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TIME-OPTIMAL CONTROL SYNTHESIS FOR NON-LINEAR SYSTEMS: A FLIGHT DYNAMIC EXAMPLE

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Abstract

The determination of optimal closed-loop control (or 'on line' control) laws is often referred to in the mathematics literature as the 'synthesis' problem. Except for the well-known case of 'linear dynamics, quadratic criteria', this problem is still largely unsolved. This paper presents a local approximation technique for time-optimal control synthesis of a class of non-linear systems: specifically, point-to-point aerodynamic flight in a resisting medium. Preliminary computational results are presented, indicating that the approximation technique is feasible.

1. INTRODUCTION

By control 'synthesis' we mean the determination of the optimal control 'on line' or 'closed loop'; that is to say, as a function of the state (as well as time) along the optimal trajectory. Excepting the case of 'linear dynamics, quadratic cost', this still continues to be the major unsolved problem in optimal control - for ordinary differential equations, at any rate. The complexity of the problem is acknowledged already in the early Pontryagin work [1]. Indeed we do not yet possess any general existence theory, let alone whether constructive or not. Computational techniques for optimal control (see [2]) yield only open-loop controls; the control is determined as a function of time for given initial and/or final conditions, and not as a function of the state. The lone exception is time-optimal bang-bang control of linear systems, where switching surfaces have been calculated for second-order and third-order systems; the general case being given up as hopelessly complicated (see [3]).

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Previous attempts at approximation have been confined mostly to linearizing the equations to conform to the linear quadratic theory [4]. In this paper we present a local approximation technique for a particular class of nonlinear dynamics: namely, rocket flight in a resisting medium, using techniques based on the Bellman equation, known to be invalid for time-optimal control of linear systems, see [1]. Bryson and Ho [4], Jacobsen [5] have also used the Bellman equation but for the purpose of obtaining an iterative technique for the optimal trajectory. Our technique is different in concept and execution from theirs. We do not in particular seek to calculate the trajectory but rather the control, directly.

2. THE PROBLEM

We begin with a more precise statement of the problem. The state vector $x(t)$ for the problem splits into two parts:

$$x(t) = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix}$$

and the dynamic equations have the form:

$$\left.\begin{aligned} \dot{x}_1(t) &= f_1(t;x_2(t)) \\ \dot{x}_2(t) &= f_2(t;x_2(t);u(t)) \end{aligned}\right\} \qquad (2.1)$$

where $u(t)$ is the control to be synthesized, and is subject to the constraint of the form:

$$u(t) \in C \qquad (2.2)$$

where C is closed and convex. Let t_i be an initial time, and $x(t_i)$ the initial state at time t_i. Assuming that it is possible to find an admissible control such that

$$\begin{aligned} &x(t_i) \text{ given} \\ &x_2(T) = 0, \quad T > t_i, \end{aligned} \qquad (2.3)$$

the 'time-optimal' control problem is that of finding $u(\cdot)$ that minimizes T. Let us assume now that an optimal control exists for every initial state in some open set $\mathscr{D}$. The synthesis problem is that of finding a function $h(t;x)$ such that the (an) optimal control can be expressed:

$$u_o(t) = h(t;x_o(t)) \quad t \geq t_i \cdots \tag{2.4}$$

where $x_o(t)$ is the corresponding optimal trajectory, so that

$$\dot{x}_o(t) = f(t;x_o(t);u_o(t)); \quad f(\cdot) = \begin{Bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{Bmatrix}$$

and satisfies the given conditions at t_i and T, and yields the minimum such T. As we have remarked earlier, no sufficient conditions of any generality are available at the present time for the existence of such a function $h(t;x)$, and of course no general algorithm for computation is known either. Indeed it does not seem likely that there will be any forthcoming in the near future. It would appear that the best we can hope for is an approximation scheme that is good in special cases, and how good being demonstrable only by computation. At any rate, the present work offers not more.

Since the Bellman equation plays a fundamental role, we shall state and prove it first in the form we shall need to use.

Theorem 2.1 (Bellman)

Let $T(t;x)$ denote the (incremental) minimal time taken to reach the origin beginning with the state x in $\mathscr{D}$ at time t, for the system described by (2.1), (2.2), (2.3). Assume that $T(t;x)$ is continuously differentiable in t and x, $x \in \mathscr{D}$. Assume further that $f_1(\cdot), f_2(\cdot)$ are also continuous in all the variables. Fix t_i and x, and let $u_o(t)$ denote an optimal control, and $x^o(t)$, $t \geq t_i$, the corresponding optimal trajectory, $x^o(t_i) = x$. Then if t_i is such that $u_o(t_i)$ is continuous from the right, or is in the Lebesgue set of the function $f_2(t;x^o(t);u_o(t))$, we have

$$\operatorname*{Min}_{u \in C} \left[\frac{\partial T}{\partial x_2}, f_2(t_i;x;u)\right] = \left[\frac{\partial T}{\partial x_2}, f_2(t_i;x;u_o(t_i))\right] \tag{2.5}$$

Proof

Define a new control to be equal to arbitrary given u in C for $t_i \leq t \leq t_i + \Delta$. Let $x(t)$ denote the corresponding trajectory. Then for all Δ sufficiently small,

$x(t_i + \Delta)$ will belong to the open set $\mathscr{D}$. Moreover

$$T(t_i + \Delta; x(t_i + \Delta)) + \Delta \geq T(t_i;x) \tag{2.6}$$

But

$$\lim_{\Delta \to 0} \frac{1}{\Delta}\Big(T(t_i + \Delta; x(t_i + \Delta)) - T(t_i;x)\Big)$$

$$= \frac{\partial T}{\partial t} + \left[\frac{\partial T}{\partial x_1}, \ f_1(t_i,x)\right] + \left[\frac{\partial T}{\partial x_2}, \ f_2(t_i;x;u)\right]$$

and is

$$\geq -1$$

by virtue of (2.6) where $\frac{\partial T}{\partial x_i}$ denotes gradient, and $[,]$ inner product. On the other hand, of course,

$$\lim_{\Delta \to 0} \frac{1}{\Delta}\Big(T(t_i + \Delta; x^o(t_i + \Delta)) - T(t_i;x)\Big)$$

$$= \frac{\partial T}{\partial t} + \left[\frac{\partial T}{\partial x_1}, \ f_1(t_i;x)\right] + \left[\frac{\partial T}{\partial x_2}, \ f_2(t_i;x;u_o(t_i))\right]$$

$$= -1 \text{ (since equality holds in (2.5) in this case),}$$

provided t_i is a point of continuity of $u_o(t)$, or more generally belongs to the Lebesgue set of the function $f_2(t;x^o(t);u_o(t))$. Hence it follows that

$$\operatorname*{Min}_{u \in C} \left[\frac{\partial T}{\partial x_2}, f_2(t_i;x;u)\right] = \left[\frac{\partial T}{\partial x_2}, \ f_2(t;x;u_o(t_i))\right]$$

at all points t_i where $u_o(t_i)$ is continuous from the right.

