ABOUT THE PROBLEM OF SYNTHESIS OF OPTIMUM CONTROL BY ELASTIC OSCILLATIONS.

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The formal procedure of obtaining Bellman equation in the problem of heat conductivity control is stated in [1].

It is shown here, that in the same way one can obtain analogous equation for problems of control processes in other systems with distributed parameters.

Here, analizing a simple problem of elastic oscillations control, the application of this method is stated.

As a result the problem reduces to solving the non-linear boundary-value problems for matrix integro-differential equations, generalizing the well-known Rikkati equations.

Let us assume that the control process is described by the function $u(t,x)$, which inside the rigion $Q=\{0\le x\le 1, t_0\le t\le T\}$ satisfies the equation

$$u_{tt}-u_{xx}=p(t)q(x)+f(t,x) \tag{1}$$

and on the boundary Q - the conditions

$$u(t_0,x)=\varphi_0(x),\ u_t(t_0,x)=\psi_0(x) \tag{2}$$

$$u(t,0)=u(t,1)=0, \tag{3}$$

where $f(t,x)$, $q(x)$ and $\varphi_0(x)$ are assigned functions from L_2, and $\psi_0(x)$ is absolutely continuous function, and control $p(t)$ belongs to $L_2(0,T)$ with the meanings at the interval (open or closed), which in future will be designated by P.

At these assumptions, each control $p(t)$ determines unique solution of the problem (1)-(3), as a function $u(t,x)\in W_2^1(Q)$, which satisfies the integral identity

$$\int_0^1 [u_t(t,x)\varphi(t,x)]_{t_1}^{t_2}dx=\int[u_t\varphi_t-u_x\varphi_x+(f+q(x)p(t))\varphi]\,dQ \tag{4}$$

at any function $\varphi\in W_2^1(Q)^{Q^2}$, which turns into zero in the neighbourhood of points $x=0$ and $x=1$.

Here t_1 and t_2 are arbitrary points from $[t_0, T]$, and $Q_1^2 = \{0 \le x \le 1, t_1 \le t \le t_2\}$.

The problem of optimal control under consideration lies in determining $p(t)$ and corresponding to it $u(t,x)$ such that the functional

$$\mathcal{J}[t_0 w(t_0,x)] = \int_0^1 [\alpha u^2(T,x) + \beta u_t^2(T,x)]dx + \gamma \int_{t_0}^{T} p^2(t)dt$$

takes the least possible meaning. Here w is the vector with components u and u_t ; α , β , and γ are positive constants, and T is fixed moment of time, exceeding t_0 .

Supposing

$$S[t, w(t,x)] = \min_{p \in P} \mathcal{J}[t, w(t,x)],$$

we find, by an ordinary method, that

$$-\frac{\partial S}{\partial t}\Delta t = \min\left\{\gamma p^2(t)\Delta t + \Phi(t, w(t,x); \Delta w(t,x)) + O\right\}, \quad (5)$$

where Φ is linear on Δw functional, obtained in point (t, w), and O is small on Δt, $\|\Delta w\|$ is value of higher order than one. Since $\Delta u(t,x)$ and $\Delta u_t(t,x)$ belong to $L_2(0,1)$ almost at all t, the vector-function $v(t,x) = \{v_1, v_2\}$ is such that

$$\Phi = \int_0^1 v^*(t,x)\Delta w(t,x)\,dx . \quad (6)$$

We have

$$\Delta u = u(t+\Delta t, x) - u(t,x) = \frac{\partial u(t,x)}{\partial t}\Delta t + O$$

$$v_2 \Delta u_t = \Delta(v_2 u_t) - u_t(t+\Delta t, x)\Delta v_2$$

and according to formula (4) we obtain

$$\int_0^1 v_2 \Delta u_t\,dx = \int_{Q_1^2} [u_t v_{2t} - u_x v_{2x} + (f + q(x)p(t))v_2]dQ - \int_0^1 u_t(t_2, x)\Delta v_2\,dx,$$

where we take $t_1 = t$ and $t_2 = t + \Delta t$, and it is supposed, that v_2 has the properties of function Φ in (4).

That is why

$$\Phi(t, w(t,x), \Delta w(t,x)) = \int_0^1 v_1(t,x)\frac{\partial u(t,x)}{\partial t}dx\,\Delta t + \int_{Q_1^2}[u_t v_{2t} - u_x v_{2x} +$$
$$+ (f + q(x)p(t))v_2]dQ - \int_0^1 u_t(t_2, x)\Delta v_2\,dx + O.$$

Substituting the obtained expression Φ in equation (5), we **have** the Bellman equation

$$-\frac{\partial S}{\partial t} = \min_{p\in P}\left\{\gamma p^2(t) + \int_0^1 \left[v_1 u_t - u_x v_{2x} + (f + q(x)\, p(t))\, v_2\right] dx\right\} .$$

