PENALTY FUNCTION METHOD AND NECESSARY OPTIMUM CONDITIONS IN OPTIMAL CONTROL PROBLEMS WITH BOUNDED STATE VARIABLES

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An approach to determine necessary optimum conditions (n.o.c.) for generalized solutions (g.s.) of mathematical programming problem based on penalty function method is considered. The results are used to derive generalized maximum principle (g.m.p.) for optimal control problem with ordinary differential equations and bounded state variables. For linear state variables problems g.m.p. has also turned out to be sufficient optimal condition and in this case generalized duality theorem (g.d.th.) takes place.

1. Mathematical Programming Problem

Let us consider the following problem

$$\underline{I}(\mathcal{U}) \to \sup, \tag{1.1}$$

$$\mathcal{U} \in U \subset B_{u}, \tag{1.2}$$

$$F(\mathcal{U}) \in K \subset B_F \tag{1.3}$$

(B_u, B_F are Banach spaces, K is a convex closed cone) and assume that $\underline{I}(\mathcal{U}) < C < \infty$ when $\mathcal{U} \in U$.

Let $\mathcal{H}$ be a set of sequences $\{\mathcal{U}^{(k)}\}$ for which: $\{\mathcal{U}^{(k)}\} \in \mathcal{H},\ \|\tilde{\mathcal{U}}^{(k)} - \mathcal{U}^{(k)}\| \to 0,\ k \to \infty \Rightarrow \{\tilde{\mathcal{U}}^{(k)}\} \in \mathcal{H}$. Designate with $\mathcal{V}$ the subset of all $\{\mathcal{U}^{(k)}\} \in \mathcal{H}$ satisfying the condition: $R(F(\mathcal{U}^{(k)})) \to 0,\ k \to \infty$, where $R(x) = \inf \|x-y\|^2\ (y \in K)$ ($\mathcal{V}$ may be called the g.s. set of the system of conditions (1.2), (1.3)). Let $\underline{I}(\{\mathcal{U}^{(k)}\}) = \underline{\lim}_{k\to\infty} \underline{I}(\mathcal{U}^{(k)})$, $\bar{\underline{I}} = \sup \underline{I}(\{\mathcal{U}^{(k)}\})\ (\{\mathcal{U}^{(k)}\} \in \mathcal{V})$. A sequence $\{\mathcal{U}^{(k)}\} \in \mathcal{V}$ will be called g.s. of the problem (1.1)-(1.3) if

$$\lim \underline{I}(\mathcal{U}^{(k)}) = \bar{\underline{I}},\quad k \to \infty.$$

Let us assume further n.o.c. of g.s. to be known for every $\mathcal{F}(\mathcal{U})$ from a sufficiently wide class of functionals in the following asymptotical form

$$P_{\mathcal{J}}\ (\mathcal{U}^{(k)}) \to 0,\ k \to \infty, \tag{1.4}$$

where $P_{\mathcal{J}}(\mathcal{U})$ is a functional. In papers /1-5/ an approach based on penalty function method and diagonal transference procedure has been developed. This approach makes it possible for a wide class of optimization problems to pass from condition (1.4) to n.o.c. of problems with additional restrictions. Generalizing the method of these papers it is possible to obtain the following results.

Let $\{\mathcal{U}^{(k)}\}$ be g.s. of the problem (1.1)-(1.3) and

$$I_{k\alpha\beta}\ (\mathcal{U}) = I(\mathcal{U}) - \alpha R(F(\mathcal{U})) - \beta \|\mathcal{U} - \mathcal{U}^{(k)}\|^2 \quad (k=1,2,\dots;\ \alpha,\beta \geq 0). \tag{1.5}$$

