

# SUFFICIENT CONDITIONS FOR ABSOLUTE MINIMUM OF THE MAXIMAL FUNCTIONAL

## IN THE MULTI - CRITERIAL PROBLEM OF OPTIMAL CONTROL

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### Introduction

One possible approach to the problem of optimization of a dynamical system

$$x(t) = f(x, u, t) , \quad (1)$$

when the simultaneous minimization of the given set  $\Omega$  of performance criteria-functionals  $J_\omega(x, u), \omega \in \Omega$  is required, consists in reducing it to a mono-criterial problem with the single functional

$$J(x, u) = \max_{\omega \in \Omega} \{ J_\omega(x, u) \} \quad (2)$$

to be minimized.

The tasks of such a type arise in technical fields (for example, the problem of maximal deviation of the regulated object coordinates minimization) and in the mathematical economics (the example is the problem of minimization of maximal time of production for components of the final product). Some problems with the nondifferentiable performance criteria can be formulated in the same way. For example, the problem of minimization of the functional

$$J(x, u) = |I(x, u)|$$

may be replaced by an equivalent problem of the minimization of the functional

$$J(x, u) = \max \{ I(x, u), -I(x, u) \}.$$

The minimax problem considered attracted attention of many authors. R.Bellman suggested treating it by means of dynamical programming [1,2] , A.Ya.Dubovitsky, A.A.Milutin, V.F.Demianov, I.V.Girsanov, V.N.Malozemov, T.K.Vinogradova [3-8] investigated various forms of the necessary optimality conditions, in works [9,10] numerical algorithms for the solution of this problem with the help

of the auxiliary functionals were constructed. V.F. Demianov and T.K. Vinogradova [4,7,8] proposed using the necessary optimality conditions for determining the stationary solutions.

In the present report the sufficient optimality conditions for optimization problem with the functional (2) are formulated. Besides the general significance of sufficient conditions for optimization problems, specifically when numerical solutions are interpreted, in the minimax problem considered they are especially interesting when the various necessary conditions (as shown in [7,8] - no equivalent) are discussed. From the results given below it follows that the necessary conditions given here, proposed for the first time by A.Ya. Dubovitsky and A.A. Milutin [3] and including Pontrjagin maximum principle as a main element, are very strong. For formulation of global sufficient conditions with their help they must be supplemented by the conditions for the mutual disposition of extremals in the studied region as a whole, but not for any characteristics of a specific separately taken extremal.

The approach suggested is the extension of the extremals field method used before for the investigation of monocriterial problem [12]. The mathematical tool of the classical calculus of variations is not applicable for treating the problem considered because the functional is not smooth and the controls are restricted. Nevertheless such analogues of an explicit nonlocal relations of the classical calculus of variations as a Hilbert invariant and exact formulas for variations of a functional can be obtained. It is the possibility to write the variation of a functional in the form of exact formula without any assumption of the nearness between the investigated trajectories that permit us to formulate the sufficient conditions for absolute minimum of functional (2) by means of the approach analogous to that of Weierstrass [13].

### Statement of the problem

Let us give an explicit formulation of the problem. In (1)  $x$  is the  $n$ -dimensional phase-coordinate vector and  $u$  is the  $2$ -dimensional control vector. We shall take the admissible controls to be piecewise-continuous functions  $u(t) \in U$ . The left end  $\{x^0, T_0\}$  of a trajectory  $x(t)$  of the system (1) is fixed, the right end  $\{x(T), T\}$  in the terminal moment of time must belong to the terminal manifold  $M$ , specified by the equations

$$M_j(x, t) = 0, \quad j = 1, \dots, m \leq n. \quad (3)$$

As the performance criteria we shall consider a finite number of functionals which are given as functions of the right end of the trajectory (in such a form integral functional may also be written)

$$J_\omega(x, u) = \Phi_\omega[x(T), T]. \quad (4)$$

Thus the problem we pose is to choose the control  $u(t) \in \mathcal{U}$ , that minimizes the functional

$$J(x, u) = \Phi[x(T), T] = \max_{\omega} \{ \Phi_\omega[x(T), T] \}. \quad (5)$$

The functions  $M_j(x, t)$  and  $\Phi_\omega(x, t)$  are assumed to be continuous together with their first-order partial derivatives and to have bounded second-order partial derivatives with respect to all arguments, and  $f(x, u, t)$  has the same properties with respect to  $x, u$  and is continuous with respect to  $t$  (the discontinuous problems also may be treated [12]).

#### Necessary optimality conditions

Necessary optimality conditions we derive by considering the region of attainability in the  $(\ell+m)$ -dimensional vector space of variations of the end-point conditions (3) and the functionals (4) [14, 15]. In that way the necessary optimality conditions for the problem discussed are reduced to the necessary minimum conditions for the auxiliary Lagrange functional

$$\psi[x(T), T] = \sum_{\omega=1}^{\ell} \lambda_\omega \Phi_\omega[x(T), T] + \sum_{j=1}^m \mu_j M_j[x(T), T] \quad (6)$$

and can therefore be derived with the help of the formula for small variations of a functional [16, 12] (this necessary optimality conditions can also be obtained from the results of the work [3]).

