RECURSIVE SOLUTIONS TO INDIRECT SENSING MEASUREMENT PROBLEMS BY A GENERALIZED INNOVATIONS APPROACH

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ABSTRACT

For a wide class of applications referred to as indirect-sensing experiments, a systematic approach yielding solutions in recursive form is established. Indirect-sensing experiments include problems of estimation, filtering, system identification, and interpolation and smoothing by splines. Our approach is based on the novel notion of a discrete-time generalized (not necessarily stochastic) innovations process. The discrete-time linear least-squares filtering problem is used to relate the new concept to the familiar one of a stochastic innovations process. An application to the problem of identifying recursively impulse responses and system parameters by using pseudo-random binary sequences as probing inputs is considered. Further, the problem of interpolation and smoothing by splines is approached by the method developed.

1 - FORMULATION OF THE PROBLEM

In order to cast many different applications in a single mathematical framework and stress their essential features, we consider an abstract version of a problem that often occurs in experimental work, for istance, in estimation, filtering, system identification, etc.. Let H be a real Hilbert space of functions defined on a set Ω of points ω. The inner product of H is denoted by $\langle \cdot,\cdot \rangle$, and the corresponding norm by $\|\cdot\|$. Let H^P be the P-fold Cartesian product of H and $R^{P\times M}$ the space of all real-valued PxM matrices. We define an indirect-sensing linear measurement, or simply a measurement, on an element $w \in H^P$ to be the values $m \in R^{P\times M}$ taken on by an ordered set of M continuous linear functionals

$$m = \{\langle w, \rho \rangle\} \triangleq \begin{bmatrix} \langle w^1, \rho^1 \rangle \cdots \langle w^1, \rho^M \rangle \\ \vdots \qquad\qquad \vdots \\ \langle w^P, \rho^1 \rangle \cdots \langle w^P, \rho^M \rangle \end{bmatrix} \tag{1}$$

$$\rho \triangleq [\rho^1, \cdots, \rho^M]' \qquad w \triangleq [w^1, \cdots, w^P]'$$

where, by the Riesz representation theorem [1], $\rho \in H^M$ will be called the measurement representator. Notice that in (1) M stands for the number of distinct measurements executed on each of the P components of w.

It is assumed that a sequence of time-indexed measurements

$$m_t = \{\langle w, \rho_t \rangle\} \quad , \qquad t \in I \triangleq \{1,2,\cdots\} \tag{2}$$

with

$$\{\rho_t^m \, , \, t \in I \, , \, m = 1,2,\cdots, M\} \text{ linearly independent} \tag{3}$$

is available.

The set $\mathcal{E}_t$ made up of the first t representators and corresponding measurements defined by

$$\mathcal{E}_t \triangleq \{\rho_\tau, m_\tau \, , \, \tau = 1,2,\cdots,t\}$$

will be referred to as the experiment up to time t. Further,

$$\mathcal{E} \triangleq \{\rho_t, m_t \, , \, t \in I\}$$

will simply be called the experiment. The problem is then to find a recursive formula for

$$\hat{w}_{|t} \triangleq [\hat{w}_{|t}^1, \cdots, \hat{w}_{|t}^P]'$$

where, for each p=1,2,...,P,

$\hat{w}_{|t}^p \triangleq$ the minimum norm element in H interpolating $\mathcal{E}_t$, or, in other words, the linear least-squares (l.l.s.) reconstruction of w^p based on the experiment up to time t.

Example 1 (l.l.s. estimation) – Let $H \triangleq L_2(\Omega, \mathcal{A}, \mathbb{P})$, the Hilbert space of all second-order random variables (r.v.), viz. r.v'.s with finite second moments. Here the inner product of $u, v \in H$ is

$$\langle u, v \rangle = E[uv] \triangleq \int_\Omega u(\omega)\, v(\omega)\, \mathbb{P}(d\omega)$$

The experiment consists of acquiring the values of the covariance

$$m_t = \{\langle w, \rho_t \rangle\} = E[w\rho_t']$$

and observing the realization of a second-order M-dimensional time-series ρ_t. For the sake of simplicity, the time series ρ_t and the P-dimensional r.v. w are assumed to have zero means. The problem is thus to obtain a recursive formula for $\hat{w}_{|t}$, the l.l.s. estimate of w based on the observations up to time t.

