ON THE PARTITIONING PROBLEM IN THE SYNTHESIS OF MULTILEVEL OPTIMIZATION STRUCTURES

G. GRATELOUP *
Professeur
I.N.S.A. Toulouse

M. RICHETIN *
Attaché de Recherche
C.N.R.S.

INTRODUCTION

Multilevel control structures are especially important when a large-scale system is to be controlled. The system is divided into interconnected sub-systems (Fig. 1) controlled by decision-making units hierarchically arranged (Fig. 2) (Mesarovic et al., (1). Multilevel structures are interesting since control is divided and thus simplified, and since the control system is much more reliable (Plander, (2)).

Up to now, most of the works have dealt with two-level optimization (Lasdon et al., (3), Brosilow et al., (4), Titli (8)). The different sub-systems are coordinated by coordination variables, calculated on the second level, and whose nature depends on the chosen coordination method. But in any case, these variables are linked with coupling equations of sub-systems. For large-scale systems, the number of coordination variables may be high. Consequently a multilevel structure is necessary if constraints exist on the size of coordinators or on the transmission capacity of channels. Several solutions are then available. The synthesis of these structures has been partly realized by Strasjak (6), Kulikowski (7).

At first, the decomposition of linear coupling equations associated with a multilevel structure is defined. The extension of classical coordination methods is done and it is shown that there is no feasible method for a n-level optimization ($n > 2$).

In a second part, the effects of couplings on the convergence of coordination algorithms are studied and are illustrated by an example.

DECOMPOSITION OF THE COUPLING EQUATIONS

In a multilevel structure, each coordination deals with the couplings between groups of sub-systems. So if $n_v = 2$ (Fig. 2), then the v^{th} level coordinator deals with the couplings between two groups of sub-systems.

The coupling equations are now decomposed according to the vertical division of the coordination and we suppose that they are linear.

* Laboratoire d'Automatique et d'Analyse des Systèmes du C.N.R.S.
B.P. 4036 - 31055 TOULOUSE CEDEX.

Let I^1 be the set of numbers of the N sub-systems. Suppose v = 2. The 2nd level-coordinator deals with n_2 groups of sub-systems and let I^2 be the n_2-partition of I^1 :

$$I^2 = \{P_1^2 \dots P_{n_2}^2\} \; ; \; P_i^2 \cap P_j^2 = \emptyset \; (i \neq j) \; ; \; \sum_{i=1}^{n_2} |P_i^2| = N$$

($|\;|$ = order of a set)

The coupling equations become :

$$X_i = \sum_j C_{ij} Z_j + \sum_k V_{ik}^2 \qquad \text{with} \quad \forall i \in I^1, \; \forall j,\, j \in P_r^2 \text{ if } i \in P_r^2, \; \forall k \,(k \neq r \text{ if } i \in P_r^2) \in \{1,\dots,n_2\} \tag{1}$$

$$V_{ik}^2 = \sum_{j \in P_k^2} C_{ij} Z_j \qquad \text{with} \quad \forall i \in I^1, \; \forall k \,(k \neq r \text{ if } i \in P_r^2) \in \{1,\dots,n_2\} \tag{2}$$

For any v, this decomposition is then achieved from the n_w-partitions of I^1 (w = 1, ..., v).

Example 1 :

$$N = 8 \; ; \; v = 3 \; ; \; n_2 = 4 \; ; \; n_3 = 2 \; ; \; i = 1$$

$$I^2 = \{(1,2),(3,4),(5,6),(7,8)\} \; ; \; I^3 = \{(1,2,3,4),(5,6,7,8)\}$$

$$X_1 = \sum_{j=1}^{2} C_{ij} Z_j + V_{12}^2 + V_{12}^3$$

$$V_{12}^2 = C_{13} Z_3 + C_{14} Z_4$$

$$V_{12}^3 = C_{15} Z_5 + C_{16} Z_6 + C_{17} Z_7 + C_{18} Z_8$$

On Fig. 3, this decomposition procedure means successive partitions of the coupling matrix C.

In general, V_{ab}^c is the input coupling vector for the sub-system "a", made up from the outputs of a group of sub-systems "b" which is obtained after the c^{th} partition of the coupling matrix.

