STRATIFIED UNIVERSAL MANIFOLDS AND TURNPIKE THEOREMS FOR A CLASS OF OPTIMAL CONTROL PROBLEMS

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Consider the following optimal control problem

$$\dot{x} = u\, Q(x) \tag{1}$$

where $x \in R_n^+$, i.e. $x = (x_1, \ldots, x_n)$, $x_i > 0$; u - is n-dimensional control vector belonging to the simplex: $u_i \geq 0$, $\sum_{i=1}^{n} u_i = 1$ for any $1 \leq i \leq n$. Q is a scalar function: $Q(x) \neq 0$ in R_+^n, $Q \in C^2(R_+^n)$. Our goal is to minimize the functional

$$R(x(T), T) \tag{2}$$

where T is the smallest value of t such that $(x(t), t) \in \mathcal{M}$, $\mathcal{M}$ being a given set: $M(x(t), t) = 0$ (called the terminal set). Here R and M are scalar functions: $R, M \in C^1(R_+^{n+1})$. For example, in the time optimal problem of transferring the point from the state (x_0, t_0) to the terminal set $M(x) = 0$ we have to minimize $R(x(T), T) = T - t_0$

A manifold is said to be singular if it is impossible to define the optimal control from the condition of maximum of the Hamiltonian form on this manifold [1]. To know singular manifolds is basic for constructing synthesis in the problems with linear vectogram. A singular manifold is said to be universal if it attracts optimal trajectories. This term was introduced by Isaacs in [2]. The universal manifolds are of great interest among the singular ones.

Isaacs put a question whether a vectogram which is linear in several controls can lead to universal manifolds of codimension more than one, and whether those manifolds are intersection of the universal hypersurfaces (see [2], ch.7). The synthesis for system

(1) is the affirmative answer to this query.

Definition. A set of measure zero is called a universal set V if for any point $(x,t)\in V$ there exists $T_0>t$ such that for any $T\in(t,T_0)$ there exists neighbourhood of point (x,t) – $U_{x,t}$ such that for any $(x_0,t_0)\in U_{x,t}$ the optimal trajectory starting from the point (x_0,t_0) will belong to V at the moment T.

The part of the state space – $\widetilde{V}$ – and the optimal synthesis in $\widetilde{V}$ is called the universal structure of the optimal synthesis if for any point $(x_0,t_0)\in\widetilde{V}$ the optimal trajectory starting from the point (x_0,t_0) comes up to the universal set, or, put it another way, the optimal structure is the behaviour of optimal trajectories in the region of attraction of the universal set. In our case the universal set is a stratified manifold [3], K-dimensional stratum of which will be called a universal K-dimensional manifold.

Theorem 1

Let the function $Q(x)$ be such that

1) The set $\frac{\partial Q}{\partial x_1}=\frac{\partial Q}{\partial x_2}=\dots=\frac{\partial Q}{\partial x_n}$ is nonempty in R_+^n,

2) On the set $\frac{\partial Q}{\partial x_i}=\frac{\partial Q}{\partial x_j}$ for any i, j $(i\neq j)$ the following condition holds:

$$\frac{\partial}{\partial x_i}\left(\frac{\partial Q}{\partial x_i}-\frac{\partial Q}{\partial x_j}\right)<0 .$$

Let us denote by $\omega_{i_1,\dots,i_K}$ the $(K-1)$-vector (see [4], p.22) which is orthogonal to the manifold

$$\frac{\partial Q}{\partial x_{i_1}}=\frac{\partial Q}{\partial x_{i_2}}=\dots=\frac{\partial Q}{\partial x_{i_K}}$$

It is required to meet the following condition:

3) For any K-tuple $(i_1,i_2,\dots,i_K)$, $(K=2,\dots,n)$ the Plukker coordinates of the projection of $\omega_{i_1,\dots,i_K}$ upon the co-

ordinate plane $(x_{i_1}, \ldots, x_{i_K})$ have the same sign and are not equal to zero. (The $(K-1)$-vector $\mathcal{N}_{i_1,\ldots,i_K}$ is formed with a normal to manifolds

$$\frac{\partial Q}{\partial x_i} = \frac{\partial Q}{\partial x_j}, \quad i \neq j, \quad i, j = i_1, \ldots, i_K)$$