Example 2 (determination of system impulse-responses) – Consider a causal linear time-invariant system with Q inputs and P outputs. Let $\{h_{pq}(\omega)\}$, $\omega \in [0,\infty)$, be its impulse-response matrix. Suppose that the given system is b.i.b.o. stable, then, for a sufficiently large $\omega_1 > 0$, $h_{pq}(\omega) = 0, \forall \omega > \omega_1$. Thus, if u_q denotes the system q-th input and m_t^p the system p-th output at time t,

$$m_t^p = \sum_{q=1}^{Q} \int_0^{\omega_1} h_{pq}(\omega)\, u_q(t-\omega)\, d\omega \, . \tag{4}$$

Setting

$$H \triangleq L_2(\Omega) \oplus L_2(\Omega) \oplus \dots \oplus L_2(\Omega) \quad (Q \text{ times}),$$

the Hilbert space of all functions $v: \Omega \to R^Q$

$$v(\omega) \triangleq [v_1(\omega), \cdots, v_Q(\omega)]$$

such that

$$\|v\|^2 = \langle v, v \rangle = \sum_{q=1}^{Q} \int_{\Omega} [v_q(\omega)]^2 d\omega < \infty \tag{5}$$

we can write (4) as

$$m_t = \{\langle w, \varrho_t \rangle\} \tag{6}$$

with

$$m_t \triangleq [m_t^1, \cdots, m_t^P]' \in R^P$$
$$w \triangleq [w^1, \cdots, w^P]' \in H^P$$
$$w^p(\omega) \triangleq [h_{p1}(\omega), \cdots, h_{pQ}(\omega)]', \qquad p = 1, \cdots, P \tag{7}$$
$$\varrho_t(\omega) \triangleq [u_1(t-\omega), \cdots, u_Q(t-\omega)] \tag{8}$$

with t fixed in I and $\omega \in [0, \omega_1]$.

Here the experiment consists of sending into the system the "inputs" or representators $\{\varrho_t\}$ and recording the values of the corresponding outputs $\{m_t\}$. The problem is thus to obtain a recursive formula for $\hat{w}_{|t}$, the l.l.s. reconstruction of the system impulse-response matrix from input-output data up to time t.

Let $\mathcal{R}_t$ be the linear manifold in H spanned by the measurement representators up to t

$$\mathcal{R}_t \triangleq \mathrm{Span}\{\varrho_\tau, \forall \tau \leq t\} \triangleq \mathrm{Span}\{\varrho_\tau^m, m = 1, \cdots, M, \forall \tau \leq t\}$$

It is well-known that $\hat{w}^P_{|t}$ coincides with the orthogonal projection of the unknown $w^P \in H$ onto $\mathcal{R}_t$

$$\hat{w}^p_{|t} = \Pi[w^p | \mathcal{R}_t]$$

Further, $\hat{w}_{|t}$ is uniquely specified by the two requirements:

$$\hat{w}^p_{|t} \in \mathcal{R}_t \qquad p = 1, \cdots, P \tag{9a}$$

$$\{\langle \tilde{w}_{|t}, \varrho_t \rangle\} = 0 \quad , \quad \forall \tau \leq t \tag{9b}$$

where

$$\tilde{w}_{|t} \triangleq w - \hat{w}_{|t} \tag{10}$$

is the error of the l.l.s. reconstruction of w based on $\mathcal{E}_t$.

Requirements (9), together with the information supplied by the experiment $\mathcal{E}_t$, enable one to write down the so-called normal equations [2]. In general, this set of equations yields the desired $\hat{w}_{|t}$ in a nonrecursive form in that, if $\hat{w}_{|t+1}$ is needed, an augmented system of normal equations has to be solved by performing the same number of computations as if $\hat{w}_{|t}$ were unknown.

