MODELING DISPERSION IN A SUBMERGED SEWAGE FIELD

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Several large urban communities located on a sea shore utilise - or consider utilizing - a deceptively simple system of disposal of their sewage water : after a rough preliminary treatment (sedimentation), the liquid is pumped to a linear diffusor anchored on the sea-floor, at several km from the shore under a submergence of some 50 m. The diffusor itself is a sparger pipe, 2 to 4 m in diameter, and pierced with equidistant side holes of 5 to 10 cm diameter. When the sea current is naught, the buoyant jets formed at the side holes unite near the diffusor into a linear vertical buoyant plume whose behaviour was studied in great detail for the case of laminar flow [1, 2, 3] and for that of turbulent flow [4, 5, 6]. It has been shown, for instance, that the maximum density difference between sea water and the plume decreases asymptotically (when the distance to the diffusor, y, increases) like $y^{-3/5}\ F_o^{4/5}$ for laminar flow, and like $y^{-1}\ F_o^{2/3}$ for turbulent flow. As the submergence is finite, these plumes are eventually deflected into horizontal buoyant plumes, either at the sea surface or at the level of a thermocline (see infra) if the flux of density difference per unit length of diffusor, F_o, is small enough.

The structure of these horizontal buoyant plumes has not yet been thoroughly investigated, and therefore prevailing design methods of marine sewage disposal systems [6] take only the dispersion in vertical plumes into account. It has occured to us that a detailed study of the dispersion properties horizontal buoyant plumes might be rated higher than an academic exercise by people willing to swim along the French Riviera. This paper then gives our main results for the case of linear laminar horizontal buoyant plumes ; the turbulent case and the axisymetric cases will be published elsewhere.

1. Asymptotic form of the conservation equations

When the Boussinesq hypothesis pertaining to natural convection in a quasi-incompressible fluid applies, the momentum and energy equations assume the following form for bi-dimensional flow [Ox is horizontal, Oy is vertical upwards] :

$$\frac{\partial(\Delta\psi,\psi)}{\partial(x,y)} = -\frac{\partial\Theta}{\partial x} + \frac{1}{Gr^{1/2}} \Delta^2 \psi \qquad (1.1)$$

$$\frac{\partial(\Theta,\psi)}{\partial(x,y)} = \frac{1}{Pr.Gr^{1/2}} \Delta\Theta \qquad (1.2)$$

The plumes studied here are in fact Prandtl boundary layers along the Ox axis. To find their asymptotic solution for $Gr \rightarrow \infty$ in the vicinity of $y = o$ (inner solution), one has to stretch y and ψ as follows :

$$Y = y.Gr^{2/10} \qquad \text{and} \qquad \Psi = \psi.Gr^{3/10} \qquad (1.3)$$

The fundamental term in the inner solution satisfies then :

$$\frac{\partial\Psi}{\partial Y}\frac{\partial^3\Psi}{\partial x\,\partial Y^2} - \frac{\partial\Psi}{\partial x}\frac{\partial^3\Psi}{\partial Y^3} = -\frac{\partial\Theta}{\partial x} + \frac{\partial^4\Psi}{\partial Y^4} \qquad (1.4)$$

$$\frac{\partial\Psi}{\partial Y}\frac{\partial\Theta}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Theta}{\partial Y} = \frac{1}{Pr}\frac{\partial^2\Theta}{\partial Y^2} \qquad (1.5)$$

The inner solution will be valid to order $O(Gr^{-2/10})$

2. Superficial horizontal buoyant plumes

When the plume is formed at the sea-surface, the solution of (1.4) (1.5) must satisfy the following boundary conditions :

$Y = o$: $\Psi = o$ (the surface is a stream-line) (2.1)

$\frac{\partial^2\Psi}{\partial Y^2} = o$ (no shear stress at the surface) (2.2)

$\frac{\partial\Theta}{\partial Y} = o$ (no heat flux to the atmosphere) (2.3)

$Y = \infty$ $\frac{\partial\Psi}{\partial Y} = o$ (no velocity in the x direction far from the surface) (2.4)

$\Theta = o$ (no effect on specific mass far from the surface) (2.5)

This problem admits the following similarity solution :

$$\Psi = \sqrt{x}\,.\,f(\eta) \quad , \qquad \Theta = \frac{1}{\sqrt{x}}\,.\,g(\eta) \qquad (2.6)$$

where the similarity variable is

$$\eta = \frac{Y}{\sqrt{x}} \tag{2.7}$$

The functions f and g satisfy the system :

$$f''' = -\frac{1}{2} f f'' - \frac{1}{2} \eta g \tag{2.8}$$

$$g' = -\frac{Pr}{2} fg \tag{2.9}$$

and the boundary conditions :

$$\eta = o \quad : \quad f = f'' = o \tag{2.10}$$

$$\eta = \infty \quad : \quad f' = o \tag{2.11}$$

Moreover, to completely determine the solution, the enthalphy flux is normalized :

$$\int_{-\infty}^{\infty} u \, \Theta \, dY = \int_{-\infty}^{\infty} f'g \, d\eta = 1 \tag{2.12}$$

