Minimal Non-simple and Minimal Non-cosimple Sets in Binary Images on Cell Complexes

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Abstract. The concepts of *weak component* and *simple* 1 are generalizations, to binary images on the n-cells of n-dimensional cell complexes, of the standard concepts of "26-component" and "26-simple" 1 in binary images on the 3-cells of a 3D cubical complex; the concepts of *strong component* and *cosimple* 1 are generalizations of the concepts of "6-component" and "6-simple" 1. Over the past 20 years, the problems of determining just which sets of 1's can be minimal non-simple, just which sets can be minimal non-cosimple, and just which sets can be minimal non-simple (minimal non-cosimple) without being a weak (strong) foreground component have been solved for the 2D cubical and hexagonal, 3D cubical and face-centered-cubical, and 4D cubical complexes. This paper solves these problems in much greater generality, for a very large class of cell complexes of dimension ≤ 4.

1 Introduction

In a binary image, the n-dimensional cells of an n-dimensional cell complex (most often, the 2D or 3D cubical complex) are labeled 1 or 0. Cells labeled 1 are referred to as 1's of the image, and cells labeled 0 are referred to as 0's.

We say that a 1 of the image is *simple* if "the topology of image is preserved" (in a sense which will be made precise in Sect. 4) when that 1 is changed into a 0. We say that a 1 is *cosimple* if the topology of the image is preserved in another, complementary, sense when the 1 is changed into a 0.

In the case of the 2D cubical complex, these are two of the oldest concepts of digital topology, and date back to the 1960's. Rosenfeld's concept of an "8-deletable" pixel in [20] is mathematically equivalent to our concept of a simple 1 in a binary image on the 2D cubical complex. The concept of a "4-deletable" pixel in [20] is similarly equivalent to our concept of a cosimple 1. Today, simple and cosimple 1's in binary images on the 2D cubical complex are often called "8-simple" and "4-simple", respectively. In binary images on the 3D cubical complex, simple 1's are often called "26-simple", cosimple 1's are often called "6-simple", and a number of non-trivial characterizations of such 1's have been published (e.g., in [2,21]).

A subset of the set of 1's of a binary image is said to be *simple* (*cosimple*) if the elements of that subset can be arranged in a sequence $D_1, \ldots, D_k$ in which each element D_i is simple (cosimple) after its predecessors $D_1, \ldots, D_{i-1}$ have all

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been changed to 0's. Such sequences were apparently first studied by Ronse [18] in the 1980's, in the case of binary images on the 2D cubical complex.

A subset $\mathcal{S}$ of the set of 1's is said to be *minimal non-simple* or *MNS* (*minimal non-cosimple* or *MNCS*) if $\mathcal{S}$ is non-simple (non-cosimple) but every proper subset of $\mathcal{S}$ is simple (cosimple). MNS and MNCS sets were first introduced by Ronse [19], for the 2D cubical complex. (In that context, Ronse referred to MNS sets as "8-MND sets" and referred to MNCS sets as "4-MND sets"; MND stood for "minimal non-deletable".)

The principal application of the concepts of simple and cosimple sets of 1's is to the theory of parallel thinning algorithms for binary images. Each iteration of such an algorithm deletes (i.e., changes to 0) all 1's for which the configuration of nearby 1's and 0's satisfies the algorithm's deletion condition. Thinning algorithms are expected to "preserve topology" in the sense that the set of 1's deleted by the algorithm should always be simple or always be cosimple.

The concepts of MNS and MNCS sets provide the basis for a systematic method of verifying that a proposed parallel thinning algorithm satisfies either of these conditions. In the types of cell complex which seem most likely to be used in applications, only a few kinds of set can ever be MNS or MNCS, and such sets can have only a few elements. (For example, in the case of the 2D cubical complex Ronse showed in [19] that a set of 1's can be MNS only if every pair of those 1's are 8-adjacent—which implies that no MNS set can contain more than four 1's.) If we can deduce from a given parallel thinning algorithm's deletion condition that the set of 1's which are changed to 0's at a single iteration can never include a non-simple (non-cosimple) set of one of the kinds that can be MNS (MNCS), then we will have proved that the set of 1's that are changed to 0's at any iteration of the algorithm is always a simple (cosimple) set, so that the thinning algorithm does indeed "preserve topology" in the corresponding sense.