Coordination methods

A static system is considered for which each sub-system has the following model and criteria : $\quad Z_i = T_i(X_i, M_i) \; ; \; \min f_i(X_i, M_i)$

The overall optimization criteria is separable : $\min F = \min \sum_i f_i$. This enables to realize a multilevel optimization (Fig. 2). With constraints being of type (1) and (2), the Lagrangian of the optimization problem is :

$$L = \sum_{i=1}^{N} \Big[f_i(X_i, M_i) + \mu_i^T (Z_i - T_i(X_i, M_i)) + \rho_i^T \big(X_i - \sum_j C_{ij} Z_j - \sum_k V_{ik}^2 - \sum_\ell V_{i\ell}^3 - \dots\big) + \sum_k \lambda_{ik}^2 \big(V_{ik}^2 - \sum_j C_{ij} Z_j\big) + \sum_\ell \lambda_{i\ell}^3 \big(V_{i\ell}^3 - \sum_j C_{ij} Z_j\big) + \dots \Big] \tag{3}$$

On the analogy of the feasible method, let us suppose that Z, V_{ik}^2, $V_{i\ell}^3$, ... are respectively calculated by 1^{st}, 2^{nd}, 3^{rd}... level coordinators.

To give the Lagrangian a separable form, it can be shown that λ^2_{ik}, $\lambda^3_{i\ell}$ must be fixed for the local optimizations and consequently calculated on coordination levels. If λ^c_{ab} is obtained by a gradient algorithm :

$$\lambda^c_{ab}(k+1) = \lambda^c_{ab}(k) + M \frac{\partial L}{\partial \lambda^c_{ab}} \quad (M > 0) \; ; \; \frac{\partial L}{\partial \lambda^c_{ab}} = V^c_{ab} - \sum_j C_{ij} Z_j$$

the variables Z_j which occur in $\frac{\partial L}{\partial \lambda^c_{ab}}$ must be known. If the information flows vertically in the structure and according to the hierarchy, the lowest coordination levels where λ^c_{ab} can be calculated is then determined (see Fig. 4 for example 1).

Consequently, all the coupling equations are satisfied only when all the coordinators have converged. Therefore, there is no feasible coordination method for a n-level optimization ($n > 2$).

Moreover, V^c_{ab} and λ^c_{ab} can be simultaneously calculated by the same coordinator. This leads to the combined method (Grateloup et al., (5)). The stationnarity equations of the Lagrangian for V^c_{ab} and λ^c_{ab} are :

$$\frac{\partial L}{\partial V^c_{ab}} = \lambda^c_{ab} - \rho_a = 0 \; ; \; \frac{\partial L}{\partial \lambda^c_{ab}} = V^c_{ab} - \sum_j C_{ij} Z_j \tag{4}$$

As V^c_{ab} and λ^c_{ab} have been defined as vectors of same dimension, from (4), they can be directly calculated for given ρ_a, Z_j. So, at any v^{th} coordination-level ($v > 1$), a direct iteration algorithm can be implemented (Grateloup et al., (11)).

COORDINATION CONVERGENCE

The effects of couplings on the coordination convergence in multilevel structures are here studied and were first pointed out by Sprague (9).

Let us suppose $v = 2$, $n_2 = 2$. V and λ are the coordination variables on the 2^{nd} coordination-level, and Z, X, M, ρ, μ, are the other variables of the optimization problem. The optimal value of a variable is marked $*$.

To determine V and λ, a combined method is used with a direct-iteration algorithm whose convergence is studied.

From (4)
$$V = V(Z) \qquad \lambda = \lambda(\rho) \qquad \text{with} \quad V(Z) = \begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix} Z \; ; \; C = \begin{pmatrix} C^{11} & C^{12} \\ C^{21} & C^{22} \end{pmatrix}, \quad \lambda(\rho) = \rho$$

C^{11}, C^{22}, C^{12}, C^{21} = submatrices of C corresponding with the partition of the N subsystems into two groups ($n_2 = 2$).