Then the sets

$$\frac{\partial Q}{\partial x_{i_1}} = \frac{\partial Q}{\partial x_{i_2}} = \cdots = \frac{\partial Q}{\partial x_{i_K}} > \frac{\partial Q}{\partial x_e} \tag{3}$$

$$(K = 2, \ldots, n; \quad \ell \neq i_1, \ldots, i_K)$$

are universal $(n-K+1)$ -dimensional manifolds.

Note, that it follows from conditions (1)-(3) that for any K there exist K -dimensional universal manifolds and that these manifolds have no singularities. In this case the universal structure can be present in the following form. There are C_n^2 $(n-1)$ -dimensional manifolds $\frac{\partial Q}{\partial x_i} = \frac{\partial Q}{\partial x_j} > \frac{\partial Q}{\partial x_K}$ (for any $K \neq i, j$) whose intersection is the manifold

$$\frac{\partial Q}{\partial x_1} = \frac{\partial Q}{\partial x_2} = \cdots = \frac{\partial Q}{\partial x_n} \tag{4}$$

These manifolds divide R_+^n into the regions - V^i - where $\frac{\partial Q}{\partial x_i} > \frac{\partial Q}{\partial x_e}$ for any $\ell \neq i$. In the region V^i the optimal control is:

$$u_i = 1, \quad u_e = 0 \qquad \text{for any } \ell \neq i$$

On the manifolds (3) the optimal control u is proportional to the vector whose coordinates are equal to the Plukker coordinates of the projection of $\mathcal{N}_{i_1,\ldots,i_K}$ upon the coordinate plane $(x_{i_1}, \ldots, x_{i_K})$.

In particular, $u_e = 0$ when $\ell \neq i_1, \ldots, i_K$. Note that this optimal control holds a state point in the universal manifold.

Conditions (1)-(3) are close to the necessary ones when the universal structure of the optimal synthesis has the form described above.

For elucidation let us note that the stratified universal manifold is diffeomorphic to the set (system) of bisector hyperplanes which cut off at the points of intersections of these hyperplanes.

Theorem II

The universal structure of optimal synthesis of the optimal control problem -(1)- is invariant to functionals from the class (2), i.e. if to attain the terminal set the optimal trajectory moves on the universal manifold for a time, then for any sufficiently far terminal set the initial part of the optimal trajectory is independent of a functional from the class (2), and in this case the optimal trajectory attains the universal manifold in the shortest run. If one knows a universal structure of optimal synthesis, then under additional conditions on a function $Q(x)$ one can construct the optimal synthesis in the whole space R^n_+. For this, one has to construct switching surfaces and to find intersection of these surfaces with the stratified universal manifold.

For example, let us consider a time-optimal problem. The terminal set is $x_n = const$. Let the function $Q(x)$ be such that the conditions (1)-(3) and these ones hold:

4) the manifold (4) have no asymptote parallelled to coordinate planes;

5) $\frac{\partial}{\partial x_i}\left(\frac{1}{Q}\frac{\partial Q}{\partial x_i} - \frac{1}{Q}\frac{\partial Q}{\partial x_j}\right) < 0$ in R^n_+ ;

6) $\frac{\partial}{\partial x_i}\left(\frac{1}{Q}\frac{\partial Q}{\partial x_j}\right) < 0$, when $i \neq j$ and $\frac{\partial}{\partial x_i}\left(\frac{1}{Q}\frac{\partial Q}{\partial x_i}\right) \leq 0$;

then a state point either doesn't attain the universal manifold, and in this case there is no more than one switch, or attains the universal manifold in the points of intersection of the latter with the switching surface and in this case it moves on the universal manifold going from strata of less dimension to strata of higher dimension one after another. In the latter case the state point can come to the region V^i at any moment. We can write out explicitly the equations of switching surface. The optimal syntheses for n = 2,3 are given in Fig. 1 and Fig. 2 respectively.