It can happen that a certain kind of set can be MNS (MNCS), but *only* in the very special case where the set is a weak (strong) component of the 1's. (Here the concepts of weak and strong components are generalizations, to sets of n-dimensional cells of nD cell complexes, of the well known concepts of 8- and 4-components, respectively, in sets of 2-cells of the 2D cubical complex.) For example, in the case of the 2D cubical complex Ronse showed in [19] that a set of two 1's that are 8-adjacent but not 4-adjacent can be MNS only if it is an 8-component of the 1's (i.e., only if neither of the 1's is 8-adjacent to any other 1 of the image). Knowing that sets of certain kinds cannot be MNS (MNCS) unless they are weak (strong) components of the 1's can considerably simplify the application of the verification method described above.

This motivates the problem of determining just which kinds of set can be MNS, just which kinds can be MNCS, and just which kinds can be MNS (MNCS) without being a weak (strong) component of the 1's. Ronse [19] solved these problems for the 2D cubical complex. Hall [6, Sect. 4] essentially solved the problems for the 2D hexagonal complex. The problems were solved for the 3D cubical complex by Ma [15] and Kong [10]. Gau and Kong [4] solved the problems for the 3D face-centered-cubical complex (whose 3-dimensional cells are rhombic dodecahedra) and, more recently, for the 4D cubical complex [5,11].

In this paper, we solve these problems for a very general class of cell complexes of dimension ≤ 4, namely the *xel-complexes* which we define in Sect. 3. The cubical, 2D hexagonal, and 3D face-centered-cubical complexes mentioned above, and most other complexes that have been considered in digital topology (such as the 3D body-centered-cubical complex [7,14], whose 3-dimensional cells are truncated octahedra), are simple examples of xel-complexes.

2 Contractibility, Polyhedra, and Polyhedral Cells

A set S in $\mathbb{R}^n$ is said to be *contractible* if S is nonempty, and S can be continuously deformed over itself to some point p in S. More precisely, S is contractible if and only if $S \neq \emptyset$ and there is a continuous mapping $h : S \times [0,1] \to S$ such that, for every point $s \in S$ and some point p in S, $h(s,0) = s$ and $h(s,1) = p$. A contractible set is necessarily connected.

Every convex set is contractible. More generally, if $\mathcal{P}$ is any nonempty collection of convex sets such that $\bigcap \mathcal{P} \neq \emptyset$, then $\bigcup \mathcal{P}$ is contractible—because if p is any point in $\bigcap \mathcal{P}$ then the map $h : \bigcup \mathcal{P} \times [0,1] \to \bigcup \mathcal{P}$ that is defined by $h(s,t) = tp + (1-t)s$ has the above-mentioned properties.

On the other hand, it is an easy consequence of basic results of algebraic topology that the boundary of a k-simplex—i.e., the set of all points that lie on one or more of the $(k-1)$-dimensional faces of the k-simplex—is not contractible.

In this paper a set in $\mathbb{R}^n$ is called a *polyhedron* if it is expressible as a union of a finite collection of simplexes (which may possibly include simplexes of different dimensions). Note that the empty set is a polyhedron, and that a polyhedron need not be connected. Evidently, the union of two polyhedra is a polyhedron. It is also not hard to prove that the intersection of two polyhedra is a polyhedron.

There is a simple characterization of contractible polyhedra in $\mathbb{R}^3$: A polyhedron P in $\mathbb{R}^3$ is contractible if and only if P is nonempty, connected, and simply connected, and $\mathbb{R}^3 \setminus P$ is connected. This characterization follows from well known results of algebraic topology—the Alexander duality theorem, and the theorems of Whitehead and Hurewicz [16, Chs. 5, 7, and 8].