2 - INNOVATIONS AS GRAM-SCHMIDT PROCESSES

As a preliminary step to the development of a systematic approach to the problem that has been posed, viz. recursive linear least-squares solution to the indirect-sensing problem, it is convenient to introduce the notion of causally equivalent experiments. We say that two experiments $\{\rho_t, m_t\}$ and $\{r_t, \mu_t\}$ are causally equivalent if

$$\forall t \in I \ , \ \mathrm{Span}\left\{ \rho_\tau , \forall \tau \le t \right\} = \mathrm{Span}\left\{ r_\tau , \forall \tau \le t \right\} .$$

This is equivalent to requiring, perhaps in more suggestive terms, the existence of a causal and causally invertible linear transformation $\mathcal{L} : H^{P\times I} \longrightarrow H^{P\times I}$ that converts the representators of the first into the representators of the second experiment in a causal way,

$$\mathcal{L}[\{\rho_\tau , \tau \le t\}] = \{ r_\tau , \tau \le t \} \quad , \quad \forall t \in I .$$

An obvious consequence of the given definitions is

Proposition 1 - Let $\hat{w}_{|t}(\mathcal{E}_i)$ be the l.l.s. reconstruction of $w \in H^P$ based on an experiment $\mathcal{E}_i$, i = 1,2. Thus,

$$\left.\begin{array}{l} \hat{w}_{|t}(\mathcal{E}_1) = \hat{w}_{|t}(\mathcal{E}_2) \\ \forall t \in I , \ \forall w \in H^P \end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l} \mathcal{E}_1 \ \& \ \mathcal{E}_2 \text{ are} \\ \text{causally equivalent.} \end{array}\right.$$

Let us now construct from the representators $\{\rho_t, \ t \in I\}$ of the given experiment (2) an orthonormal sequence $\{\nu_t, \ t \in I\}$ of the elements in H^M by the Gram-Schmidt procedure [1,2]. By orthonormality here we mean that

$$\{<\nu_t , \nu_\tau>\} \triangleq \begin{bmatrix} <\nu_t^1, \nu_\tau^1> & \cdots & <\nu_t^1, \nu_\tau^M> \\ \vdots & & \\ <\nu_t^M, \nu_\tau^1> & \cdots & <\nu_t^M, \nu_\tau^M> \end{bmatrix}$$

$$= I_M \, \delta_{t,\tau} \quad , \qquad \forall t, \tau \in I .$$

We get

$$e_t \triangleq \rho_t - \sum_{\tau=1}^{t-1} \{<\rho_t , \nu_\tau>\} \nu_\tau \quad , \quad t = 2,3,\cdots, \tag{11a}$$

$$e_1 \triangleq \rho_1 \tag{11b}$$

$$\nu_t \triangleq G_t^{-1/2} e_t \quad , \quad \forall t \in I . \tag{11c}$$

where $G_t^{-\frac{1}{2}}$ is the inverse of the positive square-root of the matrix

$$G_t \triangleq \{<e_t, e_t>\} \; .$$

The sequence $\{e_t, t \in I\}$ will be called the sequence of the <u>innovations</u> of the representators $\{\varrho_t, t \in I\}$, and $\{\nu_t, t \in I\}$ that of the <u>normalized innovations</u>.
By the way the Gram-Schmidt procedure works, the initial experiment turns out to be causally equivalent to the corresponding <u>innovations experiment</u>

$$\mathcal{J} \triangleq \{\nu_t, \mu_t, t \in I\}$$

where the ν_t's are defined by (11), and

$$\begin{aligned} \mu_t &\triangleq \{<w, \nu_t>\} \\ &= [m_t - \sum_{\tau=1}^{t-1} \mu_\tau \{<\nu_\tau, \varrho_t>\}] \, G_t^{-1/2} \quad , \quad t = 2,3,\cdots, \qquad (12a) \\ \mu_1 &\triangleq m_1 \, G_1^{-1/2} . \qquad (12b) \end{aligned}$$