At the i^{th} iteration of the 2^{nd} level-coordinator, we get :

$$\begin{aligned} (V^i - V^*) &= \frac{dV}{dZ}(Z^i - Z^*) \\ (\lambda^i - \lambda^*) &= \frac{d\lambda}{d\rho}(\rho^i - \rho^*) \end{aligned} \quad \Rightarrow \quad \begin{aligned} \varepsilon^i_V &= \frac{dV}{dZ}\varepsilon^i_Z \\ \varepsilon^i_\lambda &= \frac{d\lambda}{d\rho}\varepsilon^i_\rho \end{aligned} \tag{5}$$

For given V^i and λ^i, when local optimizations and 1^{st} level-coordinators have converged, the following equations are satisfied :

$$
\begin{aligned}
&L_X(X^{i+1}, M^{i+1}, \mu^{i+1}, \rho^{i+1}) = 0 \\
&L_M(X^{i+1}, M^{i+1}, \mu^{i+1}) = 0 \\
&L_Z(\mu^{i+1}, \rho^{i+1}, \lambda^{i}) = 0 \\
&L_\mu(X^{i+1}, M^{i+1}, Z^{i+1}) = 0 \\
&L_\rho(X^{i+1}, Z^{i+1}, V^{i}) = 0
\end{aligned}
\qquad \text{(where } L_r = \frac{\partial L}{\partial T}\text{)} \qquad (6)
$$

A first-order development of the preceeding functions give :

$$
\begin{aligned}
&\left.\frac{\partial L_X}{\partial X}\right|_* \varepsilon_X^{i+1} + \left.\frac{\partial L_X}{\partial M}\right|_* \varepsilon_M^{i+1} + \left.\frac{\partial L_X}{\partial \mu}\right|_* \varepsilon_\mu^{i+1} + \left.\frac{\partial L_X}{\partial \rho}\right|_* \varepsilon_\rho^{i+1} = 0 \\
&\left.\frac{\partial L_M}{\partial X}\right|_* \varepsilon_X^{i+1} + \left.\frac{\partial L_M}{\partial M}\right|_* \varepsilon_M^{i+1} + \left.\frac{\partial L_M}{\partial \mu}\right|_* \varepsilon_\mu^{i+1} = 0 \\
&\left.\frac{\partial L_Z}{\partial \mu}\right|_* \varepsilon_\mu^{i+1} + \left.\frac{\partial L_Z}{\partial \rho}\right|_* \varepsilon_\rho^{i+1} + \left.\frac{\partial L_Z}{\partial \lambda}\right|_* \varepsilon_\lambda^{i} = 0 \\
&\left.\frac{\partial L_\mu}{\partial X}\right|_* \varepsilon_X^{i+1} + \left.\frac{\partial L_\mu}{\partial M}\right|_* \varepsilon_M^{i+1} + \left.\frac{\partial L_\mu}{\partial Z}\right|_* \varepsilon_Z^{i+1} = 0 \\
&\left.\frac{\partial L_\rho}{\partial X}\right|_* \varepsilon_X^{i+1} + \left.\frac{\partial L_\rho}{\partial Z}\right|_* \varepsilon_Z^{i+1} + \left.\frac{\partial L_\rho}{\partial V}\right|_* \varepsilon_V^{i} = 0
\end{aligned}
\qquad \text{(where } f|_* = \text{optimal value of f).} \qquad (7)
$$

From (5) and (7), it can be pointed out :

$$
\begin{aligned}
&(\varepsilon_X^{i+1}, \varepsilon_M^{i+1}, \varepsilon_Z^{i+1}, \varepsilon_\mu^{i+1}, \varepsilon_\rho^{i+1})^T = E_* (\varepsilon_X^{i}, \varepsilon_M^{i}, \varepsilon_Z^{i}, \varepsilon_\mu^{i}, \varepsilon_\rho^{i})^T \\
&E_* = B_*^{-1} D_* A_*
\end{aligned}
\qquad (8)
$$