For any integer $k \geq 0$, a *polyhedral k-cell* is a polyhedron that is homeomorphic to a k-simplex. A *polyhedral cell* is a set that is a polyhedral k-cell for some integer k; the integer k (which is always uniquely determined) is the *dimension* of the polyhedral cell. The dimension of a polyhedral cell C is denoted by $\dim(C)$. Note that a polyhedral 0-cell consists of just one point. A polyhedral cell is closed and bounded, and is contractible because it is homeomorphic to a simplex (which is a contractible set because it is nonempty and convex).

If C is a polyhedral k-cell, and $h : \sigma \to C$ is a homeomorphism of a k-simplex σ onto C, then the image under h of the boundary of the simplex σ is called the *manifold boundary* or just the *boundary* of C, and is denoted by ∂C. (This set does not depend on our choice of h and σ.) If C is a polyhedral 0-cell then $\partial C = \emptyset$.

3 Xel-Complexes

A *xel-complex* is a collection $\mathbf{K}$ that satisfies the following conditions for some positive integer n, which we call the *dimension* of $\mathbf{K}$ and denote by $\dim(\mathbf{K})$:

1. Each member of $\mathbf{K}$ is a polyhedral k-cell for some $k \leq n$, and $\bigcup \mathbf{K} = \mathbb{R}^n$.
2. No bounded region of $\mathbb{R}^n$ intersects infinitely many members of $\mathbf{K}$.
3. For all distinct $X, Y \in \mathbf{K}$, either $X \subsetneq \partial Y$, or $Y \subsetneq \partial X$, or $X \cap Y = \partial X \cap \partial Y$.
4. For all $X, Y \in \mathbf{K}$, either $X \cap Y = \emptyset$ or $X \cap Y \in \mathbf{K}$.
5. For all $X, Y \in \mathbf{K}$ such that $\emptyset \neq Y \subsetneq X$, $\bigcup\{D \in \mathbf{K} \mid D \subsetneq X \text{ and } D \cap Y = \emptyset\}$ is a contractible polyhedron.
6. For all $X, Y \in \mathbf{K}$ such that $X \cap Y = \emptyset$, there exist $X', Y' \in \mathbf{K}$ such that $\dim(X') = \dim(Y') = n$, $X' \supseteq X$, $Y' \supseteq Y$, and $X' \cap Y' = \emptyset$.

The only places in this paper where we use conditions 5 and 6 are in the proofs of assertion 4 of our first main theorem (Theorem 3) and assertions 3 and 4 of our second main theorem (Theorem 4).

Each member of a xel-complex $\mathbf{K}$ will be called a *xel* of $\mathbf{K}$, and a xel X will be called a *k-xel* if $\dim(X) = k$. An *mD xel-complex* is a xel-complex $\mathbf{K}$ for which $\dim(\mathbf{K}) = m$. The above conditions imply that if X and Y are xels of $\mathbf{K}$ such that $X \subsetneq Y$, then $X \subsetneq \partial Y$; in such cases we say X is a *proper face* of Y. So if C_1 and C_2 are distinct intersecting xels of $\mathbf{K}$ neither of which is a proper face of the other, then $C_1 \cap C_2 = \partial C_1 \cap \partial C_2$ is a proper face of C_1 and of C_2.

A simple and important example of an nD xel-complex is the *nD cubical complex*, whose xels are the Cartesian products $E_1 \times \ldots \times E_n$ in which each set E_i either is a singleton set of the form $\{i + 0.5\}$ for some integer i, or is a closed unit interval $[i - 0.5, i + 0.5]$ for some integer i. Here $E_1 \times \ldots \times E_n$ is a k-xel of the xel-complex if $n - k$ of the n E's are singleton sets and the other k E's are closed unit intervals. (Thus a k-xel of this xel-complex is an upright closed k-dimensional unit (hyper)cube in $\mathbb{R}^n$ whose vertices are located at points each of whose coordinates differs from an integer by exactly 0.5.)