If the conditions of the Theorem are met, then we can use (2.5) to determine control synthesis, namely the u that minimizes:

$$\left[\frac{\partial T}{\partial x_2}, \ f_2(t;x;u)\right] \tag{2.7}$$

which yields a function of t and x. Of course the basic fault in this method has already been noted by Pontryagin [1]; the function $T(t;x)$ need not possess the requisite differentiability properties, indeed does not even for linear systems and bang-bang control. Hence it would be foolish to invoke this technique for the case of linear systems. Here we shall only consider nonlinear systems. Of interest in this connection is a result due to Rademacher, pointed out by Friedman [6] that uniform Lipschitz continuity implies almost everywhere differentiability. It is not clear however that this is any great help; it is not for our problem at any rate.

An Example

It may be helpful to consider first an illustrative example which although quite simple, still retains some of the salient features of the flight dynamic system we shall consider. In particular we can see what the local approximation is, and how good it is. We consider motion in a plane with fixed speed:

$$\dot{x}(t) = \cos \gamma(t)$$

$$\dot{y}(t) = \sin \gamma(t)$$

$$\dot{\gamma}(t) = \alpha$$

where α is the control variable (one-dimensional), and subject to the constraint:

$$|\alpha| \leq 1$$

We consider the problem of returning to the origin in the plane: $x = 0$, $y = 0$, in minimal time, beginning with any given x, y, γ. It can be readily verified that the optimal controls are of the form:

$$+ 1, \ - 1, \text{ or zero}$$

(The control is not bang-bang), and that the optimal trajectories are arcs of unit radius circles and straightline tangents to them. Let

$$r = \sqrt{x^2 + y^2} \quad ; \ \operatorname{Tan} \sigma = y/x; \ \cos \sigma = -y/r$$

Let $T(t;x;y;\gamma)$ denote the minimal time starting from x, y, γ at time t. Then an actual calculation shows that $T(\ldots)$ is differentiable so long as

$$r - 2|\operatorname{sine}(\gamma - \sigma)| \neq 0$$

On the other hand at points where

$$r - 2 \operatorname{sine}(\gamma - \sigma) = 0$$

the function need not even be continuous. For example at $\sigma = 0$, $\gamma = \pi/2$, $x = -2$, $y = 0$, the function is not continuous in γ. On the other hand, it is differentiable at all points along an optimal trajectory where γ does not switch from +1 to −1 or vice versa.

For a local approximation to synthesis, we proceed as follows. If $\gamma = \sigma$, then

$$\alpha = 0$$

is optimal. We now use [2.6]. We note that the optimal α is obtained by minimizing

$$\frac{\partial T}{\partial \gamma}\,\alpha \tag{2.8}$$

we do not attempt to calculate $\frac{\partial T}{\partial \gamma}$ exactly. Instead, we note first that

$$\left.\frac{\partial T}{\partial \gamma}\right|_{\gamma=\sigma} = 0$$

since the time is a minimum along a straight line path, the time being proportional to arc length. Hence also

$$\left.\frac{\partial^2 T}{\partial \gamma^2}\right|_{\gamma=\sigma} > 0$$

We now use a Taylor expansion and write

$$\frac{\partial T}{\partial \gamma} = \left.\frac{\partial T}{\partial \gamma}\right|_{\gamma=\sigma} + (\gamma - \sigma)\,\frac{\partial^2 T}{\partial \gamma^2} + \ldots$$

Stopping at the second term and substituting in (2.8) we obtain that the optimal α is given by

$$-\ \text{sign}\ (\gamma - \sigma)$$

and we define

$$\text{sign}\ 0 = 0$$

to take care of the case where $\alpha = 0$, or equivalently, $\gamma = \sigma$.

This is then the local approximation to the control law. This law yields the optimal trajectory in this example so long as

$$r - 2|\text{sine}(\gamma-\sigma)| > 0$$

It fails where

$$r - 2|\text{sine}(\gamma - \sigma)| < 0$$

For example, it does not hold when

$$\gamma - \sigma = \pi/2 \ ; \quad x = -1; \ y = 0$$

at which point the optimal choice of α is +1, rather than −1 as given by the local approximation.

3. THE FLIGHT DYNAMIC PROBLEM

We consider rocket motion in a resistive medium at fixed altitude, the dynamics now being described as follows:

$$\dot{x} = v \cos \gamma$$

$$\dot{y} = v \sin \gamma$$

$$\dot{v} = f_3(t;v;\alpha)$$

$$\dot{\gamma} = f_4(t;v;\alpha)$$

where v is the speed (magnitude of velocity vector), and α is the (one-dimensional) control variable - the 'angle of attack', γ being the flight-path angle. The functions $f_3(\cdot)$, $f_4(\cdot)$, being notable first in that they do not depend on γ, are specified as follows:

$$f_3(t;v;\alpha) = \frac{T-C}{m} \cos \alpha - \frac{N}{m} \sin \alpha$$

$$f_4(t;v;\alpha) = \frac{T-C}{m \cdot v} \sin \alpha + \frac{N}{m \cdot v} \cos \alpha$$

where T, the thrust program is a function of time assumed given,

$$C = C[\alpha;v]v^2$$

$$N = N(\alpha;v]v^2$$

with

$$N(\alpha;v) = 0 \qquad \alpha = 0$$

we assume that the functions are continuously differentiable [although in practice these are only tabulated and must be interpolated for intermediate points]. In particular

$$\frac{\partial}{\partial\alpha} f_3(t;v;\alpha) = 0 \quad \text{at} \quad \alpha = 0$$

and $f_3(t;v;\alpha)$ is a maximum at $\alpha = 0$ for all t and v, and

$$\frac{\partial^2 f_3(t;v;\alpha)}{\partial\alpha^2} < 0$$

We shall consider only the unconstrained time-optimal control problem of starting with arbitrary initial variables denoted by the subscript i:

$$x(t_i)$$
$$y(t_i)$$
$$v(t_i)$$
$$\gamma(t_i)$$

at the initial time t_i and returning to

$$x = 0$$
$$y = 0$$

in minimal time, the control variable α being unconstained.

