On Discrete Moments of Unbounded Order

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Abstract. Moment-based procedures are commonly used in computer vision, image analysis, or pattern recognition. Basic shape features such as size, position, orientation, or elongation are estimated by moments of order ≤ 2. Shape invariants are defined by higher order moments. In contrast to a theory of moments in continuous mathematics, shape moments in imaging have to be estimated from digitized data. Infinitely many different shapes in Euclidean space are represented by an identical digital shape. There is an inherent loss of information, impacting moment estimation.

This paper discusses accuracy limitations in moment reconstruction in dependency of order of reconstructed moments and applied resolution of digital pictures. We consider moments of arbitrary order, which is not assumed to be bounded by a constant.

Keywords: moments, discrete moments, accuracy of estimation, multigrid convergence, digital shapes.

1 Introduction

Moments are widely used in computer vision, image analysis, or pattern recognition (since Hu [1]). A variety of types of moments and moment-based methods has been developed and studied, for example, for object recognition [2], reconstruction of geometric properties of regions [3], or determination of invariants [4]. The (p, q)-*moment* $m_{p,q}(S)$ of a planar set S is defined by the following:

$$m_{p,q}(S) \quad = \quad \iint_S x^p y^q \, dx \, dy$$

It has the order $p + q$.

Basic shape features (e.g., size, position, orientation, elongation) are computed from moments of order less or equal to two. Higher order moments are needed for computing, for example, the orientation of 3D rotationally symmetric shapes (see [5]) or moment invariants (see [1]). In imaging applications we have to

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deal with digitized shapes (objects); consequently, exact moment computation is impossible. The accuracy of moment estimation is limited by many factors, dominated by shape complexity, applied resolution of digital pictures, and the order of reconstructed moments.

Obviously, higher picture resolution enables a higher precision in moment reconstruction. Also, if picture resolution is fixed, then accuracy would decrease if the moment's order increases. Thus, if high-order moments are needed for a particular application, reconstruction accuracy can be improved by an increase in applied picture resolution. This is formally studied as multigrid convergence in digital geometry (see [6]).

Situations, where the order of moments is bounded while picture resolution is allowed to increase (to infinity), have been discussed in [7]. The case of unboundedly increases of orders of moments remained an open problem in that publication.

This paper also covers the case where the order of moments is allowed to tend to infinity. Furthermore, for this situation we consider the special case where the order of computed moments is at most logarithmic in applied picture resolution. We prove an upper bound for the resulting error in estimation which improves the best known upper bound to date (that follows from general tools provided in [8]).

We give definitions and notations as used in this paper. Center points of grid squares are assumed to have integer coordinates (i.e., to be grid points in $\mathbb{Z}^2$). In the diversity of different models for digitizing shapes in Euclidean spaces, we decide for the set of grid points contained in the given shape (analogous to Gauss digitization in [6]). That means, for a set $S \subset \mathbb{R}^2$, its digitization $G(S)$ is defined to be the set of all grid points which are contained in S.

Let $h > 0$ be the picture resolution (i.e., the number of grid points per unit). Instead of considering a digitization of S in a picture of resolution h, we prefer here (as standard in number theory) to use a digitization of the dilated set $h \cdot S = \{(h \cdot x, h \cdot y) \mid (x, y) \in S\}$ in the grid of resolution $h = 1$. We consider $G(h \cdot S)$ to be (under number-theoretical aspects) the *shape S digitized in a binary picture of resolution h*. Gauss digitization is defined analogously in 3D. If $S \subset \mathbb{R}^3$, the Gauss digitization $G(S)$ is the set of all 3D grid points contained in S.

The exact value of $m_{p,q}(S)$ remains unknown in digital imaging (because the exact Euclidean shape of S remains unknown). The following estimation is used:

$$m_{p,q}(S) = \frac{1}{h^{p+q+2}} \cdot \iint\limits_{h \cdot S} x^p y^q \, dx \, dy \approx \frac{1}{h^{p+q+2}} \cdot \sum_{(i,j) \in G(h \cdot S)} i^p \cdot j^q \qquad (1)$$

For a given *digital planar shape* A (i.e., a finite subset of $\mathbb{Z}^2$) and non-negative integers p and q, define the *discrete moment* $\mu_{p,q}(A)$ as follows:

$$\mu_{p,q}(A) \quad = \quad \sum_{(i,j) \in A \cap \mathbb{Z}^2} i^p \cdot j^q$$