Let us consider the set of problems (1.5), (1.2). Let $I_{k\alpha\beta} \in \{\mathcal{J}\}$ Then for a g.s. $\{\mathcal{U}^{(m)}_{k\alpha\beta}\}$ of the problem $I_{k\alpha\beta}(\mathcal{U}) \to \sup$, (1.2) the n.o.c. is realized:

$$P_{k\beta}\,(\mathcal{U}^{(m)}_{k\alpha\beta}, \alpha) = P_{I_{k\alpha\beta}}\,(\mathcal{U}^{(m)}_{k\alpha\beta}) \to 0,\ m \to \infty. \tag{1.6}$$

Let us presume further that the following inequality takes place

$$|\,P_{k\beta}(\mathcal{U},\alpha) - P\ (\mathcal{U},\alpha)| \leq A\beta \quad (\forall k, \alpha) \quad (P = P_{k0}). \tag{1.7}$$

<u>Theorem 1</u>. If $\{\mathcal{U}^{(k)}\}$ is a g.s. of (1.1)-(1.3) then there exist sequences $\{\tilde{\mathcal{U}}^{(k)}\}, \{\alpha_k\}$ $(\alpha_k \to \infty,\ k \to \infty)$ for which

$$P(\tilde{\mathcal{U}}^{(k)}, \alpha_k),\ \|\tilde{\mathcal{U}}^{(k)} - \mathcal{U}^{(k)}\|,\ \alpha_k R(F(\tilde{\mathcal{U}}^{(k)})) \to 0,\ _{k\to\infty.} \tag{1.8}$$

<u>Proof (sketch)</u>. Let $\bar{I}_{k\alpha\beta} = \sup I_{k\alpha\beta}(\mathcal{U})\ (\mathcal{U} \in U)$ and $\varepsilon_k \to +0$. For $\beta > 0$ there are $\{\alpha_k\}, \{m_{k\beta}\}$ for which $\alpha_k \to \infty$, $\alpha_k R(F(\mathcal{U}^{(k)})) \to 0, k \to \infty$, $|I_{k\alpha_k\beta}(\mathcal{U}^{(k)}_\beta) - \bar{I}_{k\alpha_k\beta}| + |P_{k\beta}(\mathcal{U}^{(k)}_\beta, \alpha_k)| < \varepsilon_k$, $\mathcal{U}^{(k)}_\beta = \mathcal{U}^{(m_{k\beta})}_{k\alpha_k\beta}$. As $\bar{I}_{k\alpha_k\beta} > C_1$ (C_1 is some constant) it follows: $\alpha_k R(F(\mathcal{U}^{(k)}_\beta)) + \beta\|\mathcal{U}^{(k)}_\beta - \mathcal{U}^{(k)}\|^2 \to 0$. Now with the help of diagonal transference procedure with respect to β it is possible to show the validity of (1.8).

2. Optimal Control Problem

Let us consider now the following optimal control problem

$$\left.\begin{aligned} &I(u(\cdot)) = \varphi_0(x^1) \to \sup \quad (x^1 = x(t_1)), \\ &\frac{dx}{dt} = f(x,u,t) \quad (t \in T = [t_0, t_1]),\ x(t_0) = x^0, \end{aligned}\right\} \tag{2.1}$$

$$u(\cdot) \in U \tag{2.2}$$

$$\varphi(x^1)=0, \quad g(x(t),t)\leq 0. \qquad (2.3)$$

x, u, f, φ, g are vectors with dimensions n, m, n, q, s. Functions $f, \frac{\partial f}{\partial x}, g, \varphi_i$ are supposed to be continuous on $E^n \times U$; $\frac{\partial^2 f}{\partial x^2}$ is assumedto be restricted. With respect to t these functions are assumed to be measurable and uniformly restricted.

Let a set of $u(\cdot)$ with values in bounded set V and dense with respect to measure in the set of all measurable $u(\cdot)$ $(u(t)\in V)$ be the set of admissible controls. Suppose also that Lipschitz condition for the function f with respect to x is fulfilled uniformly with respect to $u\in V$, $t\in T$.