$$
B_* = \begin{pmatrix}
\frac{\partial L_X}{\partial X} & \frac{\partial L_X}{\partial M} & 0 & \frac{\partial L_X}{\partial \mu} & \frac{\partial L_X}{\partial \rho} \\
\frac{\partial L_M}{\partial X} & \frac{\partial L_M}{\partial M} & 0 & \frac{\partial L_M}{\partial \mu} & 0 \\
0 & 0 & 0 & \frac{\partial L_Z}{\partial \mu} & \frac{\partial L_Z}{\partial \rho} \\
\frac{\partial L_\mu}{\partial X} & \frac{\partial L_\mu}{\partial M} & \frac{\partial L_\mu}{\partial Z} & 0 & 0 \\
\frac{\partial L_\rho}{\partial X} & 0 & \frac{\partial L_\rho}{\partial Z} & 0 & 0
\end{pmatrix}_* ;\quad
D_* = \begin{pmatrix}
0 & 0 \\
0 & 0 \\
0 & -\frac{\partial L_Z}{\partial \lambda} \\
0 & 0 \\
-\frac{\partial L_\rho}{\partial V} & 0
\end{pmatrix}_* ;\quad
A_* = \begin{pmatrix}
0 & 0 & \frac{dV}{dZ} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{d\lambda}{d\rho}
\end{pmatrix}_*
$$

It can be shown that E_* has the following structure :

$$
E_* = \begin{pmatrix}
0 & 0 & X & 0 & X \\
0 & 0 & X & 0 & X \\
0 & 0 & X & 0 & X \\
0 & 0 & X & 0 & X \\
0 & 0 & X & 0 & X
\end{pmatrix}
\qquad \text{where X = non-zero submatrix}
$$

Then : $\begin{pmatrix} \varepsilon_z^{i+1} \\ \varepsilon_\rho^{i+1} \end{pmatrix} = F_* \begin{pmatrix} \varepsilon_z^{i} \\ \varepsilon_\rho^{i} \end{pmatrix}$; $F_* = \begin{pmatrix} -b_{zz} \cdot \frac{\partial L_\rho}{\partial V} \cdot \frac{dV}{dz} & -b_{z\rho} \cdot \frac{\partial L_z}{\partial \lambda} \cdot \frac{d\lambda}{d\rho} \\ -b_{\rho z} \cdot \frac{\partial L_\rho}{\partial V} \cdot \frac{dV}{dz} & -b_{\rho\rho} \cdot \frac{\partial L_z}{\partial \lambda} \cdot \frac{d\lambda}{d\rho} \end{pmatrix}_*$ (9)

where, if $x = \sum_{i=1}^{N} x_i$ and $z = \sum_{i=1}^{N} z_i$

$b_{zz}\,(z \times x)$; $b_{z\rho}\,(z \times z)$; $b_{\rho z}\,(x \times x)$; $b_{\rho\rho}\,(x \times z)$: submatrices of B_*^{-1}

Since the coupling equations are linear :

$\frac{\partial L_\rho}{\partial V} = -I$; $\frac{dV}{dz} = \begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix}$; $\frac{\partial L_z}{\partial \lambda} = -\begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix}^T$; $\frac{d\lambda}{d\rho} = I$ (I = identity matrix)

$$F_* = \begin{pmatrix} b_{zz}\begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix} & b_{z\rho}\begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix}^T \\ b_{\rho z}\begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix} & b_{\rho\rho}\begin{pmatrix} 0 & C^{12} \\ C^{21} & 0 \end{pmatrix}^T \end{pmatrix}_* \quad (10)$$

To study the coordination convergence, from equations (5) and (8), is is sufficient to study the discrete dynamic system (9). If the modulus of all the eigenvalues of F_* are less than 1, then the coordination is stable.

This stability depends on the partition of the coupling matrix, as is shown in (10), explicitly by $\frac{dV}{dz}$, $\frac{\partial L_z}{\partial \lambda}$ and implicitly in the submatrices b_{zz}, $b_{z\rho}$, $b_{\rho z}$, $b_{\rho\rho}$. Indeed, the matrix B_* contains the submatrices $\frac{\partial L_z}{\partial \rho}$ and $\frac{\partial L_\rho}{\partial z}$.

$$\frac{\partial L_z}{\partial \rho} = -\begin{pmatrix} C^{11} & 0 \\ 0 & C^{22} \end{pmatrix}^T$$

For the best coordination convergence, an optimal partition is then difficult to be found. Nevertheless the nature of F_* suggests a suboptimal partitioning of matrix C.