3D discrete moments are defined analogously. For a finite set $B \subset \mathbb{Z}^3$ and non-negative integers p, q and t, we have

$$\mu_{p,q,t}(B) \quad = \quad \sum_{(i,j,k)\in B\cap\mathbb{Z}^3} i^p \cdot j^q \cdot k^t$$

Let $\mathcal{C}(S)$ denote the content of set S, which is the area $\mathcal{A}(S)$ for 2D, or the volume $\mathcal{V}(S)$ for 3D. We have $\mu_{0,0}(A) = \mathcal{A}(S)$ and $\mu_{0,0,0}(B) = \mathcal{V}(S)$, and both values are simply defined by cardinalities $\#A$ and $\#B$, respectively. The orders of $\mu_{p,q}(A)$ or $\mu_{p,q,t}(B)$ are $p+q$ and $p+q+t$, respectively. Throughout the paper we assume that all pixels (i.e., grid points) have nonnegative coordinates (i.e., the origin of the assumed coordinate system is at the lower left corner of a considered picture).

Under these assumptions, for a real shape S, $\mu_{p,q}(G(S))$ equals the number of integer points inside of the $3D$-body $B_{p,q}(G)$ defined as

$$B_{p,q}(S) \;=\; \{(x,y,z) \;:\; (x,y) \in S \;\wedge\; 0 < z \leq x^p \cdot y^q\} \tag{2}$$

In other words,

$$\mu_{p,q}(G(S)) \;=\; \#\left(B_{p,q}(S) \cap \mathbb{Z}^3\right) \tag{3}$$

This paper is about an analysis of the maximum error in the approximation $m_{p,q}(S) \approx h^{-(p+q+2)} \cdot \mu_{p,q}(G(h \cdot S))$, when real moments are estimated by corresponding discrete moments. Obviously, this problem is equivalent [see Equation (2)] to the study of the order of magnitude of

$$|m_{p,q}(h \cdot S) - \mu_{p,q}(G(h \cdot S))| \tag{4}$$

This paper deals with planar convex shapes, but due to the given moment definition the result can easily be extended to sets which are unions, intersections or set differences of a finite number of convex sets. Also, since the estimate of (4) becomes trivial if there are any straight sections on the frontier of S, we focus on shapes that have a strictly positive curvature at all points of their frontier. Precise (formal) conditions are given below.

2 Related Results

The number of grid points, contained in convex bodies, is intensively studied in number theory. Regarding (4), a direct application of Davenport's result in [8] (to our case) says that $|m_{p,q}(h \cdot S) - \mu_{p,q}(h \cdot S)|$ is upper bounded by the total sum of projections of $B_{p,q}(h \cdot S)$ onto xy-, xz-, and yz-plane, onton x-, y-, and z-axis, and finally increased by 1. In other words, we have

$$\begin{aligned} |m_{p,q}(h \cdot S) - \mu_{p,q}(h \cdot S)| \;&=\; \left|m_{p,q}(h \cdot S) - \#\left(B_{p,q}(h \cdot S) \cap \mathbb{Z}^3\right)\right| \\ \leq\; &\left(\frac{x_{max}^{p+1} \cdot y_{max}^{q}}{p+1} + \frac{x_{max}^{p} \cdot y_{max}^{q+1}}{q+1}\right) \cdot h^{p+q+1} \;+\; h^2 \cdot x_{max} \cdot y_{max} \\ &+ x_{max}^{p} \cdot y_{max}^{q} \cdot h^{p+q} \;+\; (x_{max} + y_{max}) \cdot h + 1 \end{aligned} \tag{5}$$

A better estimate than (5) is derived in [7] for bounded orders $p+q$. This paper shows that exploiting Huxley's result in [9] allows to obtain an estimate for (4) which improves estimate (5) even for orders of unbounded values of p and q.

We assume that frontiers γ of convex shapes S are composed of finitely many *smooth arcs* γ_i, either given by an equation $y = \phi(x)$, or by $x = \theta(y)$, functions $\phi(x)$ and $\theta(y)$ have at least continuous derivatives up to the third order, also satisfying the following (for $\psi = \phi$ or $\psi = \theta$):

(i) The radius ρ of curvature and its derivative $\dfrac{d\rho}{d\psi}$ exist on each arc γ_i, and both are continuous functions of ψ on γ_i.
(ii) On each arc γ_i, the radius of curvature ρ has a maximum value and a non-zero minimum value.
(iii) On each arc γ_i, the radius of curvature has a bounded number of local maxima and minima.

