\text{mod}.1),\ x_N \geq 0\ \text{integer}\} \tag{16}$$

$i=1,2$

From (14) and (7) follows:

$$(x_N\ \varepsilon\ \Delta_{k_1} \cap \Delta_{k_2}) \Rightarrow \begin{cases} k_1\ x_B = h_1 \\ k_2\ x_B = h_2 \end{cases} \tag{17}$$

where h_1 and h_2 are positive integer m-vectors.

Then

$$x_B = \frac{h_1}{k_1} = \frac{h_2}{k_2} \tag{18}$$

and

$$k_2 h_1 = k_1 h_2 \tag{19}$$

From (19) it derives that k_1 divides each component of $k_2.h_1$.

Then k_1 and k_2 being relatively prime, for hypothesis (10), it follows

$$h_1 = k_1\, q \quad , \quad h_2 = k_2\, q$$

with q positive integer m-vector.

Therefore

$$(x_N \in \bigcap_{i=1}^{2} \Delta_{k_i}) \quad x_B \text{ is an integer m-vector} \tag{20}$$

or

$$\Delta = \bigcap_{i=1}^{2} \Delta_{k_i} \tag{21}$$

Hence the theorem is proved.

REMARK - The problem (11), with respect to the vector x_N, is in a block diagonal form and then it can be solved with respect to each $x_N^{(i)}$ indipendently.

$$\begin{aligned} &\text{minimize } z_i \\ &z_i = \tilde{c}^{(i)T} x_N^{(i)} \\ &\text{s.t. } k_i D_i x_N^{(i)} \equiv k_i p \ (\text{mod. } 1) \\ &x_N^{(i)} \geq 0 \text{ integer} \end{aligned} \tag{22}$$

where $\tilde{c}^T = |\tilde{c}^{(1)T} \vdots \tilde{c}^{(2)T}|$.

Since the components of D are rational numbers a value of k_i that satisfies (8) can be found as follows. Let:

$$d_j \triangleq j^{th} \text{ column of } D \qquad j=1,2,\ldots,(n-m) \tag{23}$$

$$\begin{aligned} &\phi_j \triangleq \{\text{set of column indices of } D_j\} \qquad j=1,2 \\ &\Gamma \triangleq \{\text{set of column indices of } D\} \end{aligned} \tag{24}$$

we can define the following linear programming problems in a single variable

$$\begin{aligned} &\text{minimize } y_j \\ &d_j y_j \equiv 0 \ (\text{mod. } 1) \qquad j=1,2,\ldots,(n-m) \\ &y_j \geq 0 \end{aligned} \tag{25}$$

The optimum solution y_j^* of each problem is integer.

If $\{k_1, k_2\}$ are relatively prime, with:

$$k_1 = \text{l.c.m.}\ \{y_j^* \mid j \epsilon \phi_2\}$$
$$k_2 = \text{l.c.m.}\ \{y_j^* \mid j \epsilon \phi_1\} \tag{26}$$

the conditions (8), (9) and (10) of the previous theorem are satisfacted, and than we can solve the problem (11).

COROLLARY - If in the previous theorem the constraint (10) is dropped

i.e. $\text{g.c.d.}\ \{k_1, k_2\} = \hat{k}$ (27)

with $\hat{k}$ positive integer

the solution of problem (11) is such that

$\hat{k}\ x_B$ is an integer vector

Proof - If:

$$k_2 = k_2' \hat{k} \tag{28}$$

from (15) we can write:

$$k_2' \hat{k}\ h_1 = k_1\ h_2 \tag{29}$$

and then:

$$\hat{k}\ h_1 = k_1 q \tag{30}$$

therefore the vector $\hat{k}\ x_B$ is an integer vector equal to q.

REMARK - The problem (11) with the hypothesis of the previous corollary is a relaxation of problem (6). In many cases the optimal solution is such that x_B become an integer vector.

4. DECOMPOSITION ALGORITHM

The decomposition procedure consists first in solving the problems (25) for $j=1,2,\ldots,(n-m)$. In order to found a partition of the set Γ,if exists, that satisfies the conditions (8), (9) and (10) with $\{k_1,k_2\}$ given by (26) we can use the following algorithm (ϕ is the empty set):

<u>Algorithm</u>:

1. Set: $\phi_1 = \phi_2 = \Omega = \phi$, $\Lambda = \{j \mid j\varepsilon\Gamma\}$

2. Take a $t\varepsilon\Lambda$, remove t from Λ, add t to the set ϕ_1

3. $\forall i\varepsilon\Lambda$ calculate:

$$g_i = \text{g.c.d. } \{y_i^*,\ y_t^*\}$$

   If $g_i \neq 1$ remove i from Λ and add i to Ω

4. If $\Lambda = \phi$ go to 7 otherwise go to 5

5. If $\Omega = \phi$ go to 7 otherwise go to 6

6. Take a $t\varepsilon\Omega$, remove t from Ω, add t to the set ϕ_1. Go to 3

7. $\phi_2 = \{i \mid i\varepsilon\Gamma,\ i\not\varepsilon\phi_1\}$ Stop.

