HYPER-AFL'S AND ETOL SYSTEMS

P. A. Christensen
Department of Computer Science
University of Aarhus
Aarhus, Denmark

The notions of K-iteration grammars and of hyper-AFL's are introduced in [57] and [103]. The notation follows that of [103].

Theorem

$ETOL = ETOL_{iter}^{(1)} = ETOL_{iter}$, i.e. ETOL is a hyper-AFL.

Sketch of a proof

It is obvious from the definition of an iteration grammar that $ETOL \subseteq ETOL_{iter}^{(1)} \subseteq ETOL_{iter}$.

To prove $ETOL_{iter} \subseteq ETOL$, let $G = (V_N, V_T, S, U)$ be an ETOL-iteration grammar with $U = \{\mathcal{T}_1, \ldots, \mathcal{T}_n\}$, where each $\mathcal{T}_j$ is an ETOL-substitution.

Assume that $\mathcal{T}_j(a_i) = L(G_{i,j})$, where $G_{i,j} = (V_N^{i,j}, V_T \cup V_N, T_{i,j}, S_{i,j})$ are synchronized versions of ETOL-systems and the alphabets $V_N^{i,j}$ are pairwise disjoint.

We define a new ETOL-system: $G' = (V'_N, V'_T, T', \bar{\bar{S}})$, where $V'_N = \{\$\} \bigcup_{i,j} (V_N^{i,j} \cup \{\bar{S}_{i,j}\}) \cup \bar{V}_T \cup \bar{V}_N \cup \bar{\bar{V}}_T \cup \bar{\bar{V}}_N$ where $ and all $\bar{S}_{i,j}$ are new symbols.

$\bar{V}_X = \{\bar{a} \mid a \in V_X\}$ and $\bar{\bar{V}}_X = \{\bar{\bar{a}} \mid a \in V_X\}$ for $X = N$ and $X = T$ are sets of new symbols.

If X is a string of symbols $X = b_1 \ldots b_k$, then $\bar{X} = \bar{b}_1 \ldots \bar{b}_k$ and $\bar{\bar{X}} = \bar{\bar{b}}_1 \ldots \bar{\bar{b}}_m$. The axiom of G' is defined as $\bar{\bar{S}}$ in exactly the same way.

Finally T' consists of the tables:

t_0:

$\bar{\bar{a}}_i \to \bar{S}_{i,j}$ for each i and each j

$A \to \$$ for any other symbol A.

For $1 \leq j \leq n$ there is the table:

t_j:

$\overline{S}_{i,j} \to \overline{S}_{i,j}$; $\overline{S}_{i,j} \to S_{i,j}$ for each i

$\bar{a} \to \bar{\bar{a}}$; $\bar{\bar{a}} \to \bar{\bar{a}}$ for each $a \in V_N \cup V_T$

$A \to \$$ for any other symbol A.

For $1 \leq j \leq n$ and $1 \leq i \leq |V_N \cup V_T|$ there is the set of tables:

$\tilde{T}_{i,j}$: which consists of all tables from $T_{i,j}$ where the table with the terminal productions ($G_{i,j}$ is synchronized) is changed to produce barred terminals instead. In all these tables we add the productions:

$\bar{\bar{a}} \to \bar{\bar{a}}$ for each $a \in V_N \cup V_T$

$\overline{S}_{k,j} \to \overline{S}_{k,j}$ for each k

$A \to \$$ for any other new symbol.

Finally there is the table with the terminal productions:

$\bar{\bar{a}} \to a$ for each $a \in V_T$

$A \to \$$ for any other symbol A.

The claim is now that $L(G) = L(G')$.

The reason for this is that rewriting a double- barred word via t_0's productions $\bar{\bar{a}}_i \to \overline{S}_{i,j}$ is the same as choosing the substitution $\mathcal{T}_j$ to be used. The substitution is then performed via the tables t_j and $\tilde{T}_{k,j}$, and when the substitution is performed, the word is again double-barred. We can choose a new substitution and so forth until we finally use the terminal table to reach a terminal word. If the tables to be used in the line of derivation are not chosen according to this scheme, a $-symbol is introduced in the string, and from this it is impossible to reach a terminal word.

Therefore ETOL = $\text{ETOL}^{(1)}_{iter}$ = ETOL_{iter}, and since it is well-known that ETOL is a full AFL, we conclude that ETOL is a hyper-AFL.

Corollary 1

If K is a family of languages such that $F \subseteq K \subseteq ETOL$ then $K_{iter} = ETOL$.

Proof

$$ETOL = F_{iter} \subseteq K_{iter} \subseteq ETOL_{iter} = ETOL.$$

Thus we have for instance proved that:

$$ETOL = F_{iter} = R_{iter} = CF_{iter} = EOL_{iter} = ETOL_{iter}.$$

Since each hyper-AFL is a full AFL and since $ETOL = R_{iter}$, we conclude:

Corollary 2

ETOL is the smallest hyper-AFL.

In [56] it is stated that the family $R^{(1)}_{iter}$ is exactly the family of languages accepted by pre-set-pushdown automata, and we are now able to prove:

Corollary 3

$$R^{(1)}_{iter} \subsetneq R_{iter} = ETOL.$$

Proof

By definition $R^{(1)}_{iter} \subseteq R_{iter}$ and we know that $R_{iter} = ETOL$. Furthermore, hyper-AFL's are easily seen to be closed under substitution; in order to prove the corollary, it suffices to prove that $R^{(1)}_{iter}$ is not closed under substitution:

Let $L_1 = \{a^{2^n} \mid n \geq 0\}$ and $L_2 = \{ab^{2^m} \mid m \geq 0\}$, then obviously $L_1, L_2 \in OL \subseteq EOL = F^{(1)}_{iter} \subseteq R^{(1)}_{iter}$.

Define the substitution $\mathcal{J}$ by $\mathcal{J}(a) = L_2$. Then $\mathcal{J}(L_1)$ is the set of all words $ab^{2^{n_1}} ab^{2^{n_2}} \ldots ab^{2^{n_k}}$, where each $n_i \geq 0$ and there exists $l \geq 0$ such that $k = 2^l$. Define the finite substitution t by $t(a) = \{a\}$ and $t(b) = \{\lambda, b\}$ then the proof in [35] of the non-closure of EOL under inverse homomorphism shows that $t(\mathcal{J}(L_1)) \notin EOL = F^{(1)}_{iter}$. It is furthermore well-known that EOL is closed under finite substitution, therefore $\mathcal{J}(L_1) \notin F^{(1)}_{iter}$. But since infinite regular sets fulfil a pumping lemma, it is obvious that $\mathcal{J}(L_1) \notin R^{(1)}_{iter}$ and the corollary is proved.

Finally we mention that we have proved the existence of full AFL's K and $\overline{K}$ such that:

$$K_{iter}^{(1)} \subsetneq K_{iter} = \overline{K}_{iter}^{(1)} = \overline{K}iter$$

and $K \subsetneq K_{iter}$ but $(K_{iter})_{iter} = K_{iter}$ namely $K = R$ and $\overline{K} = ETOL$.