The following theorem is of major importance for this paper.

Theorem 1. (Huxley 2003). *Suppose that γ consists of finitely many smooth arcs, each of which satisfies conditions (i), (ii), and (iii). Then there is a constant c, calculated from the arcs γ_i of γ (where c is independent of the chosen length unit), such that, if the minimum radius of curvature of each γ_j is at last c, then the number of grid points in S is upper bounded by*

$$\mathcal{A}(S) + \mathcal{O}\left(R^{\frac{131}{208}} \cdot (\log R)^{\frac{18627}{8320}}\right)$$

where R is the maximum radius of curvature of γ. The constant implied in the order of magnitude notation is also calculated from the arcs of γ, and it is independent of the chosen length unit.

A planar convex set S, satisfying the preconditions of Theorem 1, is said to have a *sufficiently smooth frontier*. A direct consequence of Theorem 1 is the following:

Corollary 1. *Let S be a planar convex set with a sufficiently smooth frontier. Then it follows that*

$$\#G(h \cdot S) = h^2 \cdot \mathcal{A}(S) + \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\right) \tag{6}$$

for any $\varepsilon > 0$.

This is a very strong result. It even improves the previously best known upper bound for the circle problem (i.e., if S is assumed to be a circle).

The following studies are divided into two different cases. The case where either p or q is zero, is studied in the next section. The case where both p and q are strictly positive, is studied in Section 4.

3 Error Estimate if Either $p = 0$ or $q = 0$

Obviously (due to symmetry), estimates for $\mu_{p,0}(h \cdot S)$ and $\mu_{0,q}(h \cdot S)$ can be derived in identical ways. We consider $\mu_{p,0}(h \cdot S)$.

For a compact set S, let $x_{min} = \min\{x : (x, y) \in S\}$, $x_{max} = \max\{x : (x, y) \in S\}$, $y_{min} = \min\{y : (x, y) \in S\}$, and $y_{max} = \max\{y : (x, y) \in S\}$.

Without loss of generality we can assume that the studied convex set S is a subset of $[0,1]\times[0,1]$. Consequently, we have $\{x_{min},\ x_{max},\ y_{min},\ y_{max}\}\subset[0,1]$ in what follows.

Definition 1. *For a planar set S, integer k, and real $h>0$, let*

$$(h\cdot S)(k)=\{(x,y)\ :\ (x,y)\in(h\cdot S)\ \wedge\ x\geq k\}$$

Consequently, $G((h\cdot S)(k))$ is the set of grid points in the digitization of $h\cdot S$ lying in the closed half plane determined by $x\geq k$.

Definition 2. *For a planar set S, integer k, and real $h>0$, let*

$$L(h\cdot S,k)=\{(k,j)\ :\ (k,j)\in G(h\cdot S)\}.$$

In other words, $L(h\cdot S,k)$ is the set of those grid points in the Gauss digitization of $h\cdot S$ that belong to the line $x=k$. We have the following lemma [7].

Lemma 1. *Let S be a planar convex set and k an integer. We have*

$$\#G((h\cdot S)(k))\ =\ \mathcal{A}((h\cdot S)(k))\ +\ \frac{1}{2}\cdot\#L(h\cdot S,k)\ +\ \mathcal{O}(h^{\frac{131}{208}+\varepsilon})$$

We use the following definitions of 3D-sets W_i and W_i':

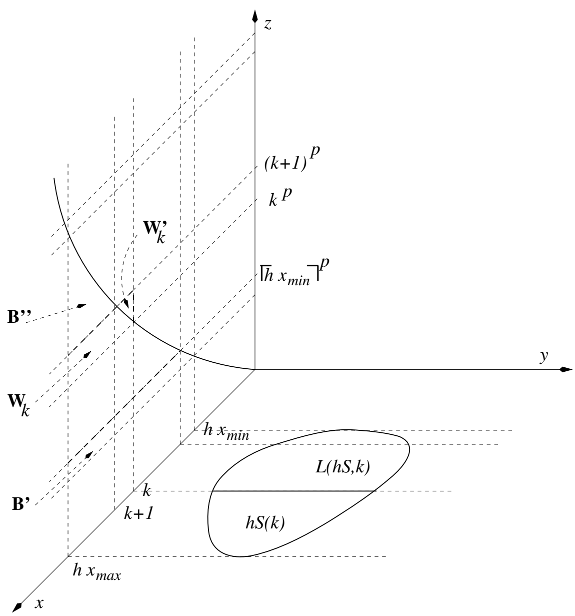

Fig. 1. Used notations in this section

Definition 3. *For planar convex set S and integer $i \in \{\lceil h \cdot x_{min} \rceil, \lceil h \cdot x_{min} \rceil + 1, \ldots, \lfloor h \cdot x_{max} \rfloor - 1\}$, we define 3D sets (see Figure 1)*