The problem (2.1)-(2.3) may be considered as a special case of (1.1)-(1.3) if, for example, we put $\mathcal{U}=u(\cdot)$, $B_{\mathcal{U}}=L_2(R^m,T)$,

$$B_F=R^q\times L_2(R^s,T), \quad K=\{y\in R^q, z(\cdot)\in L_2(R^s,T): y=0, z(t)\leq 0, t\in T\}.$$

In this case $R(y,z(\cdot))=\|y\|^2+\int_T\|\tilde{\tilde{z}}(t)\|^2dt$, where

$$\tilde{\tilde{z}}(t)=\begin{cases} z(t), & z(t)>0 \\ 0, & z(t)\leq 0. \end{cases}$$

Let conditions (2.3) correspond to the restriction (1.3) and $\mathcal{H}$ be the set of such sequences $\{u^{(k)}(\cdot)\}$ ($u^{(k)}(\cdot)$ is measurable; $u^{(k)}(t)\in V$, $t\in T$) that the corresponding $x^{(k)}(\cdot)$ in $C(R^n,T)$ converges to some $x(\cdot)$: $x^{(k)}(\cdot)\rightrightarrows x(\cdot)$. $\{^{(k)}(\cdot)\}\in\mathcal{H}\Rightarrow(2.3)$ for $x(\cdot)$.

Let $\{u^{(k)}(\cdot)\}$ be g.s. of the problem (2.1)-(2.3). Designate

$$I_{k\alpha\beta}(u(\cdot))=\varphi_0(x^1)-\alpha\left(\|\varphi(x^1)\|^2+\int_T\|\tilde{\tilde{g}}(x(t),t)\|^2dt\right)-\beta\int_T\|u(t)-u^{(k)}(t)\|^2dt. \qquad (2.4)$$

The n.o.c. in the problem (2.4), (2.1-II), (2.2) has the form (1.6) if

$$P_{k\beta}(\mathcal{U},\alpha)=\int_T(\bar{H}_{k\beta}(x,\psi,t)-H_{k\beta}(x,\psi,u,t))dt \qquad (2.5)$$

where $H_{k\beta}(x,\psi,u,t)=\psi'f(x,u,t)-\beta\|u-u^{(k)}(t)\|^2$, $\bar{H}_{k\beta}(x,\psi,t)=\sup H_{k\beta}(x,\psi,u,t)$ $(u\in V$ and $\psi(\cdot)$ satisfies the equations

$$\frac{d\psi'}{dt}=-\psi'\frac{\partial f}{\partial x}+2\alpha\tilde{\tilde{g}}'\frac{\partial g}{\partial x}, \quad \psi'(t_1)=\frac{\partial\varphi_0}{\partial x}-2\alpha\varphi'\frac{\partial\varphi}{\partial x}. \qquad (2.6)$$

„ ′ " designates transposition. This n.o.c. will be called g.m.p. For the problem (2.4), (2.1-II) g.m.p. has been proved in /2/.

The fulfilment of (1.7) follows from the boundedness of V . Now the following theorem is a simple consequence of theorem 1.

Theorem 2(g.m.p. of the problem (2.1)-(2.3)). Let $\{u^{(k)}(\cdot)\}$ be g.s. of (2.1)-(2.3). Then there exist sequences $\{\tilde{u}^{(k)}(\cdot)\}$ $\{\lambda^{(k)}\}, \{\mu^{(k)}(\cdot)\}$ $(\tilde{u}^{(k)}(\cdot) \in \mathbf{U}, \lambda^{(k)} \in R^q,$ $\mu^{(k)}(\cdot) \in C(R^s, T))$ for which

$$\int_T (\bar{H}(\tilde{x}^{(k)}, \tilde{\psi}^{(k)}, t) - H(\tilde{x}^{(k)}, \tilde{\psi}^{(k)}, \tilde{u}^{(k)}, t))dt \to 0, \quad (2.7)$$