By transforming the initial experiment $\mathcal{E}$ into the corresponding innovations experiment $\mathcal{J}$ we find immediately the desired $\hat{w}_{|t}$ in a recursive form

$$\begin{aligned} \hat{w}_{|t} = \sum_{\tau=1}^{t} \mu_\tau \nu_\tau &= \hat{w}_{|t-1} + \mu_t \nu_t \\ &= \hat{w}_{|t-1} + \mu_t G_t^{-1/2} e_t \quad , \; t \in I \qquad (13a) \\ \hat{w}_{|0} &= 0 \qquad (13b) \end{aligned}$$

<u>Theorem 1</u> - Let $\mathcal{E} = \{\varrho_t, m_t, t \in I\}$ be an indirect-sensing experiment, and $\mathcal{J} \triangleq \{\nu_t, \mu_t, t \in I\}$, with ν_t and μ_t respectively defined by (11) and (12), be the corresponding innovations experiment. Then, $\mathcal{E}$ and $\mathcal{J}$ are causally equivalent, and a recursive formula for the l.l.s. reconstruction of $w \in H^p$ based on $\mathcal{E}_t$ is given by (12) and (13).

Let us apply (13) to get

$\hat{\varrho}_{t|t-1}$ = the l.l.s. reconstruction of the representator at time t based on the experiment defined by

$$\left\{ \{ m_\tau = \{<\varrho_t, \varrho_\tau>\} \; , \; \tau = 1, \cdots, t-1 \} \right. \qquad (14)$$

We get

$$\hat{\varrho}_{t|t-1} = \sum_{\tau=1}^{t-1} \{<\varrho_t, \nu_\tau>\} \nu_\tau \; .$$

Comparing this with (11a), we arrive at justifing the term "innovations".

<u>Corollary 1</u> - The sequence of the innovations of the representators of an experiment can be written in the form

$$e_t = \rho_t - \hat{\rho}_{t|t-1} \qquad t = 2,3,\cdots,$$
$$e_1 = \rho_1 \tag{15}$$

Every term e_t of the innovations sequence is therefore obtained by substracting from the representator ρ_t its l.l.s. one-step prediction, i.e. its l.l.s. reconstruction based on the experiment (14) up to the immediate past.

Example 3 (Kalman-Bucy formulas) - Let the random vector w of Example 1 be a t-dependent random vector x_t. Eqs. (13) give at once

$$\begin{aligned}\hat{x}_{t+1|t} &= \hat{x}_{t+1|t-1} + E[x_{t+1}\nu_t'] \\ &= \hat{x}_{t+1|t-1} + E[x_{t+1}e_t']G_t^{-1}e_t\end{aligned} \tag{16}$$

Further, if x_t is the solution of the stochastic difference state-equation

$$x_{t+1} = \phi_t x_t + \xi_t \tag{17}$$
$$E[x_1] = 0 \qquad E[x_1x_1'] = \Pi$$

and the observations ρ_t are given by

$$\rho_t \triangleq z_t = C_t x_t + \varsigma_t$$

with ξ_t and ς_t zero mean vectors for every $t \in I$ uncorrelated with x_1 and

$$E[\xi_t\xi_\tau'] = Q_t\delta_{t\tau} \qquad E[\varsigma_t\varsigma_\tau'] = R_t\delta_{t\tau} \qquad E[\xi_t\varsigma_\tau'] = \Gamma_t\delta_{t\tau}$$

the discrete-time Kalman-Bucy formulas are quickly obtained

$$\hat{x}_{t+1|t} = \phi_t\hat{x}_{t|t-1} + K_te_t \tag{18}$$
$$K_t \triangleq (\phi_tP_tC_t' + \Gamma_t)(C_tP_tC_t' + R_t)^{-1}$$
$$P_{t+1} = \phi_tP_t\phi_t' - K_tG_tK_t' + Q_t \tag{19}$$
$$\hat{x}_{1|0} = 0 \qquad P_1 = \Pi\ .$$