This problem was solved with an optimization scheme (5.). For numerical integration a 4th order Runge-Kutta-Gill method was used. Figures 1 and 2 give the interesting functions $f'(\eta)$ and $g(\eta)/\sqrt{Pr}$ for $0.01 \leqslant Pr \leqslant 300$. It is possible to check that the numerical solution is correct for large values of Prandtl (the asymptotic solution is easily found) ; for instance :

$$\lim_{Pr \to \infty} g(\eta) = g(o) \exp\left(- \frac{Pr.f'(o)}{4} \eta^2\right) \tag{2.13}$$

and

$$\lim_{Pr \to \infty} \frac{g(o) \sqrt{f'(o)}}{\sqrt{Pr}} = \frac{1}{2\sqrt{\pi}} \tag{2.14}$$

Figures 3 and 4 give the streamlines and the isotherms for Pr = 1 and Pr = 10. Far from the (virtual) source, near y = o, the streamlines become parallel to Ox and equidistant : the potential energy given by the source to the fluid is completely transformed into kinetic energy and the shear stress along the surface is naught. The larger Pr, the closer the isotherms are to y = o (as indicated by 2.13). Near x = o, the similarity solution, as most boundary layer-type solutions, supports "the curse of the leading edge"...

It is worth remarking that the dilution along the surface is by no means negligible : $\Theta(o)$ varies like $x^{-1/2} F_o^{5/6}$, while the superficial velocity is independant of x (and proportional to $F_o^{1/3}$).

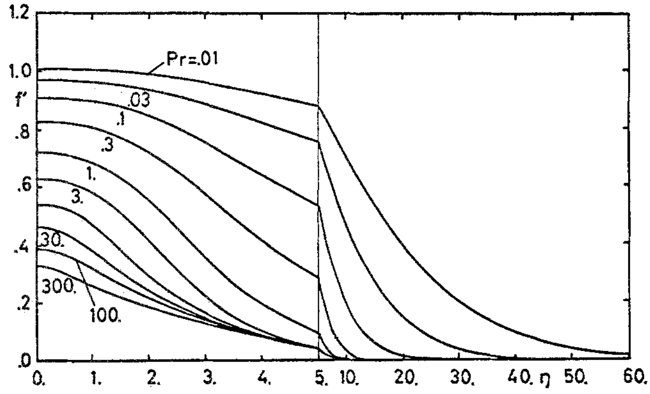

Fig. 1 : Dimensionless velocity profile (superficial plume)

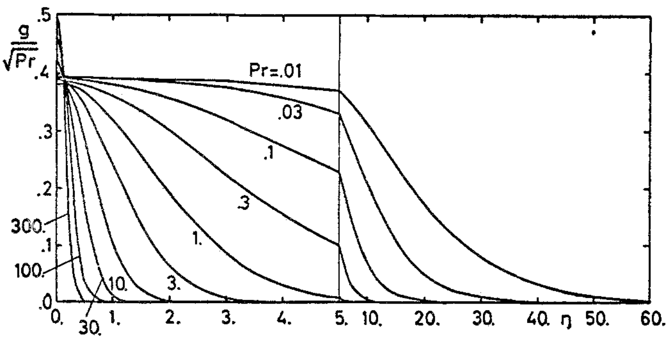

Fig. 2 : Dimensionless density-difference profile (superficial plume)

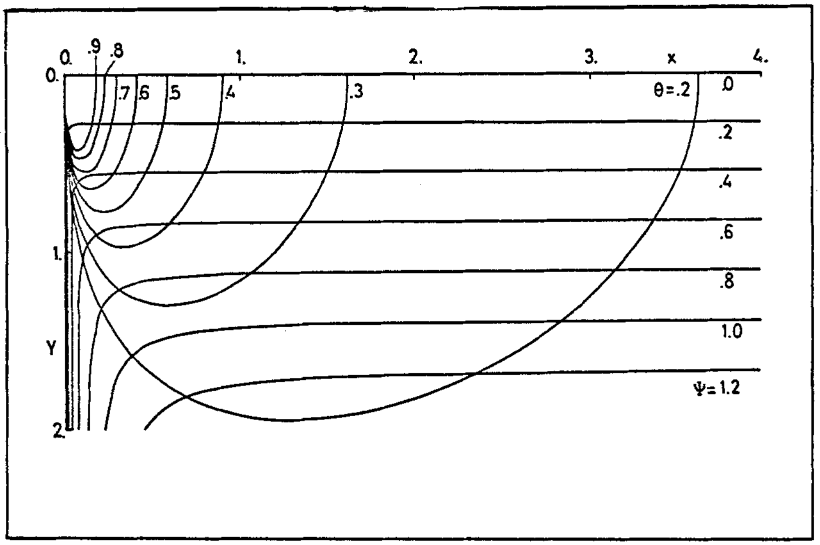

Fig. 3 : Flow pattern in a superficial plume (Pr = 1)