The joint use of the results given below with the results of [12, 14, 15] gives a possibility to formulate the necessary and sufficient optimality conditions for a broad range of minimax problems, in particular for the problems with nonfixed left end of trajectory and for the problems where the terminal points for functionals (4) is each determined by its own group of conditions of the form (3). We

shall confine ourselves here to the formulation discussed for simplicity.

Let  $J^*$  denote the minimal value of the functional (5), and  $R$  denote the set of indexes  $\omega$  for which we have an equality  $\Phi_\omega[x(T), T] = J^*$  on the optimal trajectory.

Theorem 1. If the control  $u(t)$  and trajectory  $x(t)$  are optimal in the problem (1), (3)-(5) then there exist numbers  $\lambda_\omega \geq 0$  ( $\lambda_\omega = 0$  for  $\omega \notin R$ ) and  $\mu_j$  not all zero, such that for the vector function  $\rho(t)$  determined by the system of equations

$$\dot{\rho}(t) = -\nabla_x H(x, \rho, u, t), \quad H = (p, f(x, u, t)), \quad \rho(T) = -\nabla_x \psi[x(T), T] \quad (7)$$

the following conditions are satisfied

$$H(x, \rho, u, t) = \sup_{v \in U} H(x, \rho, v, t), \quad T_0 \leq t < T, \quad (8)$$

$$-H[x(T), \rho(T), u(T-0), T] + \partial \psi[x(T), T] / \partial T = 0. \quad (9)$$

Any trajectory of the system (1) with the initial point  $\{x^0, T_0\}$  which satisfies the conditions of the theorem 1 will be denoted  $\tilde{x}(t) = \tilde{x}(x^0, T_0; t)$  and called an extremal, and corresponding control will be denoted  $\tilde{u}(t) = \tilde{u}(x^0, T_0; t)$ .

#### Field of extremals

Let a region  $A \subset E_{n+1}$  be given. We construct in  $A$  the set  $\tilde{X}$  of the extremals  $\tilde{x}(z, \tau; t)$  with the initial points  $\{z, \tau\} \in A$ . Marking all the values concerning extremals with the sign  $\sim$ , we obtain that the extremals of set  $\tilde{X}$  determine in  $A$  a reference function  $\tilde{V}(z, \tau) = \psi[\tilde{x}(z, \tau; \tilde{T}(z, \tau)), \tilde{T}(z, \tau)]$ , a synthesis control  $\tilde{u}(z, \tau) = \tilde{u}(z, \tau; \tau)$ , a function  $\tilde{R}(z, \tau)$  which indicates the set of maximal functional from (4), Lagrange multipliers - functions  $\tilde{\lambda}_\omega(z, \tau)$  and an  $(n+1)$ -dimensional incline function

$$\tilde{p}(z, \tau), \quad \tilde{H}(z, \tau) = (\tilde{p}(z, \tau), f(z, \tilde{u}(z, \tau), \tau)). \quad (10)$$

We say that the set  $\tilde{X}$  with the incline function (10) forms an  $L$ -continuous field of extremals in  $A$ , if there exist constants  $\alpha$  and  $\beta$  such that for any two extremals  $\tilde{x}'(t) = \tilde{x}(z', \tau'; t)$  and  $\tilde{x}''(t) = \tilde{x}(z'', \tau''; t)$ , where  $|\{z', \tau'\} - \{z'', \tau''\}| \leq \varepsilon$ , the corresponding controls  $\tilde{u}'(t)$  and  $\tilde{u}''(t)$  and the terminal

moments of time  $\tilde{T}'$  and  $\tilde{T}''$  satisfy the conditions (here  $\tau^* = \max\{\tau, \tau''\}$ ,  $T^* = \min\{\tilde{T}', \tilde{T}''\}$ )

$$\int_{\tau^*}^{T^*} |\tilde{u}'(t) - \tilde{u}''(t)| dt \leq \alpha \varepsilon, \quad |\tilde{T}' - \tilde{T}''| \leq \beta \varepsilon.$$

### Exact formula for variation of functional

Let  $\hat{x}(t)$  be the trajectory of the system (1) with initial and end points  $\{\hat{x}(T_0), T_0\} = \{x^0, T_0\}$  and  $\{\hat{x}(\tilde{T}), \tilde{T}\} \in M$  lying entirely in  $A$  and corresponding to the admissible control  $\hat{u}(t)$ .  
Let

$$\Delta J = \Phi[\hat{x}(\tilde{T}), \tilde{T}] - \Phi[\hat{x}(x^0, T_0; \tilde{T}), \tilde{T}]$$

denote the difference between the values of the functional (5) for  $\hat{x}(t)$  and for the extremal with the initial point  $\{x^0, T_0\}$ .