We assume that there is a bounded open set from which we can always reach the origin in the x - y plane (hereinafter simply the origin) using some control. We shall only be concerned with these points in the state space from now on. We observe now that the form of the equations conforms to (2.1), and we assume that the necessary differentiability conditions are satisfied in the state space region of interest.

In slightly different (but more convenient) notation, let $T(t_i; x_i; y_i; v_i; \gamma_i)$ denote the actual time at which the origin reached on the minimal trajectory. Our method of local approximation proceeds as follows. Let

$$r = x^2 + y^2$$
$$\cos\sigma = -x/\sqrt{x^2 + y^2}$$
$$\tan\sigma = y/x$$

Let $\alpha(t, x, y, v, \gamma)$ denote the optimal control synthesis function we are after. Then from the given properties of the functions $f_3(\cdot), f_4(\cdot)$ we can make the following crucial observations:

(i) The optimal control corresponding to $\gamma_i = \sigma_i$ is given by

$$\alpha(t_i, x_i, y_i, v_i, \sigma_i) = 0 \tag{3.8}$$

(ii) $$\frac{\partial T}{\partial \gamma_i} = 0 \quad \text{at} \quad \gamma_i = \sigma_i$$

Although strictly speaking (3.8) will be an assumption, we can offer the following explanation. At $\gamma_i = \sigma_i$, if we set the control α to be identically zero, we obtain a straightline trajectory which satisfies the initial and final conditions, since for $\alpha = 0$,

$$\dot{\gamma} = f_4(t, v, 0) = 0$$

From

$$f_3(t;v;\alpha) \leq f_3(t;v;0)$$

it follows that for the same speed, the acceleration is a maximum along the straight line. Hence we should expect $\alpha = 0$ is the optimal control for $\gamma_i = \sigma_i$. We can also show that $\alpha = 0$ satisfies the Hamilton-Jacobi system of equations. Thus let us use the notation

$$x_1 = x; x_2 = y; x_3 = v; x_4 = \gamma$$

and let ψ_i, $i = 1, 2, ..4$ denote the adjoint variables. Then for $\alpha = 0$, the adjoint equation yields the solution:

$$\psi_1 = |\psi| \cos \sigma_i$$

$$\psi_2 = |\psi| \sin \sigma_i$$

$$\psi_3 = |\psi| \int_t^T e^{\int_t^s \partial f_3/\partial v \, d\sigma} ds$$

$$\psi_4 = 0$$

(where T is the final time), and

$$\psi_3 f_3 + \psi_4 f_4 = \psi_3 f_3$$

and Ψ_3 being positive, this is clearly maximized by $\alpha = 0$. But of course, we are verifying only a necessary condition, strictly speaking. Observation (ii) is a consequence of the fact that for fixed initial velocity, the straight line trajectory is minimal time. Actually we can offer a formal proof based on (3.8) by calculating the necessary partial derivatives [see below].

We next invoke the Bellman equation for our case, and thus we must minimize with respect to α, the expression:

$$\frac{\partial T}{\partial v} f_3(t_i;v_i;\alpha) + \frac{\partial T}{\partial \gamma} f_4(t_i;v_i;\alpha) \tag{3.9}$$

We approximate the derivatives using Taylor expansion about $\gamma = \sigma_i$. We have:

$$\frac{\partial T}{\partial v} = \left(\frac{\partial T}{\partial v}\right)_{\sigma_i} + (\gamma_i - \sigma_i) \frac{\partial^2 T}{\partial \gamma \partial v} + \ldots$$

$$\frac{\partial T}{\partial \gamma} = \left(\frac{\partial T}{\partial \gamma}\right)_{\gamma=\sigma_i} + (\gamma_i - \sigma_i)\left(\frac{\partial^2 T}{\partial \gamma^2}\right)_{\sigma_i} + \ldots$$

$$= (\gamma_i - \sigma_i)\left(\frac{\partial^2 T}{\partial \gamma^2}\right)_{\sigma_i} + \ldots..$$

The first term in (3.9) being non-zero, we may neglect the second term in comparison. Hence we minimize

$$\left(\frac{\partial T}{\partial v}\right)_{\sigma_i} f_3(t_i;v_i;\alpha) + (\gamma_i - \sigma_i)\left(\frac{\partial^2 T}{\partial \gamma^2}\right)_{\sigma_i} f_4(t_i;v_i;\alpha) \tag{3.10}$$

As a further approximation, the minimum may be approximated by a Newton-Raphson step about $\alpha = 0$, yielding

$$\alpha_{opt} \doteq (\gamma_i - \sigma_i) \frac{\left|\left(\frac{\partial^2 T}{\partial \gamma^2}\right)_{\sigma_i}\right| \frac{\partial f_4}{\partial \alpha}(t_i, v_i, 0)}{\left|\left(\frac{\partial T}{\partial v}\right)_{\sigma_i}\right| \frac{\partial^2 f_3}{\partial \alpha^2}(t_i, v_i, 0)} \tag{3.11}$$

where we have exploited (3.6), (3.8). The approximate synthesis problem is thus "reduced" to calculating the indicated derivatives of the function $T(\ldots)$. For this let h_1, h_2 be fixed and let

$$T(\lambda) = T(t_i;x_i;y_i;v_i + \lambda h_1;\sigma_i + \lambda h_2)$$

All derivatives with respect to λ that are written will be understood to be the value at $\lambda = 0$. Let $x(\lambda, t)$; $y(\lambda;t)$ $v(\lambda;t)$, $\gamma(\lambda;t)$ denote the optimal state trajectory. Then we have

$$\begin{aligned} x(\lambda;T(\lambda)) &= 0 \\ y(\lambda;T(\lambda)) &= 0 \end{aligned} \tag{3.12}$$