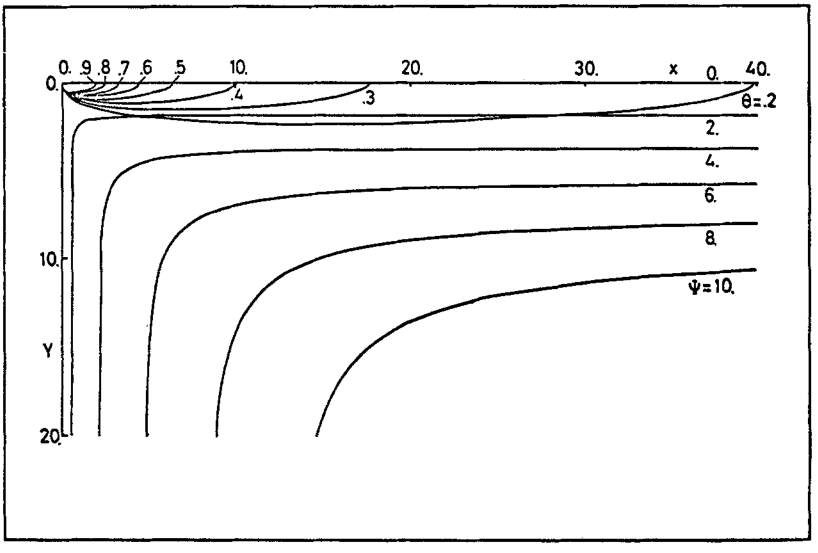

Fig. 4 : Flow pattern in a superficial plume (Pr = 10)

3. Horizontal buoyant plumes on the sea floor

If instead of urban sewage water one would pump some dense industrial effluent to the distributor, the boundary layer would now spread on the sea-floor. The velocity field and density difference field are again given by (1.4), (1.5), (2.1), (2.3), (2.4), (2.5) and (instead of (2.2)) :

$$Y = o \quad : \quad \frac{\partial \Psi}{\partial Y} = o \quad \text{(the velocity is zero on the sea floor)} \qquad (3.1)$$

The similarity solution (2.6), (2.7) still applies :

$$\Psi = \sqrt{x}\,.\, f(\eta) \qquad \text{and} \qquad \Theta = \frac{1}{\sqrt{x}}\, g\,(\eta)$$

where f and g are given by (2.8), (2.9) but under the boundary conditions :

$$\eta = o \quad : \quad f = o, \quad f' = o \qquad (3.2)$$

$$\eta = \infty \quad : \quad f' = o \qquad (3.3)$$

and with the conservation equation for the enthalpy flux :

$$\int_{-\infty}^{\infty} f' \; g \; d\eta = 1 \qquad (3.4)$$

The asymptotic solution for $Pr \to \infty$ is such that :

$$\lim_{Pr\to\infty} \; g(\eta) = g(o) \; \exp\,(- \frac{Pr \; f''(o)}{12} \; \eta^3) \qquad (3.5)$$

and
$$\lim_{Pr\to\infty} \; \frac{g(o)[\,f''(o)\,]^{1/3}}{[\,Pr\,]^{2/3}} = \frac{3}{2.(12)^{2/3}.\Gamma(\frac{2}{3})} \qquad (3.6)$$

These results were used to check the numerical results shown on fig. 5 and 6. Again, the following figures, 7 and 8 give the streamlines and the isotherms for Pr = 1 and Pr = 10. Far from the origin, near Y = o, the streamlines become approximately parabolic : the solid boundary slows the flow down and thickens the boundary layer. The dilution along Y = o is again important since Θ varies like $x^{-1/2}\; F_o^{5/6}$.

