ON THE MARGINAL VALUE OF AN ANTAGONISTIC GAME

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Let X and Y be the sets of strategies of two antagonistic players, and for $t \in [0,T] \subset \mathbb{R}$, $T > 0$,

$$f(x,y,t) : X \times Y \to \mathbb{R}$$

may be a payoff function in the parametric two-person zero-sum game

$$G_t := (X,\ Y,\ f(x,y,t)) \ , \quad t \in [0,T],$$

where it is to maximize over X and to minimize over Y.

Definition 1: If $v(t) = \mathrm{val}(G_t)$ is the value of G_t, $t \in [0,T]$, then the marginal value of the family of games $\{G_t\}_{t\in[0,T]}$ in the point $t = 0$ is defined as

$$v'_+(0) := \lim_{t\to+0} \frac{v(t) - v(0)}{t} \quad .$$

We consider conditions - different from that given for instance in [3], [5] - for the existence of the marginal value, which enable us to derive from the marginal value theorem a method for solving a wide class of constrained games.

Let X, Y be not empty closed convex subsets of real topological linear spaces.

Definition 2: A function $g: X \times Y \to \mathbb{R}$ is 'sup-inf compact', if the level sets $\{x \in X \mid g(x,y) \geq a\}$, $y \in Y$, $a \in \mathbb{R}$, and $\{y \in Y \mid g(x,y) \leq b\}$, $x \in X$, $b \in \mathbb{R}$, are compact.

Let X(t), Y(t) denote the sets of optimal strategies in G_t, $t \in [0,T]$.

Theorem 1: (marginal value theorem)

If (i) $f(x,y,t)$ is concave (convex) in x (in y) for $t \in [0,T]$,

(ii) $f(x,y,0)$ is upper (lower) semicontinuous in x (in y),

(iii) $f(x,y,0)$ has a saddle point on $X \times Y$,

(iv) $f'(x,y,0) := \frac{\partial f(x,y,t)}{\partial t}\Big|_{t=+0}$ exists, and

$$O(1,x) \leq \frac{f(x,y,t) - f(x,y,0)}{t} - f'(x,y,0) \leq O(1,y) \ , \text{ for } t \in (0,T],$$

where $O(1,x)$ is independent of y and $O(1,x) \to 0$ for $t \to +0$ and fixed $x \in X$, $O(1,y)$ is independent of x and $O(1,y) \to 0$ for $t \to +0$ and fixed $y \in Y$,

(v) $f'(x,y,0)$ and $f(x,y,t)$ for $t \in (0,T]$ are sup-inf compact on $X \times Y$,

then $v'_+(0)$ exists and

$$(1) \qquad v'_+(0) = \max_{x\in X(0)} \min_{y\in Y(0)} f'(x,y,0) = \min_{y\in Y(0)} \max_{x\in X(0)} f'(x,y,0) \quad .$$

Proof:

Because of (i), (iii) and (v) the sets $X(t)$, $Y(t)$ are not empty and there exist the values $v(t)$, for $t \in [0,T]$. Let $x_t \in X(t)$, $y_t \in Y(t)$, $t \in [0,T]$, then we have

$$(2) \qquad f(x_o,y_t,t) - f(x_o,y_t,0) \leq v(t) - v(0) \leq f(x_t,y_o,t) - f(x_t,y_o,0) \ .$$

For $t \in (0,T]$ we get from (iv)

$$(3) \qquad \bigwedge_{x\in X} \bigwedge_{y\in Y} O(t,x) + tf'(x,y,0) \leq f(x,y,t) - f(x,y,0) \leq O(t,y) + t\cdot f'(x,y,0) \ ,$$

and with (2)

$$(4) \qquad t\cdot f'(x_o,y_t,0) + O(t,x_o) \leq f(x_o,y_t,t) - f(x_o,y_t,0) \leq v(t) - v(0) \leq f(x_t,y_o,t) - f(x_t,y_o,0) \leq t\cdot f'(x_t,y_o,0) + O(t,y_o) \ .$$