Differentiation with respect to λ yields:

$$\begin{aligned} \frac{\partial x}{\partial \lambda} + \frac{\partial T}{\partial \lambda} v(T) \cos \sigma_i &= 0 \\ \frac{\partial y}{\partial \lambda} + \frac{\partial T}{\partial \lambda} v(T) \sin \sigma_i &= 0 \end{aligned} \tag{3.14}$$

where for simplicity of notation, we indicate $T(0)$ by T. Let $\alpha(\lambda;t)$ denote the control corresponding to $T(\lambda)$. Using the dynamics (3.1) and using $x(\lambda, t), y(\lambda, t)$, $v(\lambda, t)$, $\gamma(\lambda, t)$ with obvious meaning, we have

$$\begin{aligned} \frac{d}{dt} \frac{\partial x}{\partial \lambda} &= \frac{\partial v}{\partial \lambda} \cos \sigma_i - v \sin \sigma_i \frac{\partial \gamma}{\partial \lambda} \\ \frac{d}{dt} \frac{\partial y}{\partial \lambda} &= \frac{\partial v}{\partial \lambda} \sin \sigma_i + v \cos \sigma_i \frac{\partial \gamma}{\partial \lambda} \end{aligned} \tag{3.15}$$

$$\frac{d}{dt} \frac{\partial v}{\partial \gamma} = \frac{\partial f_3(t;v;0)}{\partial v} \frac{\partial v}{\partial \lambda}, \quad \text{since} \quad \frac{\partial f_3(t;v;0)}{\partial \alpha} = 0 \tag{3.16}$$

$$\frac{d}{dt} \frac{\partial \gamma}{\partial \lambda} = \frac{\partial f_4(t;v;0)}{\partial v} \frac{\partial v}{\partial \lambda} + \frac{\partial f_4(t;v;0)}{\partial \alpha} \frac{\partial \alpha}{\partial \lambda} \tag{3.17}$$

Equations (3.14-3.17) together yield for $h_1 = 0$, $h_2 = 1$

$$\int_{t_i}^{T} v(t) \frac{\partial \gamma}{\partial \lambda} dt = 0; \quad \frac{\partial v}{\partial \lambda} \equiv 0; \quad v(T) \frac{\partial T}{\partial \lambda} = 0 \tag{3.18}$$

(where for simplicity of notation we use $v(t)$ for $v(0;t)$),

and for $h_1 = 1$, $h_2 = 0$, we obtain:

$$\frac{\partial T}{\partial \lambda} = -\frac{1}{v(T)} \int_{t_i}^{T} \frac{\partial v}{\partial \lambda} \, dt$$

or,

$$\frac{\partial T}{\partial v}_{\sigma_i} = -\frac{1}{v(T)} \int_{t_i}^{T} e^{\int_{t_i}^{t} \frac{\partial f_3}{\partial v} ds} \, dt \tag{3.19}$$

The first equation in (3.18) yields

$$\int_{t_i}^{T} v(t)[1 + \int_{t_i}^{t} \frac{\partial f_4}{\partial \alpha} \frac{\partial \alpha}{\partial \lambda} ds]dt = 0 \tag{3.20}$$

This is a condition then that $\frac{\partial \alpha}{\partial \lambda}$ must satisfy. This relation can be simplified by noting that

$$r = \int_{t_i}^{T} v(t)dt \; ; \; \frac{\partial f_4}{\partial \alpha} = \dot{v}(t)/v(t)$$

or,

$$r + \int_{t_i}^{T} v(t) \int_{t_i}^{t} \frac{\dot{v}(s)}{v(s)} \frac{\partial \alpha}{\partial \lambda} ds \quad dt = 0 \tag{3.21}$$

Remembering that all derivatives written are to be taken at $\alpha = 0$, we can now indicate the second derivative equations, dropping derivatives that are zero at $\alpha = 0$. We shall only calculate them for the case $h_1 = 0$, $h_2 = 1$.

$$\frac{\partial}{\partial t} \frac{\partial^2 (x(\lambda;t)}{\partial \lambda^2} = \frac{\partial^2 v}{\partial \lambda^2} \cos\gamma - v \cos\gamma \left(\frac{\partial \gamma}{\partial \lambda}\right)^2 - v \sin\lambda \frac{\partial^2 \gamma}{\partial \lambda^2} \tag{3.22}$$

$$\frac{\partial}{\partial t} \frac{\partial^2 y(\lambda;T)}{\partial \lambda^2} = \frac{\partial^2 v}{\partial \lambda^2} \sin\gamma - v \sin\gamma \left(\frac{\partial \gamma}{\partial \lambda}\right)^2 + v \cos\gamma \frac{\partial^2 \gamma}{\partial \lambda^2} \tag{3.23}$$

$$\frac{\partial}{\partial t} \frac{\partial^2 v}{\partial \lambda^2} = \frac{\partial f_3}{\partial v} \frac{\partial^2 v}{\partial \lambda^2} + \frac{\partial^2 f_3}{\partial \alpha^2} \left(\frac{\partial \alpha}{\partial \lambda}\right)^2 \tag{3.24}$$

$$\frac{\partial}{\partial t} \frac{\partial^2 \gamma}{\partial \lambda^2} = \frac{\partial f_4}{\partial v} \frac{\partial^2 v}{\partial \lambda^2} + \frac{\partial^2 f_4}{\partial \alpha^2} \frac{\partial \alpha}{\partial \lambda}^2 + \frac{\partial f_4}{\partial \alpha} \frac{\partial^2 \alpha}{\partial \lambda^2} \tag{3.25}$$