$$W_i = \{(x, y, z) \; : \; (x, y) \in h \cdot S \;\wedge\; x \geq i \;\wedge\; i^p < z \leq (i+1)^p\}$$

and

$$W_i' = \{(x, y, z) \; : \; (x, y) \in h \cdot S \;\wedge\; x \geq i \;\wedge\; x^p < z \leq (i+1)^p\}$$

Now we calculate $\mu_{p,0}(h \cdot S)$. As a reminder, $\mathcal{V}(B)$ is the volume of a 3D set B, and $\mathcal{A}(S)$ is the area of a $2D$ set S.

Lemma 2. *Let S be a convex set. Then*

$$\sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \mathcal{V}(W_i')$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \#L(h \cdot S, i) \cdot \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1} \right) + \mathcal{O}\left(\frac{h^p}{p+1} \cdot \binom{p+1}{\lceil \frac{p+1}{2} \rceil} \right)$$

Proof. The frontier of $h \cdot S$ can be divided into two arcs of the form $y = y_1(x)$ and $y = y_2(x)$, such that $y_1(x) \leq y_2(x)$. Then we have that

$$\sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \mathcal{V}(W_i') = \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{x^p}^{(i+1)^p} dz \int_{y_1(x)}^{y_2(x)} dy$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{x^p}^{(i+1)^p} dz \left(\int_{y_1(x)}^{y_1(i)} dy \; + \; \int_{y_1(i)}^{\lceil y_1(i) \rceil} dy \; + \; \int_{\lceil y_1(i) \rceil}^{\lfloor y_2(i) \rfloor} dy \right.$$

$$\left. + \quad \int_{\lfloor y_2(i) \rfloor}^{y_2(i)} dy \quad + \quad \int_{y_2(i)}^{y_2(x)} dy \right)$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \int_i^{i+1} dx \int_{x^p}^{(i+1)^p} \left(\int_{\lceil y_1(i) \rceil}^{\lfloor y_2(i) \rfloor} dy \quad + \quad \mathcal{O}(1) \right) dz \; + \; \mathcal{O}(h^p)$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \int_i^{i+1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \cdot ((i+1)^p - x^p) dx \quad + \quad \mathcal{O}(h^p)$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \cdot \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1} \right) \; +$$

$$+ \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} (\lfloor y_2(i) \rfloor - \lceil y_1(i) \rceil) \left(i^p + \frac{p}{2} \cdot i^{p-1} - \frac{(i+1)^{p+1} - i^{p+1}}{p+1} \right) + \mathcal{O}(h^p)$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \#L(h \cdot S, i) \cdot \left((i+1)^p - i^p - \frac{p}{2} \cdot i^{p-1} \right) + \mathcal{O}\left(\frac{h^p}{p+1} \cdot \binom{p+1}{\lceil \frac{p+1}{2} \rceil} \right)$$

The following estimate was used:

$$\frac{(i+1)^{p+1} - i^{p+1}}{p+1} - i^p - \frac{p}{2} \cdot i^{p-1}$$
$$= \frac{1}{p+1} \cdot \left(\binom{p+1}{3} \cdot i^{p-2} + \binom{p+1}{4} \cdot i^{p-3} + \ldots + \binom{p+1}{p+1} \cdot i^0 \right)$$
$$\leq \frac{p-1}{p+1} \cdot \binom{p+1}{\lceil \frac{p+1}{2} \rceil} \cdot i^{p-2}$$

□

Finally, Lemma 3 evaluates the discrete moments $\mu_{p,0}(h \cdot S)$ and $\mu_{0,q}(h \cdot S)$.