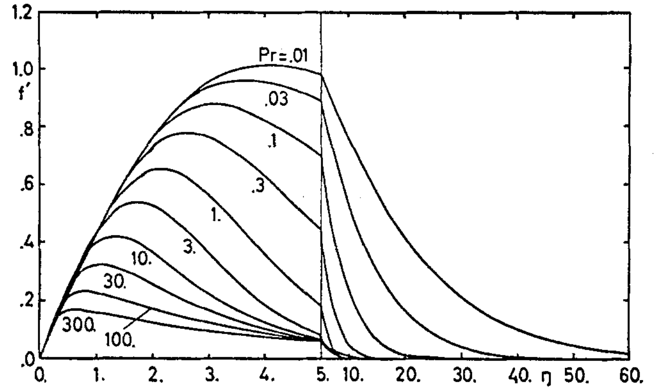

Fig. 5 : Dimensionless velocity profile (floor plume)

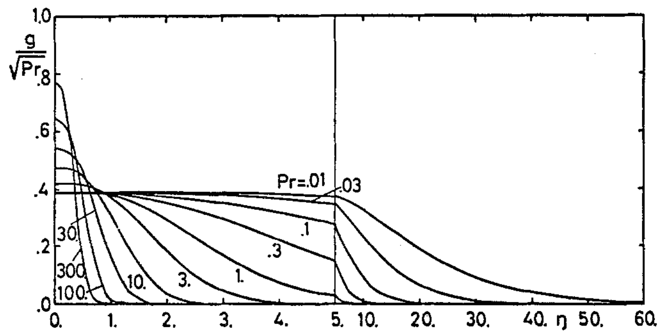

Fig. 6 : Dimensionless density-difference profile (floor plume)

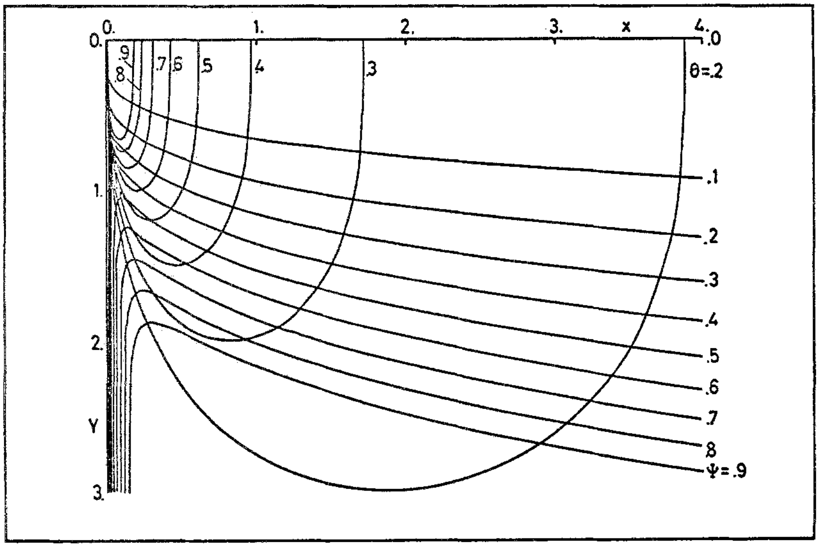

Fig. 7 : Flow pattern in a floor plume (Pr = 1)

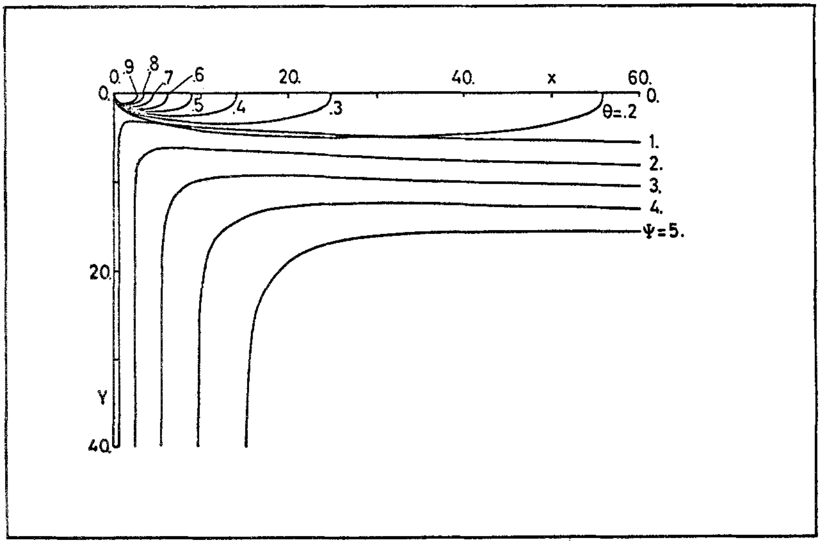

Fig. 8 : Flow pattern in a floor plume (Pr = 10)