Substituting into:

$$\frac{\partial^2 x}{\partial \lambda^2} + v(T) \cos \gamma(T) \frac{\partial^2 T}{\partial \lambda^2} = 0$$

$$\frac{\partial^2 y}{\partial \lambda^2} + v(T) \sin \gamma(T) \frac{\partial^2 T}{\partial \lambda^2} = 0$$

we obtain:

$$\frac{\partial^2 T}{\partial \lambda^2} = -\frac{1}{v(T)} \int_{t_i}^{T} \left(\frac{\partial^2 v}{\partial \lambda^2} - v\left(\frac{\partial \gamma}{\partial \lambda}\right)^2 \right) dt = \left(\frac{\partial^2 T}{\partial \gamma^2} \right)_{\sigma_i} \tag{3.26}$$

$$\int_{t_i}^{T} v(t) \frac{\partial^2 \gamma}{\partial \lambda^2} dt = 0 \tag{3.27}$$

$$\frac{\partial^2 v}{\partial \lambda^2} = \int_{t_i}^{t} e^{\int_s^t \frac{\partial f_3}{\partial v} d\sigma} \frac{\partial^2 f_3}{\partial \alpha^2} \left(\frac{\partial \alpha}{\partial \lambda} \right)^2 ds \tag{3.28}$$

substituting (3.28) into (3.26) and noting that

$$\frac{\partial \gamma}{\partial \lambda} = 1 + \int_{t_i}^{t} \frac{\partial f_4}{\partial \alpha} \frac{\partial \alpha}{\partial \lambda} ds$$

we have thus evaluated all the quantities required in (3.11) except for $\frac{\partial \alpha}{\partial \lambda}$. We note that $\frac{\partial \alpha}{\partial \lambda}$ must satisfy (3.20). Further we must also have

$$\left. \frac{\partial \alpha}{\partial \lambda} \right|_{t=t_i} = \text{Factor of } (\gamma_i - \sigma_i) \text{ in } (3.11),$$

yielding us a second condition it must satisfy. We do not know whether these two conditions can uniquely specify $\frac{\partial \alpha}{\partial \lambda}$. To obtain an approximation we let

$$\frac{\partial \alpha}{\partial \lambda} = a \quad t_i \le t \le b \quad [\text{or} = a(t-b)/(t_i-b)]$$
$$= 0 \quad b < t < T$$

where a and b are unknown, and determine them from the conditions that $\frac{\partial\alpha}{\partial\lambda}$ must satisfy. Of course other choices are possible. The best choice will depend on comparison with the optimal open loop solutions. This completes our approximation procedure.

4. NUMERICAL RESULTS

Calculations were made for a specific example with the functions $m(t)$, $T(t)$ shown in figure 1. The functions $C(\alpha, v)$, $N(\alpha, v)$ were taken in the form:

$$C(\alpha, v) = (32400)\,(f(\alpha)h(v) - H(v))$$
$$N(\alpha, v) = (32400)\,(N(\alpha)M(v))$$

and $f(\alpha)$, $N(\alpha)$, $h(v)$, $H(v)$, and $M(v)$ are shown in figures 2, 3, 4, 5, 6, respectively. The time optimal problem was considered for

$$t_i = 0$$
$$v_i = 800 \text{ ft/sec}$$
$$\gamma_i = 0$$
$$\sigma_i = 23.7^o$$
$$y_i = -9133$$

The optimal open loop control for this case is indicated in figure 7, curve 1 . Curve 2 is for the case $\sigma_i = 60^o$. Both were obtained by the epsilon technique. For the synthesis, we note first of all that when

$$\frac{\partial\alpha}{\partial\lambda} \approx 0$$

which we expect to hold as $t_i \to T$, we can calculate the coefficient of $(\gamma_i - \sigma_i)$ in (3.11) as

$$= \frac{\int_{t_i}^{T} v(t)dt}{\int_{t_i}^{T} e^{\int_{t_i}^{t} \partial f_3/\partial v\, ds}\, dt} \cdot \frac{\dot{v}(t_i)}{v(t_i)} \cdot \frac{1}{\left(\frac{\partial^2 f_3}{\partial\alpha^2}\right)}$$

$$\approx - 1$$

$$\text{(since} \quad \frac{\partial^2 f_3}{\partial \alpha^2} \approx -\dot{v} \text{)}$$

Thus $a = -1$ is a first approximation. The corresponding control is shown in figure 8, where we also show the open loop control for comparison. It is seen that the approximation is already reasonable, (although it systematically underestimates the actual value) demonstrating the feasibility of the technique. Further computer studies are in progress.