Example 4 (recursive system identification by PRBS's) - Hereafter, the problem of determining impulse responses and system parameters is considered. To this end the setting of Example 2 will be used throughout. Our first comment is that, though solution (13) is completely general and hence can immediately be applied to the problem posed in Example 2, the proposed algorithm becomes very complicated for large t unless some special input is used. This is so because: first, the number of computations required by (11) to get e_t increases linearly with t; and second, an ever expanding Span $\{ \rho_\tau , \forall \tau \leq t\}$ makes eventually the reconstructed impulse response extremely sensitive to measurement noise [3,5]. On the other hand, the given solution becomes particularly convenient if the system output is uniformly sampled every Δ sec. and a periodic input with period $L\Delta > \omega_1$ is used. In this way, if the measurements start at least $L\Delta$ sec. after the test input has been applied to the system, there are only L lineraly independent representators to consider, and ideally, the experiment is completed in the next $L\Delta$ sec.

In the single-input single-output case, attractive input signals are the pseudorandom binary sequences (PRBS) [6] of length

$$L = 2^i - 1, \qquad\qquad i = 2,3,\ldots$$

and amplitude +V and -V. They look attractive essentially because of the following property of their autocorrelation function

$$\langle \rho_t , \rho_\tau \rangle = \begin{cases} \|\rho\|^2 & t = \tau + mL\Delta \\ -\|\rho\|^2/L & \text{elsewhere} \end{cases}$$

where, for a system with an input excited by a PRBS of period $L\Delta$, $\|\rho\|^2 = V^2 L\Delta$. This feature greatly semplifies Eqs. (11) - (13). In fact, after some further manipulations, we get the recursive l.l.s. reconstruction of the system impulse response according to the following steps:

$$\begin{aligned} e_t(\omega) &= \rho_t(\omega) - \rho_{t-1}(\omega) + \alpha_t\, e_{t-1}(\omega) \\ \epsilon_t &= m_t - m_{t-1} + \alpha_t\, \epsilon_{t-1} \\ \hat{w}_{|t}(\omega) &= \hat{w}_{|t-1}(\omega) + L(L+1)^{-1} \|\rho\|^{-2} \alpha_{t+1}\, \epsilon_t\, e_t(\omega) \end{aligned} \tag{20}$$

where: $\alpha_t \triangleq (L-t+3)\,(L-t+2)^{-1}$; $t = 1,2,\ldots,L$; and the initial values are

$$\rho_o(\omega) = 0 \qquad e_o(\omega) = 0 \qquad \hat{w}_{|0}(\omega) = 0$$

$$\epsilon_o = 0 \qquad m_o = 0 \; .$$

PRBS's have been used for a long time as probing inputs for identifying systems [7,8]. However, all previous algorithms used in connection with the identification experiment of this section essentially relied on the PRBS resemblance to white noise and were based on crosscorrelation-type arguments. Our success in getting in a neat way the recursions (20) has been due to the systematic procedure developed in this paper and based on the notion of a generalized innovation process.

4 - RECURSIVE INTERPOLATION AND SMOOTHING

Let $K(t,\tau)$ be a real-valued nonnegative definite kernel defined for t and on some interval T of the real line. Hereafter, the Hilbert space H of Sect. 2 will be identified with the reproducing kernel Hilbert space (RKHS) H(K) with reproducing kernel (RK) $K(t,\tau)$. As for RKHS theory and applications, the reader is referred to [9] and [10]. The only property of H(K) that will be repeatedly used in the sequel is the so-called reproducing property, viz.

$$y(t) = \langle y, K(\cdot,t)\rangle \qquad \forall y \in H(K).$$

The interpolation problem we intendo to pose can be formulated as follows. Given a sequence of numbers

$$y_i \triangleq y(t_i) = \langle y, K(\cdot,t_i)\rangle, \quad i \in I, \; t_i \in T,$$

find

$\hat{y}_{|n} \triangleq$ the minimum-norm element in H(K) interpolating $y_1, y_2, \ldots, y_n$,

in a recursive form. This problem is clearly a particular version of the indirect-sensing measurement problem formulated in Sect. 2.