4. Horizontal buoyant plume submerged at the level of a thermocline

When the sea density increases with depth, one says that the sea is stably stratified. A vertical density profile then typically displays two or more plateaus, some 10 to 100 m deep, separated by transition zones of only 1 m depth, the thermoclines. If a vertical plume reaching a thermocline has lost enough buoyancy underway it will be deflected horizontally and feed a so-called submerged sewage field. To model this field, suppose that a known flow of liquid of density equal to the mean density between those of the adjacent plateaus is injected at the level of the (infinitely thin) thermocline. The resulting plume will be symmetric with respect to Ox and the equations describing this free shear boundary layer are again (1.4), (1.5) with the boundary conditions :

$Y = o \qquad \Psi = o$ (Ox is a streamline) (4.1)

$\dfrac{\partial^2\Psi}{\partial Y^2} = o$ (the horizontal velocity profile is symmetric with respect to Ox) (4.2)

$\Theta = o$ (by symmetry) (4.3)

$Y = \infty \qquad \dfrac{\partial\Psi}{\partial Y} = o$ (far from the plume, the velocity is purely vertical) (4.4)

$\Theta = 1$ (density is given) (4.5)

Let us adapt Schlichting's [7] solution for the linear isothermal jet and look for a Blasius-Howarth [8] expansion of the form :

$$\Psi = x^{1/3} \sum_{i=o}^{\infty} x^{4i/3} f_i(\eta) \tag{4.6}$$

$$\Theta = \sum_{i=o}^{\infty} x^{4i/3} g_i(\eta) \tag{4.7}$$

where the similarity variable is :

$$\eta = \frac{Y}{x^{2/3}} \tag{4.8}$$

The fundamental terms of these expansions are

$$f_o = 6\,\alpha \tanh \alpha\,\eta \tag{4.9}$$

$$g_o = \frac{\int_o^{\eta} \frac{d\eta}{\cosh^{2\,Pr} \alpha\eta}}{\int_o^{\infty} \frac{d\eta}{\cosh^{2\,Pr} \alpha\eta}} \tag{4.10}$$

where α is related to the momentum flux by :

$$M = 2\,\rho \int_o^{\infty} u^2\, dY = 48\,\rho\,\alpha^3 \tag{4.11}$$

and the enthalpy flux in the x direction is naught (by antisymmetry). This first term gives an impression of the first part of the development of the plume (figs 9, 10).

For the practical case of disposal of urban sewage in sea-water, the density difference is essentially due to the concentration difference in sodium chloride. The interesting pollutants might be present in minute concentrations and would then diffuse through this plume, but without disturbing its density or its velocity-field, If the concentration of such a pollutant is c, a solution of the following form exists :

$$c = x^{-1/3} \sum_{i=o}^{\infty} x^{4i/3}\, h_i(\eta) \tag{4.12}$$

satisfying

$$\frac{\partial c}{\partial x}\frac{\partial \Psi}{\partial Y} - \frac{\partial c}{\partial Y}\frac{\partial \Psi}{\partial x} = \frac{1}{Sc}\frac{\partial^2 c}{\partial Y^2} \tag{4.13}$$

$$Y = o \quad : \quad \frac{\partial c}{\partial Y} = o \quad \text{(by symmetry)} \tag{4.14}$$

$$Y = \infty \quad : \quad c = o \quad \text{(the dilution far from the plume is complete)} \tag{4.15}$$

It is easy to show that the fundamental term in (4.12) is :

$$h_o = \frac{h_o(o)}{\cosh^{2\,Sc} \alpha\eta} \tag{4.16}$$

Once again, the dispersion in this horizontal plume, less important than in a vertical one, is however not negligible : c(Y = o) varies like $x^{-1/3}\, P_o$. Figs. 11 and 12 show, for Sc = 100, the sharp pollutant tongue at the level of the thermocline.

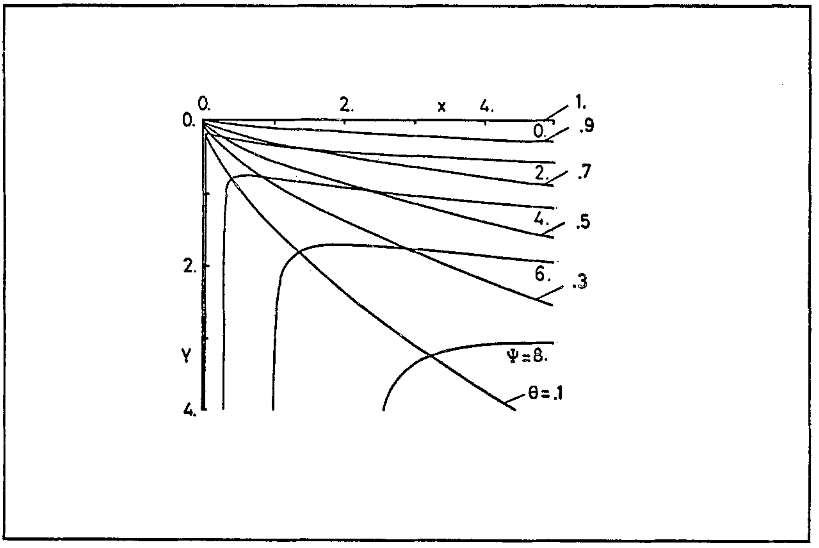

Fig. 9 : Flow pattern in a submerged plume (Pr = 1)