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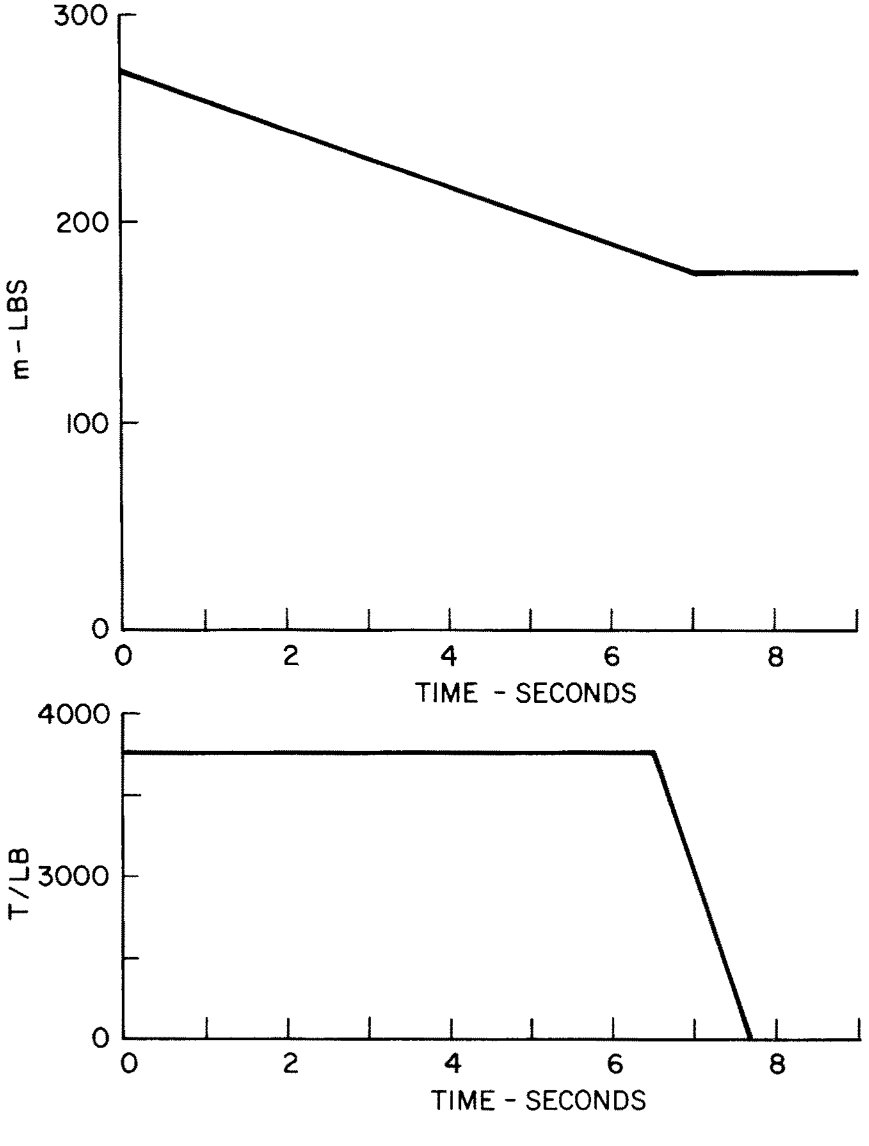
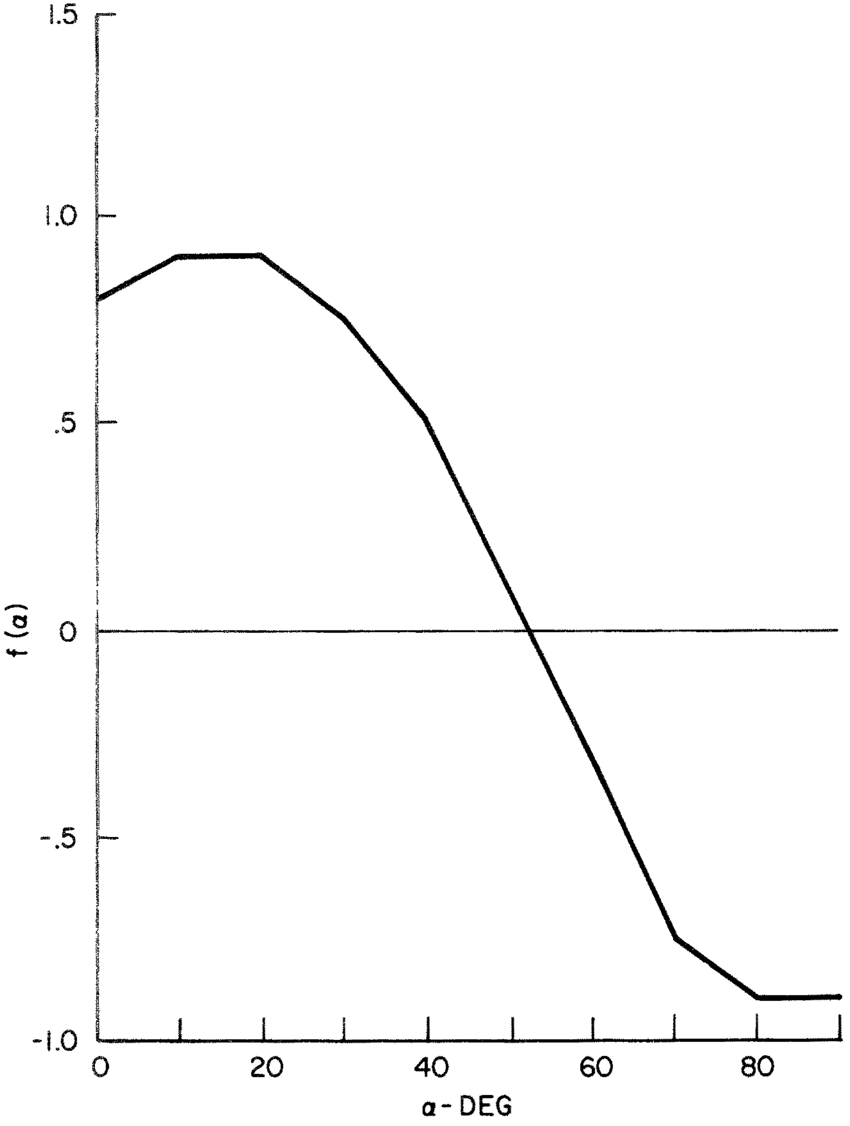
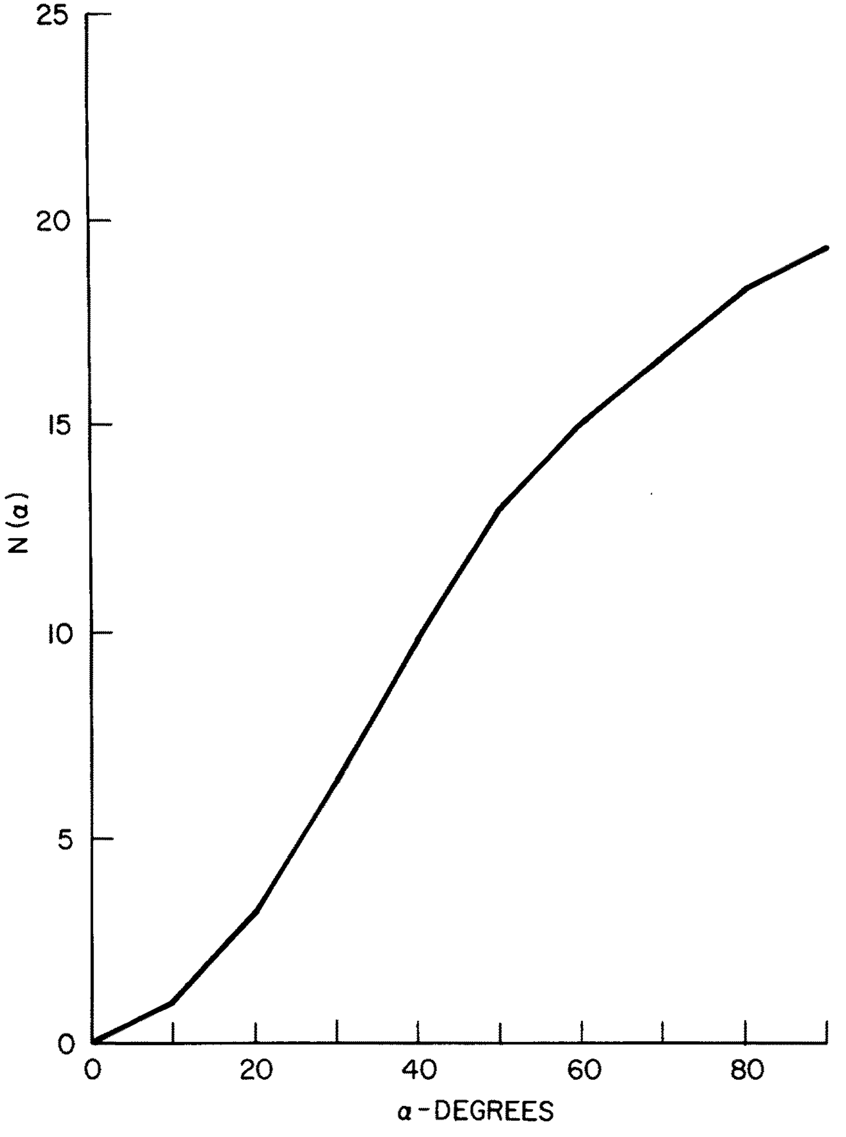