From (v) we get

$$(5) \qquad f'(x,y,0) \geq \min_{y\in Y} f'(x,y,0) =: f'(x,y(x),0) > -\infty$$

and

$$f'(x,y,0) \leq \max_{x\in X} f'(x,y,0) =: f'(x(y),y,0) < +\infty \quad ,$$

such that with (4):

$$(6) \qquad t\cdot f'(x_o,y(x_o),0) + O(t,x_o) \leq v(t) - v(0) \leq t\cdot f'(x(y_o),y_o,0) + O(t,y_o) \ ,$$

which means that

$$(7) \qquad \lim_{t\to+0} v(t) = v(0) \ .$$

Dividing in (4) by t, (5) yields

$$(8) \qquad f'(x_o,y_t,0) + O(1,x_o) \leq f'(x(y_o),y_o,0) + O(1,y_o) \;, \quad \text{and}$$
$$f'(x_t,y_o,0) + O(1,y_o) \geq f'(x_o,y(x_o),0) + O(1,x_o) \;,$$

which by (v) means, that the x_t, y_t are elements of compact sets independent of t. Therefore $\{x_t\}_{t\to+0}$, $\{y_t\}_{t\to+0}$ have accumulation points $\hat{x} \in X$, $\hat{y} \in Y$ and convergent subsequences $\{x_{t_n}\}_{n\in\mathbb{N}} \subset \{x_t\}_{t\to+0}$, $\{y_{t_n}\}_{n\in\mathbb{N}} \subset \{y_t\}_{t\to+0}$, such that

$$\lim_{n\to\infty} x_{t_n} = \hat{x} \;,$$

$$\lim_{n\to\infty} y_{t_n} = \hat{y} \;.$$

By (ii) we get for all $x \in X$

$$(9) \qquad f(x,\hat{y},0) \leq \underline{\lim}_{t_n\to+0} f(x,y_{t_n},0) \leq \overline{\lim}_{t_n\to+0} f(x,y_{t_n},0) \;,$$

and with (iv) and (5)

$$(10) \qquad v(t_n) - f(x,y_{t_n},0) \geq f(x,y_{t_n}) - f(x,y_{t_n},0)$$
$$\geq t_n\cdot f'(x,y(x),0) + O(t_n,x) \;,$$

$$(11) \qquad \underline{\lim}_{t_n\to+0} (v(t_n) - f(x,y_{t_n},0)) \geq 0 \;,$$

and by (7)

$$(12) \qquad \overline{\lim}_{t_n\to+0} f(x,y_{t_n},0) \leq v(0) \;.$$

Because of (9) this gives

$$(13) \qquad f(x,\hat{y},0) \leq v(0) \quad \text{for all } x \in X \;,$$

i.e. $\hat{y}$ is an optimal strategy in G_o. Analogously you show $\hat{x} \in X(0)$. From inequality (4) it follows

$$(14) \qquad f'(x_o,y_t,0) + O(1,x_o) \leq \frac{v(t) - v(0)}{t}$$
$$\leq f'(x_t,y_o,0) + O(1,y_o) \;,$$
$$\text{for all } x_o \in X(0),\; y_o \in Y(0) \;,$$

such that

$$(15) \qquad \sup_{x\in X(0)} f'(x,y_t,0) + O(1,x) \leq \frac{v(t) - v(0)}{t}$$
$$\leq \inf_{y\in Y(0)} f'(x_t,y,0) + O(1,y) \;.$$

Since $y \to f'(x,y,0)$ is lower semicontinuous, also $y \to \sup_{x\in X(0)} f'(x,y,0)$ is lower semicontinuous. Similarly, $x \to \inf_{y\in Y(0)} f'(x,y,0)$ is upper semicontinuous.