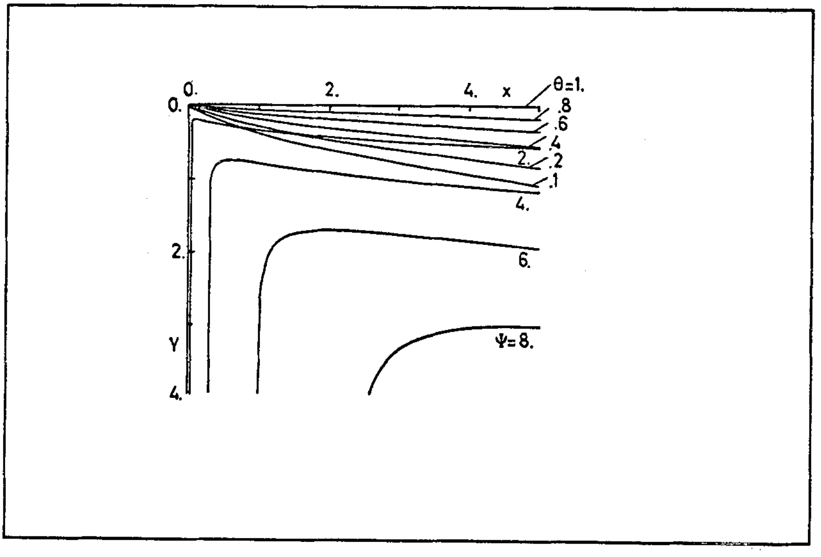

Fig. 10 : Flow pattern in a submerged plume (Pr = 10)

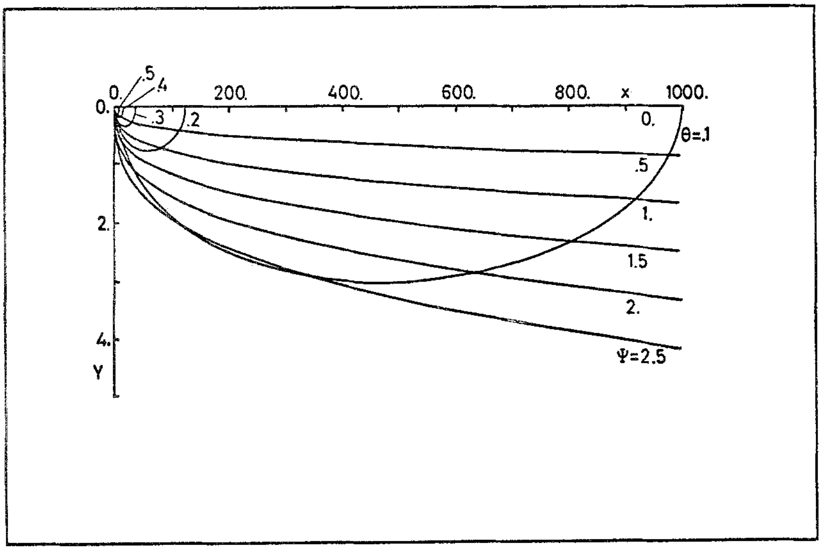

Fig. 11 : Pollutant dispersion in a submerged plume (Sc = 100)

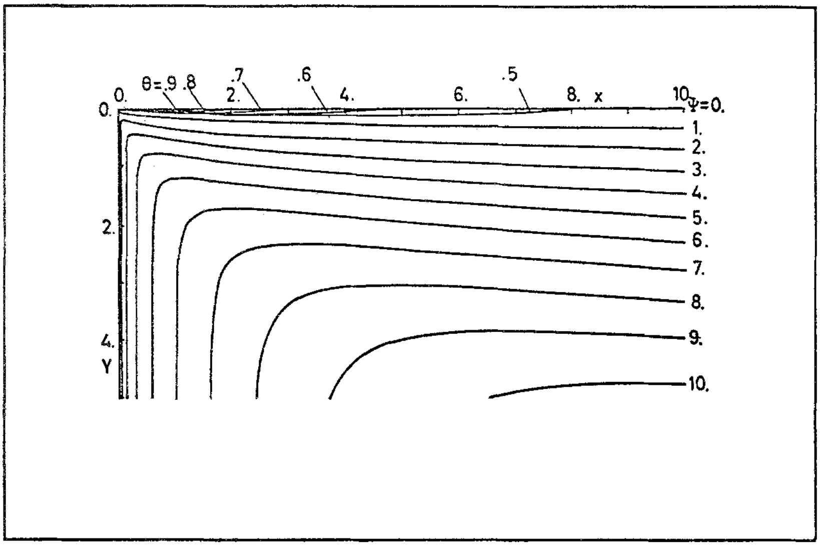

Fig. 12 : Pollutant dispersion in a submerged plume (Sc = 100)