Taking into account the reproducing property of H(K), from (11) -(13) we get at once

$$\begin{aligned}
e_n(\cdot) &= K(\cdot,t_n) - \sum_{i=1}^{n-1} e_i(t_n)\|e_i\|^{-2} e_i(\cdot)\\
\|e_n\|^2 &= K(t_n,t_n) - \sum_{i=1}^{n-1}[e_i(t_n)]^2\|e_i\|^{-2}\\
\mu_n &= \|e_n\|^{-1}\Big[y_n - \sum_{i=1}^{n-1}\mu_i\|e_i\|^{-1}e_i(t_n)\Big]\\
\hat{y}_{|n}(\cdot) &= \hat{y}_{|n-1}(\cdot) + \mu_n\|e_n\|^{-1}e_n(\cdot)
\end{aligned} \tag{21}$$

Example 5 (interpolation by splines) - Let y be the output of a one-input one-output finite-dimensional linear system [1)]

$$\mathcal{S}: \begin{cases} \dot{x}(t) = A(t)x(t) + b(t)u(t)\\ x(t_0) = 0\\ y(t) = c(t)x(t)\end{cases}$$

Thus, the set of all outputs y on $T \triangleq [t_0, t_f]$ corresponding to all possible square-integrable inputs u on T, coincides [12] with the RKHS H(K) with RK given by

$$K(t,\tau) = \int_{t_0}^{t\wedge\tau} H(t,\sigma)H(\tau,\sigma)\,d\sigma \tag{22}$$

where $\wedge$ denotes minimum, $H(t,\sigma) \triangleq c(t)\phi(t,\sigma)b(\sigma)$ and $\phi(t,\sigma)$ is the state-transition matrix of $\mathcal{S}$. Moreover, the transformation $\mathcal{S}: u \to y$ from $L_2(T)$ onto

1) The results that follow can be generalized [11] to the case of unknown initial state $x(t_0)$

H(K) is a congruence (isometric isomorphism), i.e.

$$\mathcal{J}u = y \implies \|y\|^2 = \int_T u^2(t)\,dt \tag{23}$$

In particular, if

$$\mathcal{J}: \begin{cases} [Ly](t) = u(t) \\ x(t_o) \triangleq [y(t_o), y^{(1)}(t_o), \cdots, y^{(m-1)}(t_o)]' = 0 \end{cases}$$

with L a differential operator ($D \triangleq d/dt$)

$$L \triangleq D^m + a_{m-1}D^{m-1} + \cdots + a_1 D + a_o$$

(23) yields an explicit formula for the H(K)-norm of y, viz.

$$\|y\|^2 = \int_T [Ly(t)]^2 dt \tag{24}$$

and [2]) $\hat{y}_{|n}$ is [11, 13] the <u>L-spline</u> interpolating $x(t_o)$, $y_1, y_2, \ldots, y_n$. If $L \triangleq D^m$, $\hat{y}_{|n}$ is called the <u>polynomial spline</u> of order m interpolating $x(t_o), y_1, y_2, \ldots, y_n$.

Strictly related to the above interpolation problem, we now consider the following <u>smoothing problem</u>. Let $K(t, \tau)$ be again a nonnegative definite kernel, H(K) the associated RKHS and $\|.\|$ the corresponding norm. Given a sequence of real numbers

z_i, $i \in I$,

find $\hat{y}_{|n} \triangleq$ the element in H(K) minimizing

$$\sum_{i=1}^{n} \sigma_i^{-2} (z_i - y_i)^2 + \|y\|^2 \tag{25}$$
$$y_i \triangleq y(t_i), \qquad t_i \in T,$$

in a recursive form. This is essentially a problem of <u>smoothing by generalized splines</u>. It has been shown [12] that (25) is equivalent to the following problem of <u>statistical smoothing</u>. Given the discrete-time observations

$$z_i = y_i + \xi_i \quad , \quad i \in I, \tag{26}$$

where $y_i \triangleq y(t_i)$ are samples from a stochastic process $y(t)$ with zero mean and covariance kernel

$$K(t, \tau) \triangleq E[y(t)\,y(\tau)]$$

and ξ_i r.v.'s uncorrelated with $y(t)$ with zero mean and covariance

2) The L-spline interpolating $y_1, y_2, \ldots, y_n$, is the function passing through $y_1, y_2, \ldots, y_n$ and minimizing (26).