Let P and Q be the sets of accumulation points of $\{x_t\}_{t\to+0}$ and $\{y_t\}_{t\to+0}$, respectively, and for $p \in P$, $q \in Q$

$$\{x_{t(p)}\}_{t(p)\to+0} \subset \{x_t\}_{t\to+0} , \quad \{y_{t(q)}\}_{t(q)\to+0} \subset \{y_t\}_{t\to+0}$$

may be convergent subsequences such that

$$\lim_{t(p)\to+0} x_{t(p)} = p , \qquad \lim_{t(q)\to+0} x_{t(q)} = q .$$

From (15) we get now

(16)
$$\begin{aligned}
&\inf_{q\in Q} \sup_{x\in X(0)} f'(x,q,0) \\
&\quad \leq \inf_{q\in Q} \underline{\lim}_{t(q)\to+0} \sup_{x\in X(0)} f'(x,y_{t(q)},0) \\
&\quad \leq \underline{\lim}_{t\to+0} \sup_{x\in X(0)} f'(x,y_t,0) \\
&\quad \leq \underline{\lim}_{t\to+0} \frac{v(t) - v(0)}{t} \\
&\quad \leq \overline{\lim}_{t\to+0} \frac{v(t) - v(0)}{t} \\
&\quad \leq \overline{\lim}_{t\to+0} \inf_{y\in Y(0)} f'(x_t,y,0) \\
&\quad \leq \sup_{p\in P} \overline{\lim}_{t(p)\to+0} \inf_{y\in Y(0)} f'(x_{t(p)},y,0) \\
&\quad \leq \sup_{p\in P} \inf_{y\in Y(0)} f'(p,y,0) .
\end{aligned}$$

By (13) we have

(17)
$$P \subset X(0) , \quad Q \subset Y(0) ,$$

and (16) yields

(18)
$$\begin{aligned}
&\inf_{y\in Y(0)} \sup_{x\in X(0)} f'(x,y,0) \\
&\quad \leq \inf_{y\in Q} \sup_{x\in X(0)} f'(x,y,0) \\
&\quad \leq \underline{\lim}_{t\to+0} \frac{v(t) - v(0)}{t} \\
&\quad \leq \overline{\lim}_{t\to+0} \frac{v(t) - v(0)}{t} \\
&\quad \leq \sup_{x\in P} \inf_{y\in Y(0)} f'(x,y,0) \\
&\quad \leq \sup_{x\in X(0)} \inf_{y\in Y(0)} f'(x,y,0) .
\end{aligned}$$

On the other hand there holds the sup-inf inequality

(19) $$\sup_{x\varepsilon X(0)} \inf_{y\varepsilon Y(0)} f'(x,y,0) \leqslant \inf_{y\varepsilon Y(0)} \sup_{x\varepsilon X(0)} f'(x,y,0) .$$

Thus $v'_+(0) = \lim_{t\to+0} \frac{v(t) - v(0)}{t}$ exists and by the sup-inf compactness of $f'(x,y,0)$ we get

$$v'_+(0) = \max_{x\varepsilon X(0)} \min_{y\varepsilon Y(0)} f'(x,y,0)$$

$$= \min_{y\varepsilon Y(0)} \max_{x\varepsilon X(0)} f'(x,y,0) .$$

<u>Theorem 2:</u>

Let $x_t \varepsilon X(t)$, $y(t) \varepsilon Y(t)$, $t \varepsilon (0,T]$.
For any accumulation points $\hat{x}$, $\hat{y}$ of $\{x_t\}_{t\to+0}$, $\{y_t\}_{t\to+0}$, respectively, it holds then under the assumptions of Theorem 1:

(20) $$f'(\hat{x},\hat{y},0) = v'_+(0) .$$

<u>Proof:</u>

Let $x_{t_n} \to \hat{x}$, for $t_n \to +0$, then we get from (15) with the results of Theorem 1:

(21) $$\sup_{x\varepsilon X(0)} f'(x,\hat{y},0) \leqslant \underline{\lim}_{t_n\to+0} \sup_{x\to X(0)} f'(x,y_{t_n},0)$$

$$\leqslant v'_+(0) = \text{val}(X(0),Y(0),f'(x,y,0)) .$$

Since $\hat{y} \varepsilon Y(0)$ that means, $\hat{y}$ is an optimal strategy in the game $(X(0),Y(0),f'(x,y,0))$. Similarly, $\hat{x}$ is an optimal strategy in this game.