If  $\lambda_\omega > 0$  for  $\omega \in \tilde{R}(z, \tau)$  for each  $\tilde{x}(z, \tau; t) \in \tilde{X}$  in the conditions of theorem 1, then we can normalize the multipliers  $\lambda_\omega, \mu_j$  so that to obtain

$$\sum_{\omega \in \tilde{R}(z, \tau)} \tilde{\lambda}_\omega(z, \tau) = 1, \quad \{z, \tau\} \in A. \quad (11)$$

Then from (5), (6) and determination of the reference function  $\tilde{V}(z, \tau)$  we have

$$\begin{aligned} J(x, u) &= \Phi[\tilde{x}(z, \tau; \tilde{T}(z, \tau)), \tilde{T}(z, \tau)] = \\ &= \Psi[\tilde{x}(z, \tau; \tilde{T}(z, \tau)), \tilde{T}(z, \tau)] = \tilde{V}(z, \tau). \end{aligned} \quad (12)$$

The representation (12) and the condition (11) permit us to employ the tool of exact formulas for variations of functional for studying the functional in the region  $A$  as a whole. For the problem considered all the results of lemma 1 [12] under the conditions of theorem 2 are valid. We shall confine ourselves to the result that is necessary for the discussion of sufficient optimality conditions given below.

Theorem 2. Let the field of extremals in  $A$  be  $L$ -continuous, the condition (11) be true and  $\tilde{R}(\hat{x}(t), t)$  as a function of  $t$  be piecewise-continuous. Then the following formula is true

$$\Delta J = - \int_{T_0}^{\tilde{T}} \{H[\hat{x}, \hat{p}(\hat{x}, t), \hat{u}(t), t] - H[\hat{x}, \hat{p}(\hat{x}, t), \tilde{u}(\hat{x}, t), t]\} dt. \quad (13)$$

### Sufficient optimality conditions

It follows from the formula (13) and maximum condition (8) that under the conditions of theorem 2 for any trajectory  $\hat{x}$  of the system (1) lying entirely in  $A$

$$\phi[\hat{x}(\hat{T}), \hat{T}] \geq \phi[\tilde{x}(\tilde{T}), \tilde{T}].$$

This inequality proves the following statement.

**Theorem 3.** Let the field of extremals in  $A$  be  $L$ -continuous,  $\tilde{\lambda}_0(x, t) > 0$  for  $\omega \in \tilde{R}(x, t)$  and  $\tilde{R}(x(t), t)$  as a function of  $t$  be piecewise-continuous along any admissible trajectory  $x(t)$ . Then each extremal belonging entirely to  $A$  for any of its points in the role of fixed initial data at the left end furnishes in  $A$  an absolute minimum to the functional (5) subject to the fulfillment of conditions (3).

Theorem 3 determines the properties of principle under which the global optimality takes place. Its statement is valid also for the piecewise  $L$ -continuous fields of extremals. Such generalization of the theorem 3 proved as in [12] has a wide range of applications. If set (4) consists of a unique functional, then the conditions of theorems 1 and 3-4 transform to the necessary and sufficient conditions [11, 12] for mono-criterial problem.

### R e f e r e n c e s

1. R. Bellman, "Dynamic Programming", N.Y., 1957.
2. R. Bellman, "Adaptive Control Processes: a Guided Tour", N.Y., 1961.
3. A. Ya. Dubovitsky, A. A. Milutin, Zh. Vychisl. Mat. i Mat. Fiz., v. 5, N 3, 1965.
4. V. F. Demianov, Vestnik Leningradskogo Univers., N 7, 1966.
5. I. V. Girsanov, "Lectures on the Mathematical Theory of the Extremal Problems", M., 1970 (Russian).
6. V. F. Demianov, V. N. Malozemov, "Introduction into Minimax", M., 1972 (Russian).
7. T. K. Vinogradova, V. F. Demianov, Dokl. Akad. Nauk SSSR, v. 213, N 3, 1973.
8. -----, Zh. Vychisl. Mat. i Mat. Fiz., v. 14, N 1, 1974.
9. V. V. Velichenko, "Numerical method for solving the optimal control problems", dissertation, M., 1966.
10. -----, Kosmicheskie Issledov., v. X, N 5, 1972.
11. L. S. Pontrjagin, V. G. Boltjanskii, R. V. Gamkrelidze, E. F. Mishenko, "The Mathematical Theory of Optimal Processes", N.Y., 1962.
12. V. V. Velichenko, Zh. Vychisl. Mat. i Mat. Fiz., v. 14, N 1, 1974.
13. G. A. Bliss, "Lectures on the Calculus of Variations", M., 1950.
14. V. V. Velichenko, Dokl. Akad. Nauk SSSR, v. 174, N 5, 1967.
15. -----, Issledovanie Operatsii, Comp. Cent. Akad. Nauk SSSR, N4, 1974.
16. L. I. Rozonoer, Avtomat. i Telemekh., v. 20, N 10, 1959.