Let $\begin{cases} A = [a_{ij}] \\ |A| = \sum_i \sum_j |a_{ij}| \end{cases}$

Let $\tau[A]$ be the spectral radius of A

It is known that (Varga, (12)):

$$\tau[A] \leqslant \min\left[\max_j \sum_i |a_{ij}|,\ \max_i \sum_j |a_{ij}|\right]$$

and also (Coviello, (13)) :

$$\tau[A]^2 \geqslant \max\left[\max_i A_i^T A_i,\ \max_j A^{jT} A^j\right]$$

(A_i = i^{th} line of A)
A_j = j^{th} column of A)

Let us consider the dynamic linear system (9). In the expression of elements f_{ij} of F_*, all the elements of C^{12}, C^{21}, are multiplicative terms.

By looking for a partition of C which minimizes $|C^{12}| + |C^{21}|$ we tend to decrease the modulus of f_{ij} and thus we tend to decrease the upper and lower bounds of $\mathcal{T}[F_*]$ whose value mainly determine the dynamic of the system. Consequently, we tend to have a faster coordination convergence.

It can be noted that the effects of the partition arise in the submatrices b_{zz}, $b_{z\rho}$, $b_{\rho z}$, $b_{\rho\rho}$, but are difficult to estimate.

Example 2 : Consider N identical sub-systems (linear models, quadratic criteria)
$N = 6$; $v = 2$; $n_2 = 2$; $|P_1^2| = |P_2^2| = 3$

The coupling matrix C is given by Fig. 5.

There are 10 partitions of the N sub-systems into 2 equal groups. A direct iteration algorithm is implemented at the 2nd coordination-level. It is unstable for all the cases. So we look for k such as for $V = kV(Z)$ and $\lambda = k\lambda(\rho)$ the algorithm will be stable.

Let $K = |C^{12}| + |C^{21}|$. Results appear in table 1.

	K	k
(123)(456)	9	0,94
(124)(356)	16	0,48
(125)(346)	23	0,24
(126)(345)	16	0,36
(134)(256)	15	0,33
(135)(246)	24	0,15
(136)(245)	19	0,18
(145)(236)	15	0,44
(146)(235)	22	0,22
(156)(234)	21	0,14

Table 1 - Effects of the partition on coordination convergence

There is a good correlation between k and K (Fig. 6). For slightly different values of K, this correlation is not so strong. This comes from the suboptimality of the partitioning criteria. But these results show that less interactive groups of sub-systems give a faster coordination convergence.

CONCLUSION

In multilevel optimization, all the decomposition-coordination methods deal with the coupling equations of the sub-systems. The partition of the system is then an important factor in the synthesis of multilevel structures. In the study of the effects of couplings on coordination convergence, a suboptimal partitioning criteria has been proposed for linear coupling equations. For large scale systems, one will be faced with the partition of large coupling matrices and graphical methods seems to be useful.

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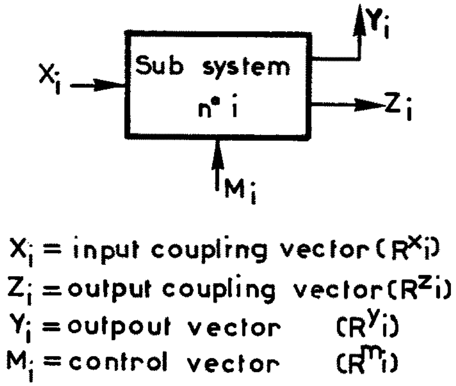