Balakrishnan
Figure 1



Balakrishnan
Figure 4



Balakrishnan
Figure 2

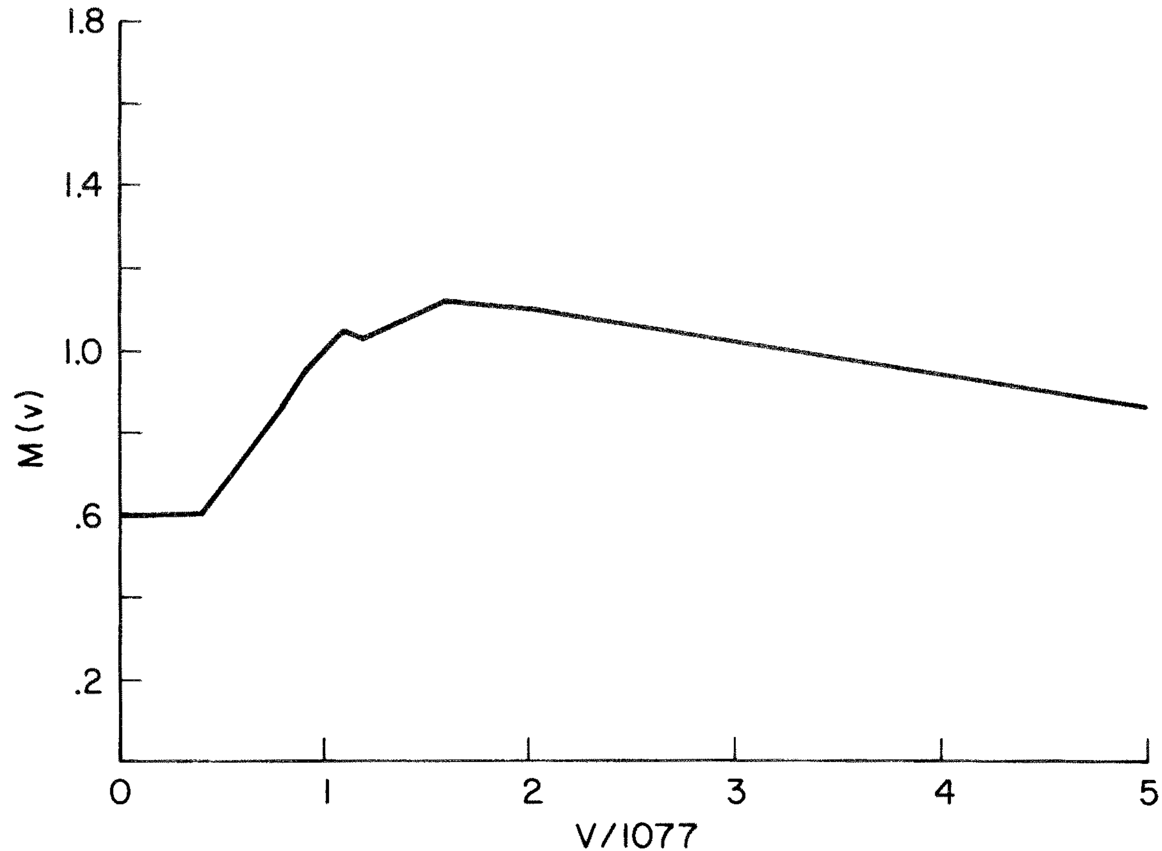


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Figure 3

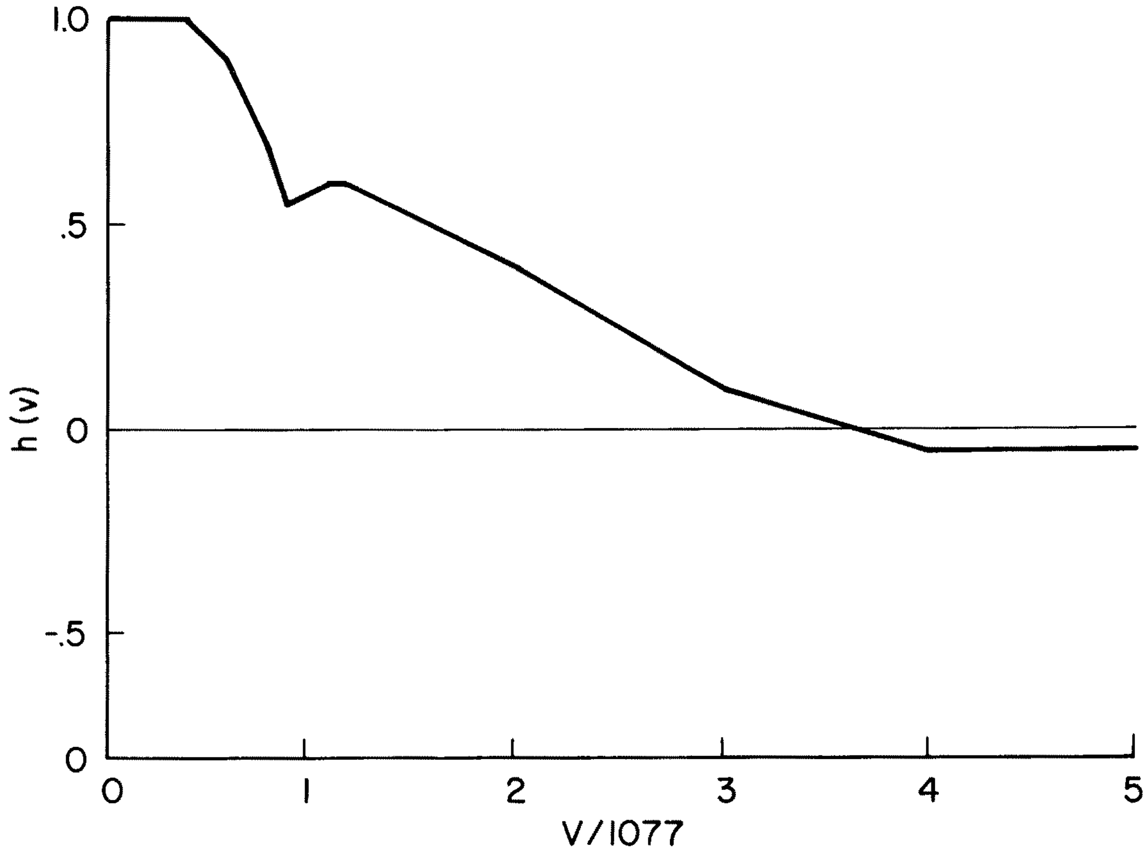

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Figure 5