Lemma 3. *The following asymptotic expressions are satisfied:*

$$\mu_{p,0}(h \cdot S) = \sum_{(i,j) \in G(h \cdot S)} i^p \;=\; \iint_{h \cdot S} x^p dx dy \;+\; \mathcal{O}\left(h^p \cdot \left(\binom{p}{\lceil \frac{p}{2} \rceil} \;+\; h^{\frac{131}{208}+\varepsilon} \right) \right)$$

$$\mu_{0,q}(h \cdot S) = \sum_{(i,j) \in G(h \cdot S)} j^q \;=\; \iint_{h \cdot S} y^q dx dy \;+\; \mathcal{O}\left(h^q \cdot \left(\binom{q}{\lceil \frac{q}{2} \rceil} \;+\; h^{\frac{131}{208}+\varepsilon} \right) \right)$$

Proof. According to (3), $\mu_{p,0}(G(h \cdot S))$ equals the number of grid points belonging to the 3D set B given by

$$B \;=\; \{(x,y,z) \;:\; (x,y) \in h \cdot S \;\wedge\; 0 < z \leq x^p\} \;=\; B' \;\cup\; B''$$

where B' and B'' are defined as follows:

$$B' = \{(x,y,z) \;:\; (x,y) \in h \cdot S \;\wedge\; 0 < z \leq \lceil h \cdot x_{min} \rceil^p\}$$
$$B'' = \{(x,y,z) \;:\; (x,y) \in h \cdot S \;\wedge\; \lceil h \cdot x_{min} \rceil^p < z \leq x^p\}$$

First, consider the number of grid points which belong to B'. It follows that

$$\#G(B') \;=\; \lceil h \cdot x_{min} \rceil^p \cdot \left(\mathcal{A}(h \cdot S) + \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon} \right) \right) \;=\; \mathcal{V}(B') + \mathcal{O}\left(h^{p+\frac{131}{208}+\varepsilon} \right)$$

Now we calculate the number of grid points which belong to B''. By Definition 3 and also using the (obvious) estimate

$$\mathcal{V}\left(\{\; (x,y,z) \;:\; (x,y) \in h \cdot S \;\wedge\; x \geq \lfloor h \cdot x_{max} \rfloor \;\wedge\; z \leq x^p \;\}\right) \;=\; \mathcal{O}(h^p)$$

we derive

$$\mathcal{V}(B'') = \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \left(\mathcal{V}(W_i) - \mathcal{V}(W_i') \right) + \mathcal{O}(h^p)$$

$$= \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \mathcal{V}(W_i) - \sum_{i=\lceil h \cdot x_{min} \rceil}^{\lfloor h \cdot x_{max} \rfloor - 1} \mathcal{V}(W_i') + \mathcal{O}(h^p)$$

$$= \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot \mathcal{A}((h\cdot S)(i)) - \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} \mathcal{V}(W_i') + \mathcal{O}(h^p)$$

(by using Lemata 1 and 2, it follows)

$$= \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot \left(\#G((h\cdot S)(i)) - \frac{1}{2}\cdot \#L(h\cdot S, i) + \mathcal{O}(h^{\frac{131}{208}+\varepsilon})\right)$$

$$- \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} \#L(h\cdot S, i)\left((i+1)^p - i^p - \frac{p}{2}\cdot i^{p-1}\right) \;+\; \mathcal{O}\left(\frac{h^p}{p+1}\cdot \binom{p+1}{\lceil\frac{p+1}{2}\rceil}\right)$$

$$= \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot \left(\#G((h\cdot S)(i)) - \#L(h\cdot S, i) + \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\right)\right)$$

$$- \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} \frac{\#L(h\cdot S, i)}{2}\cdot \left((i+1)^p - i^p - p\cdot i^{p-1}\right) + \mathcal{O}\left(\frac{h^p}{p+1}\cdot \binom{p+1}{\lceil\frac{p+1}{2}\rceil}\right)$$

$$= \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot (\#G((h\cdot S)(i)) - \#L(h\cdot S, i))$$

$$+\mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\cdot ((\lfloor h\cdot x_{max}\rfloor)^p \;-\; (\lceil h\cdot x_{min}\rceil)^p)\right)$$

$$- \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} \frac{1}{2}\cdot \#L(h\cdot S, i)\cdot \left(\binom{p}{2}\cdot i^{p-2} + \binom{p}{3}\cdot i^{p-3} + \ldots + \binom{p}{p}\cdot i^0\right)$$

$$+\mathcal{O}\left(\frac{1}{p+1}\cdot \binom{p+1}{\lceil\frac{p+1}{2}\rceil}\cdot h^p\right) \;=\; \#G(B'') + \mathcal{O}\left(h^p\cdot \left(\binom{p}{\lceil\frac{p}{2}\rceil} \;+\; h^{\frac{131}{208}+\varepsilon}\right)\right)$$