5. Optimization technique

Boundary value problems can be solved usingan optimization scheme. The expression :

$$f = \alpha_1 \left(\int_{-\infty}^{+\infty} f'g \, d\eta - 1 \right)^2 + \alpha_2 f'^2(\infty)$$

for instance is a suitable objective function for the solution of (2.8), (2.9), (2.10), (2.11), (2.12). For the case of analog integration in a hybrid configuration, machine noise disturbs the correct evaluation of the criterium function. If the partial derivatives cannot be determined analytically, the numerical evaluation of the derivatives is jeopardized by noise. A good and fast direct search technique is preferable . The method chosen here is a modified rotating coordinate technique. The algorithm has been provided of an efficient line search for determining the minimum point in a given direction.

line search

The line search is a combination of direct search and curve fitting in such a way that under fairly general conditions convergence to the minimum is guaranteed [9].

Let $\underline{x}_k$ be the present point, $\underline{d}_k$ the direction of search and α_k a given step. Following function evaluations are done

$$f(\underline{x}_k + \alpha_k \underline{d}_k), \; f(\underline{x}_k + 2\alpha_k \underline{d}_k), \; f(\underline{x}_k + 4\alpha_k \underline{d}_k) \ldots\ldots$$

till three points

$$\underline{x}_1 = \underline{x}_k + \alpha_1 \underline{d}_k ,$$
$$\underline{x}_2 = \underline{x}_k + \alpha_2 \underline{d}_k ,$$
$$\underline{x}_3 = \underline{x}_k + \alpha_3 \underline{d}_k$$

are obtained

which satisfy the condition

$$f(\underline{x}_1) > f(\underline{x}_2) < f(\underline{x}_3)$$

If the function $f(\underline{x})$ is strictly unimodal in the given direction the coordinate α_m of the minimum point $\underline{x}_1 + \alpha_m \underline{d}_k$ will be in the interval (α_1, α_3). Then a curve fitting procedure is started which does not require derivatives.

A quadratic

$$q(\alpha) = \sum_{i=1}^{3} f(\underline{x}_i) \frac{\prod_{j\neq i} (\alpha - \alpha_j)}{\prod_{j\neq i} (\alpha_i - \alpha_j)}$$

is passed through the three points and the coordinate of the extremum

$$\alpha_e = \frac{1}{2} \frac{(\alpha_2^2 - \alpha_3^2) f(\underline{x}_1) + (\alpha_3^2 - \alpha_1^2) f(\underline{x}_2) + (\alpha_1^2 - \alpha_2^2) f(\underline{x}_3)}{(\alpha_2 - \alpha_3) f(\underline{x}_1) + (\alpha_3 - \alpha_1) f(\underline{x}_2) + (\alpha_1 - \alpha_2) f(\underline{x}_3)}$$

is warranted to be a minimum and contained in the interval (α_1, α_3) ; $f(\underline{x}_k + \alpha_e \underline{d}_k)$ is evaluated. If $\alpha_e < \alpha_2$ a new point $\underline{x}_1 = \underline{x}_k + \alpha_e \underline{d}_k$ is introduced reducing (α_1, α_3) to (α_e, α_3).

If $\alpha_e > \alpha_2$, $\underline{x}_3 = \underline{x}_k + \alpha_m \underline{d}_k$ is calculated and (α_1, α_3) reduces to (α_1, α_e). A new quadratic fit is performed on the reduced interval.

If $\alpha_1 = \alpha_2$, the interval $(\alpha_2, \alpha_i) - \alpha_i$ is the coordinate of $\underline{x}_i$ being the argument of $f_i = \max(f(\underline{x}_1), f(\underline{x}_2))$-is divided to obtain a new point $\underline{x}_n$ in such a way that the new interval is smaller than the preceding one. It can be proved by the Global Convergence Theorem [9] that this algorithm converges to the solution if the objective function is continuous and unimodal in α. The order of convergence is known to be about 1.3 [9]. In practice the search procedure has to be terminated before it has converged. For these problems α_m is determined to within a fixed percentage of its true value. A constant c, $o < c < 1$ is selected ($c = 0.01$) and α is found so as to satisfy $|\alpha - \bar{\alpha}| \leqslant c|\bar{\alpha}|$ where $\bar{\alpha}$ is the lower bound α_1 on the true minimizing value of the parameter if α_1 is different from zero or equal to the termination value for the complete algorithm if α equals zero.