$$\int_T \| \tilde{u}^{(k)}(t) - u^{(k)}(t)\| dt \to 0, \quad (2.8)$$

$$|(\lambda^{(k)})'\varphi(\tilde{x}^{(k)}(t_1))| + \int_T (\mu^{(k)})' g(\tilde{x}^{(k)}(t), t)dt \to 0 \quad (2.9)$$

if $k \to \infty$ and $\tilde{\psi}^{(k)}(\cdot), \mu^{(k)}(\cdot)$ satisfy the equations

$$\frac{d\psi'}{dt} = -\psi' \frac{\partial f}{\partial x} + \mu' \frac{\partial g}{\partial x}, \quad \psi'(t_1) = \frac{\partial \varphi_0}{\partial x} + \lambda' \frac{\partial \varphi}{\partial x} \quad (2.10)$$

(with transpositions $\mu^{(k)}(\cdot), \lambda^{(k)}, \tilde{\psi}^{(k)}(\cdot), \tilde{u}^{(k)}(\cdot), \tilde{x}^{(k)}(\cdot)$),

$$\mu_i^{(k)}(t) \geq 0, \ \mu_i^{(k)}(t) = 0 \quad if \quad g_i(\tilde{x}^{(k)}(t), t) < 0. \quad (2.11)$$

In (2.7) $\quad H(x, \psi, u, t) = \psi' f(x, u, t). \quad (2.12)$

$\lambda^{(k)}, \mu^{(k)}(\cdot)$ being normed, the following theorem immediately follows from theorem 2.

Theorem 3. Let $\{u^{(k)}(\cdot)\}$ be g.s. of the problem (2.1)-(2.3) and $x^{(k)}(\cdot) \rightrightarrows x(\cdot)$. Then there exist $\lambda_0, \lambda, \mu(\cdot)$ $(\lambda_0 \geq 0,$ $\lambda \in R^q, \mu_i \ (i = \overline{1, s})$ is a nonnegative measure concentrated on the set $\{t : t \in T, g_i(x(t), t) = 0\}$) for which

$$\int_T (\bar{H}(x, \psi^{(k)}, t) - H(x, \psi^{(k)}, u^{(k)}, t)) dt \to 0, \quad (2.13)$$

$$\lambda_0 + \|\lambda\| + \int_T \sum_{i=1}^{s} d\mu_i(t) > 0 \quad (2.14)$$

where $\psi^{(k)}(\cdot)$ satisfies (2.10-I) (with transposition $u^{(k)}(\cdot), x(\cdot)$) and

$$\psi'(t_1) = \lambda_0 \frac{\partial \varphi_0}{\partial x} + \lambda' \frac{\partial \varphi}{\partial x}. \quad (2.15)$$

The theorem similar to theorem 3 has been proved in /6/ (analogous theorem with the replacement of asymptotical equality (2.13) by the precise one is valid for classical solutions of the problem (2.1)-(2.3) /7/ and for stationary g.s., the last assertion being a simple corollary of theorem 3).

Let us emphasize that n.o.c. of theorem 3 are weaker than those of theorem 2 (see the example given below).

3. Linear state variable problem

Let

$$f(x,u,t)=a(t)x+b(u,t),\quad g(x,t)=c(t)x+d(t),$$
$$\varphi_i(x)=p_i'x+q_i,\quad i=\overline{0,q}. \tag{3.1}$$

In this case g.m.p. gives also sufficient optimal condition.

Theorem 4. Suppose $\{\tilde{u}^{(k)}(\cdot)\}, \{\lambda^{(k)}\}, \{\mu^{(k)}(\cdot)\}$ satisfying (2.7)-(2.12) exist for a given $\{u^{(k)}(\cdot)\}$. Then $\{u^{(k)}(\cdot)\}$ is the g.s. of the problem (2.1)-(2.3).