The following inequalities are used:

a)
$$\sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} \left(\binom{p}{2}\cdot i^{p-2} + \binom{p}{3}\cdot i^{p-3} + \ldots + \binom{p}{p}\cdot i^0\right) \;\le$$

$$\le \sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} (p-1)\cdot \binom{p}{\lceil\frac{p}{2}\rceil}\cdot i^{p-2} \;=\; \mathcal{O}\left(\binom{p}{\lceil\frac{p}{2}\rceil}\cdot h^{p-1}\right)$$

b) for a large p :
$$\binom{p}{\lceil\frac{p}{2}\rceil}\cdot h^{p-1} \le \frac{1}{p+1}\cdot \binom{p+1}{\lceil\frac{p+1}{2}\rceil}\cdot h^{p-1}$$

Note that, if an integer i with $h\cdot x_{min} \le i \le h\cdot x_{max}$ is fixed, then

$$((i+1)^p - i^p)\cdot (\#G((h\cdot S)(i)) - \#L(h\cdot S, i))$$

equals the number of grid points contained in W_i, and, consequently,

$$\sum_{i=\lceil h\cdot x_{min}\rceil}^{\lfloor h\cdot x_{max}\rfloor -1} ((i+1)^p - i^p)\cdot(\#G((h\cdot S)(i)) - \#L(h\cdot S, i)))$$

equals the number of grid points contained in B''.

Finally, the sum of $\#G(B')$ and $\#G(B'')$ is the number of grid points in B. Together with the already derived expression for $\#G(B')$, we have

$$\mu_{p,0}(G(h\cdot S)) = \#G(B) = \#G(B') + \#G(B'') \;=\; \mathcal{V}(B') + \mathcal{O}\left(h^p\cdot h^{\frac{131}{208}+\varepsilon}\right)$$
$$+\mathcal{V}(B'') + \mathcal{O}\left(h^p\cdot\left(\binom{p}{\lceil\frac{p}{2}\rceil} + h^{\frac{131}{208}+\varepsilon}\right)\right) \;=\; \mathcal{V}(B) \;+$$
$$\mathcal{O}\left(h^p\cdot\left(\binom{p}{\lceil\frac{p}{2}\rceil} + h^{\frac{131}{208}+\varepsilon}\right)\right) = m_{p,0}(h\cdot S) + \mathcal{O}\left(h^p\cdot\left(\binom{p}{\lceil\frac{p}{2}\rceil} + h^{\frac{131}{208}+\varepsilon}\right)\right) \;\square$$

4 Error Estimate if $p > 0$ and $q > 0$

It remains to estimate $\mu_{p,q}(h\cdot S)$, if $p > 0$ and $q > 0$. (The next definition and lemma are analogous to Definition 1 and Lemma 1.)

Definition 4. *For a convex set S, integers k, p, q, and a real $r > 0$, let*

$$(h\cdot S)(k,p,q) = \{(x,y) \;:\; (x,y)\in(h\cdot S) \;\wedge\; x^p\cdot y^q \geq k\}$$

$G((h\cdot S)(k,p,q))$ is the set of grid points in the digitization of $h\cdot S$ lying in the closed part of the plane determined by $x^p\cdot y^q \geq k$. Since both S and $(h\cdot S)(k,p,q)$ satisfy the preconditions of Theorem 1, we have the following lemma:

Lemma 4. *For a convex set S with a sufficiently smooth frontier, and integers r, p, q, we have*

$$\#G((h\cdot S)(p,q,k)) \;=\; \mathcal{A}((h\cdot S)(k)) \;+\; \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\right) \tag{7}$$

Lemma 5. *Let S be a convex set with a sufficiently smooth frontier, and p, $q > 0$. Then we have the following:*

$$\mu_{p,q}(h\cdot S) = \iint\limits_{h\cdot S} x^p\cdot y^q dxdy + \mathcal{O}\left(h^{p+q}\cdot h^{\frac{131}{208}+\varepsilon}\right) \tag{8}$$

Proof. Note that $\mu_{p,q}(h\cdot S)$ is equal to the number of grid points belonging to the 3D set E given by

$$E = \{(x,y,z) \;:\; (x,y)\in h\cdot S \;\wedge\; 0 < z \leq x^p\cdot y^q\} \;=\; E' \;\cup\; E''$$