Let us assume now, that $P = (-\infty, +\infty)$ **and there exists, integrable** with square, derivative v_{2xx} . Then from Bellman equation we have

$$p(t) = -\frac{1}{2\gamma}\int_0^1 q(x)\, v_2(t,x)\, dx \tag{7}$$

$$-\frac{\partial S}{\partial t} = \int_0^1 \left[v_1 u_t + u v_{2xx} + f v_2\right] dx - \frac{1}{4\gamma}\left(\int_0^1 q(x)\, v_2(t,x)\, dx\right)^2 \tag{8}$$

We find the solution of equation (8) in the form

$$S[t, w(t,x)] = \int_0^1\!\!\int_0^1 w^*(t,x) K(t,x,s) w(t,s)\, ds\, dx + \int_0^1 \varphi^*(t,x) w(t,x)\, dx + \eta(t), \tag{9}$$

where matrix K , vector φ and scalar function $\eta(t)$ must be determined.

From (9) we find, that

$$\Phi(t,w,h) = \int_0^1\!\!\int_0^1 w^*(t,s) N(t,s,x) h(t,x)\, ds\, dx + \int_0^1 \varphi^*(t,x) h(t,x)\, dx ,$$

where

$$N(t,x,s) = K(t,x,s) + K^*(t,s,x) \tag{10}$$

and vector $v = \{v_1, v_2\}$, a component part of formula (6), is determined as follows

$$v(t,x) = \int_0^1 N^*(t,s,x) w(t,s)\, ds + \varphi(t,x). \tag{11}$$

Suppose N_{ij} are elements of matrix N , and φ_i **is the i-** component of vector φ . Then, substituting the meanings of S and v from (9) and (11) into equation (8), we obtain

$$-K_t(t,x,s) = L(t,x,s) - \frac{1}{4\gamma} M(t,x,s), \tag{12}$$

$$-\varphi_{1t}(t,x) = \varphi_{2xx}(t,x) + \int_0^1 N_{12}(t,x,s) f(t,s)\, ds - \frac{1}{2\gamma}\int_0^1\!\!\int_0^1 q(y)\, q(s) N_{12}(t,x,y) \varphi_2(t,s)\, dy\, ds \tag{13}$$

$$-\varphi_{2t}(t,x) = \varphi_1(t,x) + \int_0^1 N_{22}(t,x,s) f(t,s)\, ds - \frac{1}{2\gamma}\int_0^1\!\!\int_0^1 q(s)\, q(y) N_{22}(t,x,y) \varphi_2(t,s)\, dy\, ds \tag{14}$$

$$\eta'(t) = \frac{1}{4\gamma}\Big[\int_0^1 q(x)\varphi_2(t,x)\,dx\Big]^2 \tag{15}$$

where

$$L(t,x,s) = \begin{pmatrix} N_{21\,xx}(t,x,s) & N_{22\,xx}(t,x,s) \\ N_{11}(t,x,s) & N_{12}(t,x,s) \end{pmatrix},$$

and $M(t,x,s)$ is matrix with components

$$M_{ij}(t,x,s) = \int_0^1\!\!\int_0^1 q(y)q(z)N_{2i}(t,y,s)N_{2j}(t,z,x)\,dy\,dz$$
$$M_{12}(t,x,s) = M_{21}(t,s,x).$$

Since we assumed that $v_2(t,0) = v_2(t,1) = 0$ at any vector $w(t,x)$, then from formula (11) we obtain

$$N_{22}(t,0,s) = N_{22}(t,1,s) = N_{21}(t,0,s) = N_{21}(t,1,s) = 0 \tag{16}$$

$$\varphi_2(t,0) = \varphi_2(t,1) = 0. \tag{17}$$

Besides, directly from the definition of functional S it follows that

$$S[T, w(T,x)] = \int_0^1 [\alpha u^2(T,x) + \beta u_t^2(T,x)]\,dx$$

and from formula (9) we obtain

$$\left.\begin{aligned} K_{ij}(T,x,s) &= 0, \quad i \neq j \\ K_{11}(T,x,s) = \alpha\,\delta(s-x), &\quad K_{22}(T,x,s) = \beta\,\delta(s-x) \end{aligned}\right\} \tag{18}$$

$$\varphi_1(T,x) = \varphi_2(T,x) = \eta(T) = 0, \tag{19}$$

where $\delta(x)$ is Dirac function.

The boundary-value problem (12), (16) is the generalization of the analized case of the known Rikkati equation from the theory of optimal stabilization of systems with the finite number of the degrees of freedom. Taking into consideration that matrices K and N are connected in ratio (10), we find that (12) represents the system of non-linear integro-differential equations relative to K_{ij}. Here we shall not solve this boundary-value problem. However, we shall mention, that two last conditions in (18) show the necessity to analyze it in the space of distributions (the generalized functions). After the solution of Rikkati boundary-value problem (12), (16), (18) it is necessary to determine vector φ from (13), (14), (17) and (18). This will give the possibility to find the control according to formula (7). However the problem of belonging $P(t)$ to space $L_2(0,T)$ requires special investigations.

References

1. Egorov A.I. Optimal stabilization of systems with distributed parameters. In this volume.