Since:

$$(\text{g.c.d.}\{y_i^*,y_j^*\}=1;\ \forall i\varepsilon\phi_1,\ \forall j\varepsilon\phi_2) \Rightarrow (\text{g.c.d.}\{k_1,k_2\}=1)$$

with k_1 and k_2 calculated by (26) the partition obtained with this algorithm satisfies the conditions of the previous theorem.

REMARK - In the previous algorithm it is sufficient to consider only the different values of y_j^*.

REMARK - The subproblem defined by the set ϕ_1 cannot be further decomposed. On the other hand the same decomposition procedure can be applied to the subproblem defined by the set ϕ_2.

5. CONCLUSIONS

In this work, using the group theoretical approach we point out some conditions on the $B^{-1}N$ matrix, often verified in practice, that make it possible to transform the system of linear congruences (constraints of problem 2) in a block diagonal form. In some cases, using this procedure, the number of constraints can increase with respect to the number of constraints of problem 2. However, the problem can be solved indipendently for the variables associated with each block.

This procedure leads to the indipendent solution of a number of subproblems in a smaller number of variables.

In the worst case each subproblem requires the same number of constraints as the original problem, but generally this number is smaller.

REFERENCES

|1| J.J.H. FORREST, J.P.H. HIRST, J.A. TOULIN, Practical solution of large mixed integer programming problems with UMPIRE. Management Science, vol. 20, n. 5, January 1974.

|2| G. MITRA, Investigation of some branch and bound algorithms for (0-1) mixed integer linear programming. Mathematical Programming 4, pp. 155-170, 1973.

|3| M. SHAW, Review of computational experience in solving large mixed integer programming problems. pp. 406-412. Applications of Mathematical Programming Techniques, English Universities Press, London 1970.

|4| A.M. GEOFFRION, G.W. GRAVES, Multicommodity distribution system design by Benders decomposition. Management Science, vol.20, n. 5, January 1974.

|5| G. GALLO, E. MARTINO, B. SIMEONE, Group optimization algorithms and some numerical results via a branch and bound approach. Giornate AICA su "Tecniche di Simulazione ed Algoritmi". Milano, Nov. 1972.

|6| J.F. SHAPIRO, Dynamic programming algorithms for the integer programming problem I: the integer programming problem viewed as a knapsack-type problem. Operation Research, 16 January 1968.

|7| J.F. SHAPIRO, Group theoretic algorithms for the integer programming problem II: extension to a general algorithm. Operation Research 16, September 1968.

|8| J.A. TOMLIN, Branch and bound methods for integer and non-convex programming. Integer and non-linear programming, cap. 21, North-Holland, Amsterdam 1970.

|9| L.A. WOLSEY, Extensions of the group theoretic approach in integer programming. Management Science, vol. 18, n. 1, September 1971.

|10| S. ZIONTS, Linear and integer programming, Prentice-Hall, 1974.

|11| T.C. HU, Integer programming and network follows. Addison-Wesley Publishing Company, 1969.

|12| R.E. GOMORY, Some polyhedra related to combinatorial problems. Linear Algebra and Its Applications, n. 2, 1969.

|13| D.E. BELL, Improved bounds for integer programs: a supergroup approach. Research Memorandum of IIASA, November 1973.

|14| H. GREENBERG, Integer programming. Academic Press, 1971.

|15| A.M. GEOFFRION, Lagrangean relaxtion and its uses in integer programming. Western Management Science Institute, Working Paper n. 195.

|16| M.L. FISHER, J.F. SHAPIRO, Constructive duality in integer programming. Massachusetts Institute of Technology. Working Paper OK 008-72, April 1972.

|17| R.S. GARFINKEL, G.L. NEMHAUSER, Integer programming. John Wiley and Sons, 1972.

|18| G.S. MOSTOW J.H SAMSON, I.P. MEYER, Fundamental structure of algebra. Mc Graw-Hill, New York, 1963.