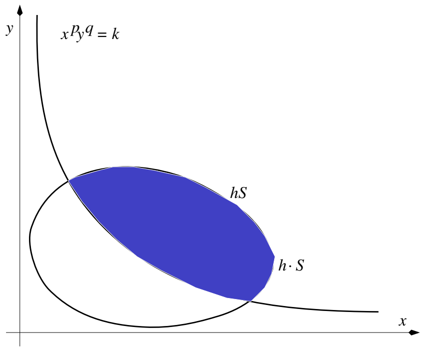

Fig. 2. The shaded area is $(h \cdot S)(k, p, q)$

where E' and E'' are defined as follows:

$$E' = \{(x, y, z) \; : \; (x, y) \in h \cdot S \; \wedge \; 0 < z < h^{p+q} \cdot z_{min}\}$$
$$E'' = \{(x, y, z) \; : \; (x, y) \in h \cdot S \; \wedge \; h^{p+q} \cdot z_{min} \leq z \leq x^p \cdot y^q\}$$

where $z_{min} = \min\{z : z = x^p \cdot y^q \; \wedge \; (x, y) \in S\}$ and $z_{max} = \max\{z : z = x^p \cdot y^q \; \wedge \; (x, y) \in S\}$.

Furthermore, from (9) we have

$$\#G(E') = (\lceil h^{p+q} \cdot z_{min} \rceil - 1) \cdot \left(\mathcal{A}(h \cdot S) + \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\right)\right) = \mathcal{V}(E')$$
$$- \;\; h^{p+q} \cdot z_{min} \cdot \mathcal{A}(h \cdot S) + (\lceil h^{p+q} \cdot z_{min} \rceil - 1) \cdot \left(\mathcal{A}(h \cdot S) + \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\right)\right)$$
$$= \;\; \mathcal{V}(E') + \mathcal{A}(h \cdot S) \cdot \left(\lceil h^{p+q} \cdot z_{min} \rceil - h^{p+q} \cdot z_{min}\right) + \mathcal{O}\left(h^{p+q} \cdot h^{\frac{131}{208}+\varepsilon}\right)$$

(Note that $\mathcal{A}(h \cdot S) = \mathcal{O}(h^2)$ and $p + q \geq 2$ have been used in this derivation.)

Now, let us calculate the number of grid points belonging to the set E''. What follows is a definition of 3D-sets ω_i and ω_i', for $i \in \{\lceil h^{p+q} \cdot x_{min} \rceil, \lceil h^{p+q} \cdot x_{min} \rceil + 1, \ldots, \lfloor h^{p+q} \cdot x_{max} \rfloor\}$:

$$\omega_i = \{(x, y, z) | (x, y) \in h \cdot S \; \wedge \; x^p \cdot y^q \geq i \; \wedge \; i < z < \min\{x^p \cdot y^q, i + 1\}\}$$
$$\omega_i' = \{(x, y, z) | (x, y) \in h \cdot S \; \wedge \; i < x^p \cdot y^q \leq i + 1 \; \wedge \; x^p \cdot y^q < z < i + 1\}$$

Now, we can estimate the volume of E''. By using $\mathcal{O}(h^2)$ as a trivial upper bound for the volume of

$$\{(x, y, z) \; : \; (x, y) \in h \cdot S \wedge x^p \cdot y^q \leq \lceil h^{p+q} \cdot z_{min} \rceil \; \wedge \; x^p \cdot y^q \leq z \leq \lceil h^{p+q} \cdot z_{min} \rceil\}$$