Fig. 1 : Sub-system n° i

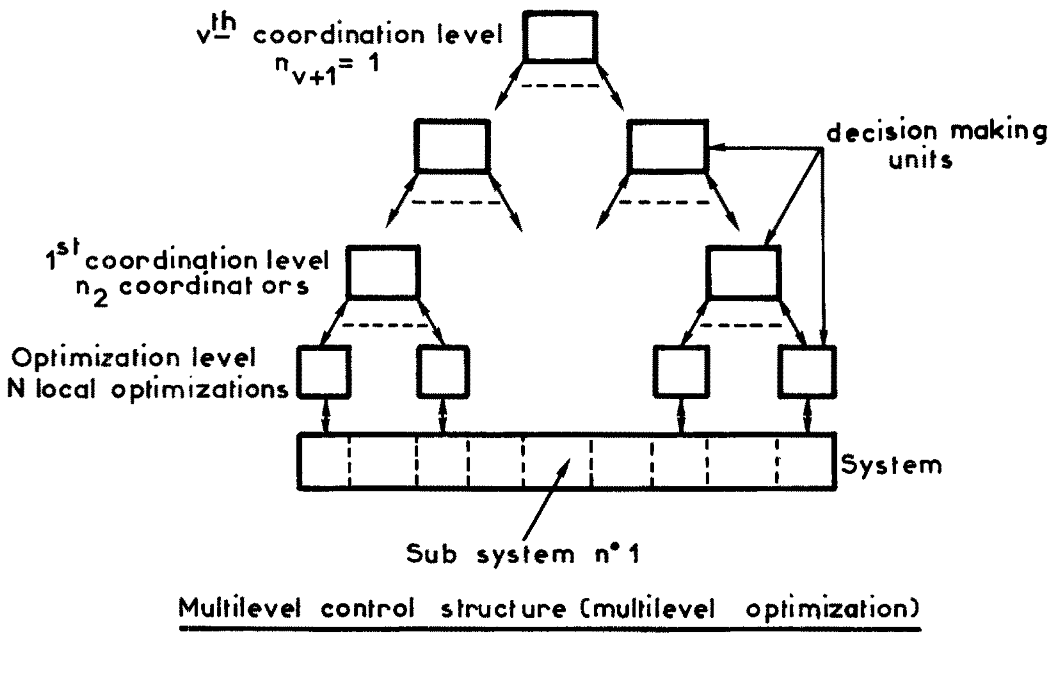

Fig. 2 : A coordination method

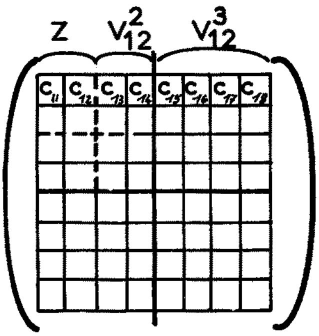

Fig. 3 : **Partition of coupling matrix C**

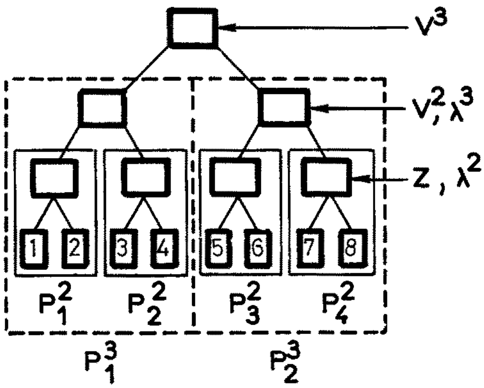

Fig. 4 : **A coordination method**

0	3	4	2	0	0
2	0	0	0	0	2
0	3	0	1	0	0
2	0	0	0	2	0
0	0	0	3	0	2
1	0	1	0	2	0

Fig. 5 : Coupling matrix C.

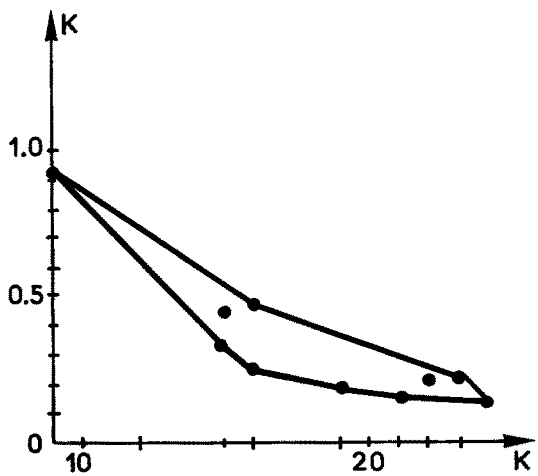

Fig. 6 : **Coupling effects on coordination convergence**