Note that the conditions on a choice of a function $Q(x)$ are not restrictive. For instance, the conditions (1)-(6) are fulfilled for functions $Q(x) = f_1(x_1) \cdot \ldots \cdot f_n(x_n)$ where $f_i(x_i)$ is a monotonic and logarithmically convex function, i.e. $f_i' > 0$, $(\ln f_i(x_i))'' < 0$.

One can obtain the similar results for a case when there are uncontrollable variables in system (1), i.e. for a system

$$\dot{x} = u\, Q(x, y)$$
$$\dot{y}_i = F_i(y)$$

In this case, if $Q(x,y) = Q_1(x)\, Q_2(y)$, then the universal manifolds are cylindric in y. One can investigate similarly the case in which the manifold (4) is empty, but at least one of the manifolds (3) is nonempty.

The problem (I) arises naturally for an economical problems of optimal allocating of sources among the sectors, producing several factors of production $(x_1, \ldots, x_n)$, where $Q(x)$ is a production function. Theorems I and II have in this case a direct relation to turnpike theorems. The manifold (4) plays a role of von-Neumann's ray of maximal balanced growth.

The calculation of universal manifolds and optimal control on it has made it possible to derive a variety of economic facts of great importance. Namely, the universal manifolds are defined by the condition of equality of relating norm of efficiency of product relative to factors of production. For a Cobb-Duglas's type production functions $Q(x) = x_1^{\alpha_1} \cdot \ldots \cdot x_n^{\alpha_n}$ we have that the optimal controls on the universal manifolds (3) are proportional to α_i - the elasticity of product relative to the factors of production $-x_i$ $(i = i_1, \ldots, i_k)$.

As a consequence of theorem II, this assertion can be derived: the shadow prices (a gradient of a Bellmann function) for a problem of time optimal control are invariant related to a choice of any sufficiently far terminal set. In this case the discounting rate on the universal manifold is equal to the optimal rate of growth and to a norm of efficiency. It is constant only for a first order homogeneous function, i.e. for a case of a constant return to scale.

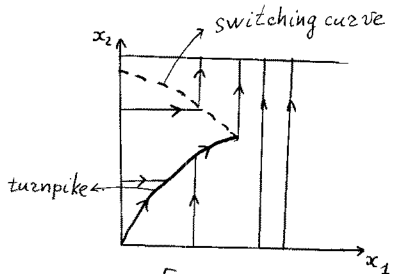
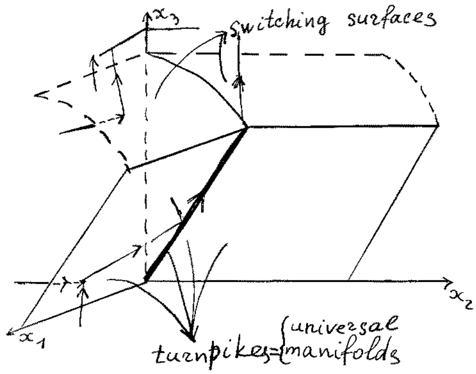


Fig. 1



References

[1] L.S. Pontrjagin, V.G. Boltjanskiī, R.V. Gamkrelidze and E.F. Miščenko, The mathematical theory of optimal processes, Fizmatgiz, Moscow, 1961; English transl., Wiley, New York, 1962, and Macmillan, New York, 1964.

[2] Isaacs, R., Differential games, Wiley, New-York, 1965, Russian transl., Mir, Moscow, 1967.

[3] Thom,R., Local topological properties of differentiable mappings, Differential Analysis (Papers presented at the Bombay Colloquium, 1964).

[4] Картан Э. Внешние дифференциальные системы и их геометрические приложения. Издательство Московского Университета, 1962.