it follows that

$$
\begin{aligned}
&\mathcal{V}(E'') \\
&= \sum_{i=\lceil h^{p+q}\cdot z_{min}\rceil}^{\lfloor h^{p+q}\cdot z_{max}\rfloor} \mathcal{V}(\omega_i) \;+\; \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) + \mathcal{O}(h^2) \\
&= \sum_{i=\lceil h^{p+q}\cdot z_{min}\rceil}^{\lfloor h^{p+q}\cdot z_{max}\rfloor} \left(\mathcal{A}((h\cdot S)(i,p,q)) - \mathcal{V}(\omega_i')\right) \\
&\qquad\qquad + \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) \;+\; \mathcal{O}(h^2) \\
&= \sum_{i=\lceil h^{p+q}\cdot z_{min}\rceil}^{\lfloor h^{p+q}\cdot z_{max}\rfloor} \mathcal{A}((h\cdot S)(i,p,q)) \quad - \sum_{i=\lceil h^{p+q}\cdot z_{min}\rceil}^{\lfloor h^{p+q}\cdot z_{max}\rfloor} \mathcal{V}(\omega_i') \\
&\qquad\qquad + \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) \quad + \quad \mathcal{O}(h^2) \\
&\quad \Bigg(\text{note that } \sum_{i=\lceil h^{p+q}\cdot z_{min}\rceil}^{\lfloor h^{p+q}\cdot z_{max}\rfloor} \mathcal{V}(\omega_i') \le h^2\cdot \mathcal{A}(S) \quad \text{because the projections of} \\
&\qquad\quad \omega_i' \text{ onto the xy-plane belong to } h\cdot S\Bigg) \\
&= \sum_{i=\lceil h^{p+q}\cdot z_{min}\rceil}^{\lfloor h^{p+q}\cdot z_{max}\rfloor} \left(\#G((h\cdot S)(i,p,q)) \;+\; \mathcal{O}\left(h^{\frac{131}{208}+\varepsilon}\right)\right) \\
&\qquad\qquad + \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) + \mathcal{O}(h^2) \\
&= \#G(E'') + \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) + \mathcal{O}\left(h^{p+q+\frac{131}{208}+\varepsilon}\right).
\end{aligned}
$$

Thus,

$$
\#G(E'') = \mathcal{V}(E'') - \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) + \mathcal{O}\left(h^{p+q+\frac{131}{208}+\varepsilon}\right).
$$

The proof of the lemma is finished by summing up $\#G(E')$ and $\#G(E'')$:

$$
\begin{aligned}
&\mu_{p,q}(h\cdot S) \quad = \quad \#G(E') \quad + \quad \#G(E'') \\
&= \mathcal{V}(E') + \mathcal{A}(h\cdot S)\cdot \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right) + \mathcal{O}\left(h^{p+q+\frac{131}{208}+\varepsilon}\right) \\
&\quad + \mathcal{V}(E'') - \left(\lceil h^{p+q}\cdot z_{min}\rceil - h^{p+q}\cdot z_{min}\right)\cdot \mathcal{A}(h\cdot S) + \mathcal{O}\left(h^{p+q+\frac{131}{208}+\varepsilon}\right) \\
&= \mathcal{V}(E) \quad + \quad \mathcal{O}\left(h^{p+q+\frac{131}{208}+\varepsilon}\right) \quad = \quad m_{p,q}(h\cdot S) + \mathcal{O}\left(h^{p+q+\frac{131}{208}+\varepsilon}\right). \qquad \Box
\end{aligned}
$$

Our theorem summarizes the accuracy in estimating real moments of an arbitrary order based on digitized sets.

Theorem 2. *Let S be a convex set that satisfies the preconditions of Theorem 1. Then we have the following:*

$$\left| m_{p,q}(S) - \frac{\mu_{p,q}(h \cdot S)}{h^{p+q+2}} \right| = \begin{cases} \mathcal{O}\left(h^{-\frac{285}{208}+\varepsilon} + \frac{1}{h^2} \cdot \binom{p}{\lceil \frac{p}{2} \rceil}\right) & \textit{for } p = 0 \textit{ or } q = 0 \\ \mathcal{O}\left(h^{-\frac{285}{208}+\varepsilon}\right) & \textit{for } p > 0 \textit{ and } q > 0. \end{cases}$$

Stirling's formula gives $\binom{p}{\lceil \frac{p}{2} \rceil} = \mathcal{O}(2^n)$ and implies the following:

Corollary 2. *Let S be a convex set with sufficiently smooth frontier, and let $p + q = o(\log h)$. Then we have the following:*

$$\left| m_{p,q}(S) - \frac{\mu_{p,q}(h \cdot S)}{h^{p+q+2}} \right| = \mathcal{O}\left(h^{-\frac{285}{208}+\varepsilon}\right) \qquad \textit{for any} \quad \varepsilon > 0.$$

Corollary 2 shows that the error in approximating $m_{p,q}(S) \approx \frac{\mu_{p,q}(h \cdot S)}{h^{p+q+2}}$ can be reduced to any fraction of the pixel size (what is $1/h$) if a moment's order $p + q$ is not to large compared to the applied picture resolution. The assumed relation $p + q = o(\log h)$ is reasonable for practical applications. In such a case, Corollary 2 gives a better estimate than the estimate $\frac{1}{c^{o(\log h)} \cdot o(h \cdot \log h)}$ ($c > 0$ is computable from x_{max} and y_{max}) that follows from (5) (i.e., from Davenport's result).

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