<u>optimization algorithm</u> [10]

In a simple coordinate descent method the coordinate directions $(\underline{e}_1, \underline{e}_2, \underline{e}_3 \dots \underline{e}_n)$ are cyclically used to provide the directions for individual line searches. If the objective functions has continuous partial derivatives this method is globally convergent [9], and the convergence rate is affected by rotation of the coordinates. However if the first partial derivatives are not continuous objective functions and coordinate directions can be found so that the algorithm will not find the minimum. By rotating the coordinate system after n line searches an attempt is made to solve this problem. If at the same time one axis is oriented towards the direction of the valley, locally estimated in a way analoguous to the method used in the parallel tangent algorithm it has been found by some trial objective functions that the convergence rate is improved. An efficient method for obtaining a new orthonormal set is that of Powell [11], which requires $O(n^2)$ multiplications instead of $O(n^3)$.

The final algorithm is the following :

Given $\underline{x}_o$ and the current set of orthogonal directions $D = (\underline{d}_o, \underline{d}_1, \dots d_{n-1})$ a set of β_j's are computed using n line searches :

$$\beta_j = \min_{\beta} f(\underline{x}_j + \beta \underline{d}_j) \qquad \text{with } \underline{x}_{j+1} = \underline{x}_j + \beta_j \underline{d}_j$$

$$\text{for } j = 0, 1, 2 \dots, n-1$$

The orders of the directions d_j is changed yielding $D' = (\underline{d}'_o, \underline{d}'_1, \dots \underline{d}'_{n-1})$ so that the first k directions have β-values different from zero $(\beta_o, \beta_1, \beta_2, \dots, \beta_k, o, o, \dots o)$. Then a new set of directions is computed.

1 Set $j = k$

$\tau = (\beta_k)^2$

$\underline{\sigma} = \beta_k \underline{d}'_k$

2 if $j = o$ terminate the process, otherwise compute

$$d^n_j = \frac{(\tau \underline{d}'_{j-1} - \beta_{j-1} \underline{\sigma})}{[\tau(\tau + \beta^2_{j-1})]^{1/2}}$$

3 Set $j = j - 1$

$\tau = \tau + (\beta_j)^2$

$\underline{\sigma} = \underline{\sigma} + \beta_j \underline{d}'_j$; go to 2

4 The remaining vectors are obtained as follows

$$\underline{d}^n_o = \frac{\underline{\sigma}}{\sqrt{\tau}} \qquad \underline{\sigma} = \sum_{j=o}^{k} \beta_j \underline{d}'_j \qquad \tau = \sum_{j=o}^{k} (\beta_j)^2$$

$$\underline{d}^n_k = \underline{d}'_k \qquad \text{for } j = k+1, k+2, \dots, n-1$$

We now have a new set $D^n = (\underline{d}^n_o, \underline{d}^n_1, \dots, \underline{d}^n_{n-1})$ to repeat the procedure.

To minimize the number of objective function evaluations a suitable step for the line search is necessary. If the step is too small ; the initial value has to be doubled too many times. If the step is too large, too many curve fittings have to be performed. Therefore the step is adjusted during the optimization. For every coordinate relaxation (n line searches) $a = \frac{1}{n} \sum_{j=o}^{n-1} \beta_j$ is computed. The series $\{a_k\}$ converges at least linearly for the quadratic case [9] . The convergence rate is dependent of the special objective function under study but experimentally it has been found that if a fraction of a (say a/8) is used as step for the next coordinate search an improvement in overall computation time is observed for the different objective functions encountered in the problem.

6. Conclusion

The classical methods of boundary layer theory allow us to accurately model linear laminar horizontal buoyant plumes. Using the modern developments of the theory (matched asymptotic expansions) we could even produce still better solutions of the non-linear problems considered. However, for any reasonable design, the unit flux F_o is likely to be so large that the flow would be turbulent rather than laminar. The results of our forthcoming study of turbulent horizontal plumes are more qualitative, but the general approach to modeling closely resembles the one used here.

Notations

u	horizontal velocity component
x	horizontal distance
y	vertical distance
F_o	density difference flux per unit length of diffusor
Gr	GRASHOF-number
P_o	mass flux of pollutant per unit length of diffusor
Pr	PRANDTL-number
α	thermal expansion coefficient
ψ	streamfunction
Θ	reduced density difference
ρ	specific mass
Γ	GAMMA-function

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