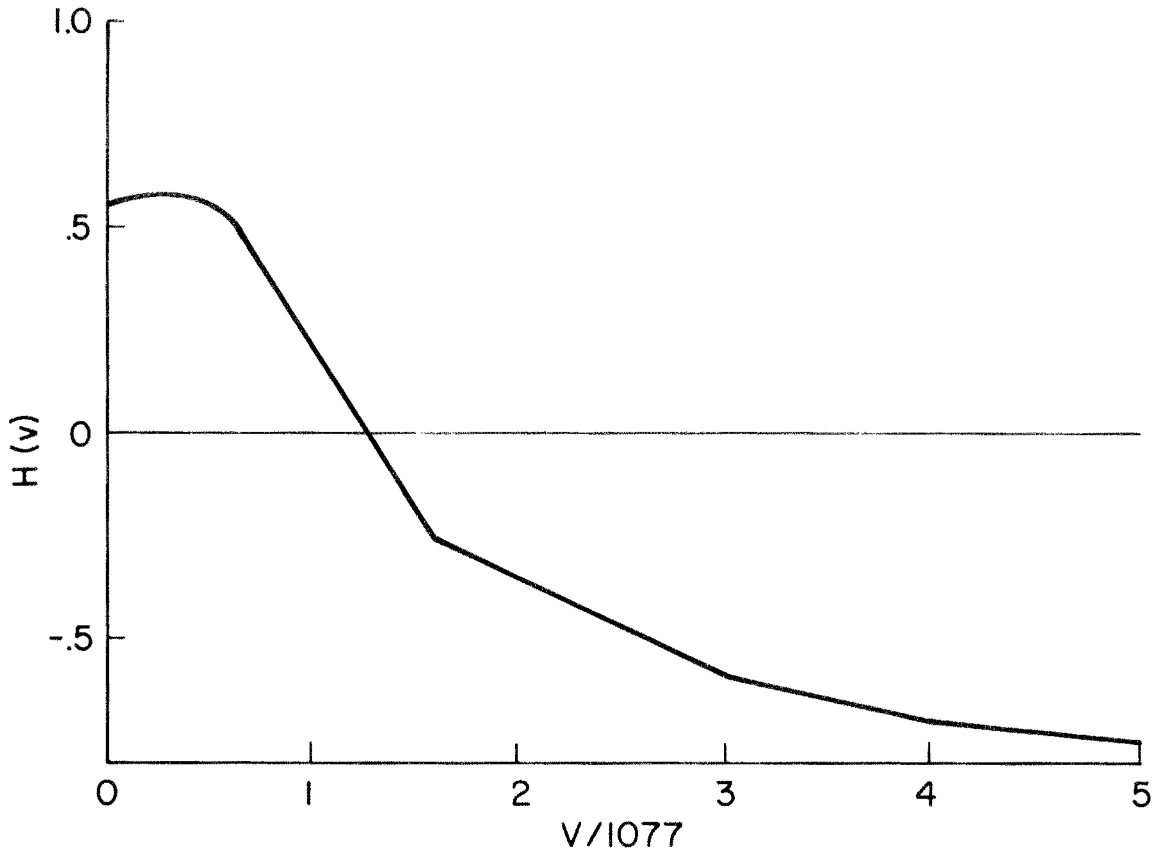

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Figure 6

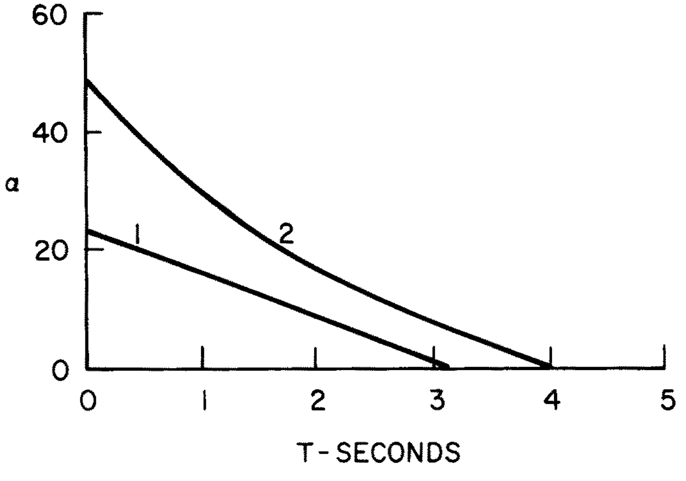

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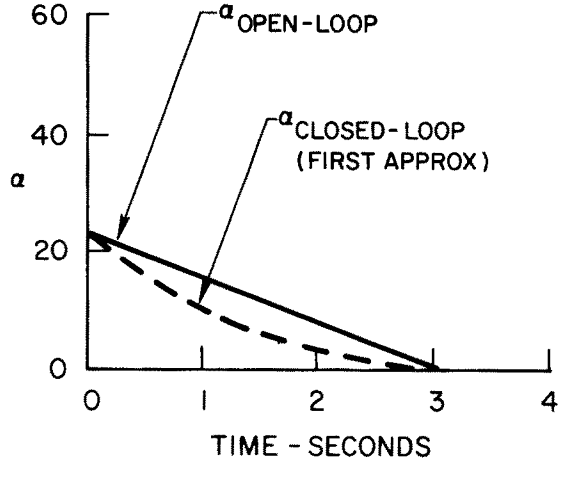
Figure 7



Balakrishnan
Figure 8