Proof (sketch). Let $\{\bar{u}^{(k)}(\cdot)\}$ be a g.s. of (2.1-II), (2.2), (2.3). Without decreasing the generality of the results it is possible to assume that

$$\overline{\lim_{k\to\infty}}\left(|(\lambda^{(k)})'\varphi(\bar{x}^{(k)})| + \int_T (\mu^{(k)})' g(x^{(k)},t)\,dt\right) \le 0. \tag{3.2}$$

As f, g and φ_i are linear with respect to x it is valid (see /2,3/) that

$$I_{\lambda^{(k)},\mu^{(k)}}(\bar{u}^{(k)}(\cdot)) - I_{\lambda^{(k)},\mu^{(k)}}(\tilde{u}^{(k)}(\cdot)) = \tag{3.3}$$
$$\int_T \left(H(\tilde{x}^{(k)},\tilde{\psi}^{(k)},\bar{u}^{(k)},t) - H(\tilde{x}^{(k)},\tilde{\psi}^{(k)},\tilde{u}^{(k)},t)\right)dt$$

where

$$I_{\lambda,\mu}(u(\cdot)) = \varphi_0(x^1) + \lambda'\varphi(x^1) - \int_T \mu' g(x,t)\,dt. \tag{3.4}$$

From (2.9), (3.2)-(3.4) follows the assertion of the theorem.

Example.

$$x(2)\to\sup,\quad \frac{dx}{dt}=u\ \ (t\in[0,2]),\ x(0)=-0{,}5;$$
$$|u|\le 1,\quad g(x,t)=(1-t)x\le 0. \tag{3.5}$$

Here $\frac{d\psi}{dt}=(1-t)\mu$, $H(x,\psi,u,t)=\psi u$ and (2.13),(2.14) are valid for every $u(\cdot)$ satisfying the condition $x(1)=0$ (with $\lambda_0 = \lambda = 0$, $\mu(t)=\delta(t-1)$). But g.m.p. is valid only for the optimal $u(\cdot)$.

4. Generalized Duality Theorem

Different results concerning g.d.th. for mathematical programming problem have been obtained by Golstein /8/. However it is interesting to show that for the problem (2.1)-(2.3), (3.1) g.d.th. is a very simple corollary of g.m.p.

Theorem 4. Let f, g, φ_i satisfy (3.1) and (see (3.4))

$$J(\lambda, \mu(\cdot)) = \sup_{u(\cdot)\in U} I_{\lambda,\mu}(u(\cdot)). \qquad (4.1)$$

Then $\bar{I} = I(\{u^{(k)}(\cdot)\}) = \inf J(\lambda, \mu(\cdot)) = \bar{J}$ (4.2)

where $\lambda \in R^q$, $\mu(\cdot) \in C(R^s, T)$, $\mu_i(t) \geq 0 \; (i=\overline{1,s})$.

Proof. $I_{\lambda,\mu}(\{u^{(k)}(\cdot)\}) \geq I(\{u^{(k)}(\cdot)\})$. So $\bar{J} \geq \bar{I}$. Let $\{\lambda^{(k)}\}$, $\{\mu^{(k)}(\cdot)\}$, $\{\tilde{u}^{(k)}(\cdot)\}$ be those given in theorem 2. From (3.3) (for an arbitrary $\tilde{u}^{(k)}(\cdot) \in U$), (2.7) and (2.9) $J(\lambda^{(k)}, \mu^{(k)}(\cdot)) - I_{\lambda^{(k)},\mu^{(k)}}(\tilde{u}^{(k)}(\cdot)) \to 0$, $I_{\lambda^{(k)},\mu^{(k)}}(\tilde{u}^{(k)}(\cdot)) - I(\tilde{u}^{(k)}(\cdot)) \to 0$, $k \to \infty$. So $J(\lambda^{(k)}, \mu^{(k)}(\cdot)) \to \bar{I}$, $k \to \infty$ and $\bar{J} = \bar{I}$.

Theorem 4 makes it possible to get solution algorithms for the problem (2.1)-(2.3), (3.1).

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