Now we come to an application of the results above to constrained games.

Let U, V be normed real vector spaces, $C \subset U$, $K \subset V$ not empty closed convex cones, $g: X \to U$, $h: Y \to V$ continuous and concave relative to the cones, and $\phi: X \times Y \to \mathbb{R}$ be upper-lower semicontinuous, concave-convex and sup--inf compact.
We consider the constrained game

(22) $$(CG) := (\{x\varepsilon X \mid g(x)\varepsilon C\}, \{y\varepsilon Y \mid h(y)\varepsilon K\}, \phi(x,y)) .$$

With

$$d(g(x),C) := \inf_{c\varepsilon C} \|g(x) - c\| ,$$

$$d(h(y),K) := \inf_{k\varepsilon K} \|h(y) - k\| ,$$

we define for $t \geqslant 0$

(23) $$f(x,y,t) := d(h(y),K) - d(g(x),C) + t\cdot\phi(x,y) .$$

A solution method for (CG) is given by the following

Theorem 3:

Let $\{t_n\}_{n\varepsilon N}$ be a positive real nullsequence and (CG) may have admissable strategies.

Then (i) the unconstrained games $(X,Y,f(x,y,t_n))$ have optimal strategies x_{t_n}, y_{t_n} .

(ii) $\{x_{t_n}\}_{n\varepsilon N}$, $\{y_{t_n}\}_{n\varepsilon N}$ have accumulation points $\hat{x}$, $\hat{y}$ and $\hat{x}$, $\hat{y}$ are optimal strategies for (CG).

(iii) $\lim\limits_{n\to\infty} \{\phi(x_{t_n},y_{t_n}) + \frac{1}{t_n}(d(h(y),K) - d(g(x),C)\} = \phi(\hat{x},\hat{y}) = \mathrm{val(CG)}$

Proof:

First we show that $d(g(x),C)$ is continuous and convex in x.
$d(.,C)$ is continuous on U, $g(.)$ is continuous on X, thus $d(g(.),C)$ is continuous on X. g is concave on C, i.e. for $x_1,x_2 \varepsilon X$ and $0 \leqslant \alpha \leqslant 1$ it holds $g(\alpha x_1 + (1-\alpha)x_2) - \alpha g(x_1) - (1-\alpha)g(x_2) \varepsilon C$. Then for all $c \varepsilon C$:
$g(\alpha x_1 + (1-\alpha)x_2) - \alpha g(x_1) - (1-\alpha)g\ x_2) + c \varepsilon C$, which gives

$$\begin{aligned}
&\|\alpha g(x_1) + (1-\alpha)g(x_2) - c\| \\
&\quad = \|g(\alpha x_1 + (1-\alpha)x_2) - (g(\alpha x_1 + (1-\alpha)x_2) - \\
&\qquad - \alpha(g(x_1) - (1-\alpha)g(x_2) + c)\| \\
&\quad \geqslant \inf_{d\varepsilon C} \|g(\alpha x_1 + (1-\alpha)x_2) - d\| \quad .
\end{aligned}$$

Theorem 3.4 in [6] states, that $d(.,C)$ is convex. Thus we have

$$\begin{aligned}
d(g(\alpha x_1 + (1-\alpha)x_2),C) &\leqslant d(\alpha g(x_1) + (1-\alpha)g(x_2),C) \\
&\leqslant \alpha d(g(x_1),C) + (1-\alpha)d(g(x_2),C).
\end{aligned}$$