$$E[\varsigma_i \varsigma_j] = \sigma_i^2 \delta_{ij}$$

find the l.l.s. smoothed estimate $\hat{y}_{|n}(t)$ of $y(t)$, $t \in T$, based on $z_1, z_2, \dots z_n$, in a recursive form. To solve this problem without resorting to a dynamic representation of the process y, we rephrase it in a suitable form. First, notice that by the reproducing property of $H(K)$ the unknown $y \in H(K)$ must be such that

$$y_i = \langle y, K(\cdot, t_i) \rangle, \qquad i \in I$$

From (21a) on the other hand we get

$$K(\cdot, t_i) = \sum_{j=1}^{i} \alpha_{ij}\, e_j(\cdot)$$

where

$$\alpha_{ij} \triangleq \begin{cases} \| e_j \|^{-2}\, e_j(t_i), & j < i \\ 1, & j = i \end{cases}$$

Therefore,

$$y_i = \sum_{j=1}^{i} \alpha_{ij} \langle y, e_j \rangle$$

Hence, setting

$$\theta_i = \theta \triangleq [\langle y, e_1 \rangle, \langle y, e_2 \rangle, \cdots]'$$

$$c_i \triangleq [\alpha_{i1}, \alpha_{i2}, \cdots, \alpha_{ii}, 0, 0, \cdots]'$$

we have

$$\begin{cases} \theta_{i+1} = \theta_i \\ z_i = c_i \theta_i + \varsigma_i, \quad i \in I \end{cases} \tag{27}$$

from which the Kalman-Bucy formulas (18) and (19) give the l.l.s. estimate $\hat{\theta}_{|n}$ of θ based on $z_1, z_2, \dots, z_n$, viz.

$$\begin{aligned} \hat{\theta}_{|n} &= \hat{\theta}_{|n-1} + F_n [z_n - c_n \hat{\theta}_{|n-1}] \\ F_n &= (c_n P_n c_n' + \sigma_n^2)^{-1} P_n c_n' \\ P_{n+1} &= P_n - (c_n P_n c_n' + \sigma_n^2) F_n F_n' \end{aligned} \tag{28}$$

with P_1 equal to a symmetric nonnegative definite matrix, e.g. $P_1 = \sigma_0^2 I$ with a sufficiently large σ_0^2. Finally, we obtain the desired recursive formula for $\hat{y}_{|n}$,

$$\hat{y}_{|n} = G \hat{\theta}_{|n} = \hat{y}_{|n-1} + G F_n [z_n - c_n \hat{\theta}_{|n-1}] \tag{29}$$

where

$$G \triangleq [\| e_1 \|^{-2} e_1(\cdot), \| e_2 \|^{-2} e_2(\cdot), \cdots].$$

5 - CONCLUSIONS

Indirect sensing experiments are defined and shown to encopass a large class of applications such as estimation, filtering, system identification, and interpolation and smoothing by splines. When a recursive solution to the indirect-sensing experiment problem is desired, the notion of a discrete-time generalized innovations process, or innovations experiment, appear to be a natural and effective one to use. The problem of estimating the state of a finite-dimensional linear system from discrete-time noisy measurements appears to be but one of the possible applications of the theory developed. The problem of determining the impulse response of a Q-input P-output system is approached by the use of the notion of an innovations experiment. When PRBS's are used as probing inputs, attractive formulas of recursive type are obtained by the proposed method easily and in a direct way. Finally, it is shown that problems of interpolation and smoothing by splines can be approached by the theory developed.

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