It is $f(x,y,0) = d(h(y),K) - d(g(x),C)$. We show $X(0) = \{x \varepsilon X \mid g(y) \varepsilon C\}$, $Y(0) = \{y \varepsilon Y \mid h(y) \varepsilon K\}$. Let $x_o \varepsilon X(0)$, $y_o \varepsilon Y(0)$, then for all $x \varepsilon X$, $y \varepsilon Y$ it is valid

$$\begin{aligned}
d(h(y_o),K) - d(g(x),C) &\leqslant d(h(y_o),K) - d(g(x_o),C) \\
&\leqslant d(h(y),K) - d(g(x_o),C) \quad .
\end{aligned}$$

It follows $d(h(y_o),K) \leqslant d(h(y),K)$ for all $y \varepsilon Y$.
Since (CG) should have admissible strategies, there exists a $y \varepsilon Y$ with $h(y) \varepsilon K$. So $d(h(y_o),K) \leqslant 0$, i.e. $h(y_o) \varepsilon K$. Analogously it can be shown that $g(x_o) \varepsilon C$, under the assumption that there exists an $x \varepsilon X$ with $g(x) \varepsilon C$. Now let $\hat{x} \varepsilon X$, $\hat{y} \varepsilon Y$ such that $g(\hat{x}) \varepsilon C$, $h(\hat{y}) \varepsilon K$. Then $d(g(\hat{y}),C) = d(h(\hat{y}),K) = 0$ and for all $x \varepsilon X$, $y \varepsilon Y$ we have

$$-d(g(x),C) \leq 0 \leq d(g(y),K) \ ,$$

i.e. $\hat{x} \in X(0)$, $\hat{y} \in Y(0)$. Further we have $v(0) = 0$.
$f(x,y,t)$ is sup-inf compact for $t > 0$, because $f'(x,y,0) = \phi(x,y)$ has this property and $f(x,y,0)$ is bounded from above (below) in x (in y). From this fact part (i) of the theorem follows.

We have shown that the assumptions of theorem 1. are fulfilled, such that (ii) and (iii) follow from the theorems 1. and 2., respectively from the proofs.

To solve the problem of finding optimal strategies of the games $(X,Y,f(x,y,t))$, you often have to take algorithms which need for convergency the function $f(.,.,t)$ to be strictly concave-convex, as for example the successive approximation method given in [1]. If f does not possess this property, we can do the following:

Let $\psi(x,y)$ be a strictly concave-convex upper-lower semicontinuous real valued function on $X \times Y$, which is bounded from above (below) in x (in y) by some $a(y) \in \mathbb{R}$ ($b(x) \in \mathbb{R}$). Then we define

$$F(x,y,t) := f(x,y,t) + t^2\psi(x,y) \ .$$

If the conditions of theorem 1. are fulfilled for f, then also for F. We show this for the condition (iv):

If $O_f(1,x) \leq \frac{f(x,y,t) - f(x,y,0)}{t} - f'(x,y,0) \leq O_f(1,y) \ , \ t > 0$,

then we define $O_F(1,x) := O_f(1,x) + t \cdot b(x)$, $O_F(1,y) := O_f(1,y) + t \cdot a(y)$ and get

$$O_F(1,x) \leq \frac{f(x,y,t) - f(x,y,0)}{t} + t \cdot \psi(x,y) - f'(x,y,0) \leq O_F(1,y) .$$

Since $F(x,y,0) = f(x,y,0)$, $F'(x,y,0) = f'(x,y,0)$ this states 1.(iv) for F. Furthermore, let $V(t) := \text{val}(X,Y,F(x,y,t))$, then $V(0) = v(0)$ and $V'_+(0) = v'_+(0)$. Thus in order to compute $v'_+(0)$, instead of $(X,Y,f(x,y,t))$ we can solve the games $(X,Y,F(x,y,t))$, which have unique solutions (for $t > 0$).

Under certain conditions it can be shown that the accumulation points of the corresponding optimal strategies for $t \to +0$ are uniquely determined, such that the whole sequences are converging to optimal strategies of the game $(X(0), Y(0), f'(x,y,0))$.

This kind of regularization is particularly interesting for the above given method for solving constrained games.

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