If X and Y are n-xels of an nD xel-complex $\mathbf{K}$, then X is said to be *weakly adjacent* to Y if $X \neq Y$ and $X \cap Y \neq \emptyset$, and X is said to be *strongly adjacent* to Y if $X \cap Y$ is an $(n-1)$-xel of $\mathbf{K}$. If $\mathcal{T}$ is any set of n-xels of $\mathbf{K}$, then each equivalence class of the reflexive transitive closure of the restriction to $\mathcal{T}$ of the "is weakly adjacent to" relation is called a *weakly connected component* of $\mathcal{T}$. Similarly, each equivalence class of the reflexive transitive closure of the restriction to $\mathcal{T}$ of the "is strongly adjacent to" relation is called a *strongly connected component* of $\mathcal{T}$. We say $\mathcal{T}$ is *weakly connected* if $\mathcal{T} = \emptyset$ or if there is just one weakly connected component of $\mathcal{T}$. Similarly, we say $\mathcal{T}$ is *strongly connected* if $\mathcal{T} = \emptyset$ or if there is just one strongly connected component of $\mathcal{T}$. (In the 2D (3D) cubical complex, a set of 2-(3-)xels is strongly connected if and only if it is 4-(6-)connected, and is weakly connected if and only if it is 8-(26-)connected.) Evidently, every strongly connected set is weakly connected.

We now state (without proof) a number of properties of xel-complexes which will be used in proving our main theorems. Readers are encouraged to at least convince themselves that the 2D and 3D cubical complexes have these properties.

Property 1. If X is a xel of a xel-complex $\mathbf{K}$, then ∂X is a union of xels of $\mathbf{K}$. $\square$

Property 2. If X_1 and X_2 are xels of a xel-complex $\mathbf{K}$ such that $X_1 \subsetneq X_2$, then $\dim(X_1) < \dim(X_2)$. $\square$

Property 3. If X is a xel of a xel-complex $\mathbf{K}$, and $\dim(X) > 0$, then ∂X contains at least $\dim(X) + 1$ distinct 0-xels of $\mathbf{K}$. $\square$

Property 4. If Z is an $(n-1)$-xel of an nD xel-complex $\mathbf{K}$, then there are n-xels $X_1, X_2 \in \mathbf{K}$ such that $X_1 \cap X_2 = \partial X_1 \cap \partial X_2 = Z$, and no other xel of $\mathbf{K}$ intersects $Z \setminus \partial Z$. $\square$

Property 5. If X and X' are distinct n-xels of an nD xel-complex $\mathbf{K}$ such that $X \cap X' \neq \emptyset$, then there exists a sequence $X_0, X_1, \ldots, X_k$ of n-xels of $\mathbf{K}$ such that $X_0 = X$, $X_k = X'$, and, for $1 \leq i \leq k$, $X_{i-1} \cap X_i$ is an $(n-1)$-xel of $\mathbf{K}$ that contains $X \cap X'$. $\square$

Property 6. If X and C are xels of a xel-complex $\mathbf{K}$ such that $X \subsetneq \partial C$, then there is a $(\dim(C) - 1)$-xel Y of $\mathbf{K}$ such that $X \subseteq Y \subsetneq \partial C$. $\square$

4 MNS and MNCS Sets in Binary Images

Let $\mathbf{K}$ be an nD xel-complex, for some positive integer n, and let $\mathcal{G}$ be the set of all n-xels of $\mathbf{K}$. A function $\mathbb{I} : \mathcal{G} \to \{0, 1\}$ for which either $\mathbb{I}^{-1}[\{1\}]$ is finite or $\mathbb{I}^{-1}[\{0\}]$ is finite will be called a *binary image on* $\mathbf{K}$ or, more briefly, a $\mathbf{K}$*-image*. Note that $\mathbb{I}(X)$ is only defined if $X \in \mathcal{G}$ (i.e., if X is an n-xel of $\mathbf{K}$)—$\mathbb{I}(X)$ is undefined if X is a xel of lower dimension in $\mathbf{K}$.

If $\mathbb{I}$ is a $\mathbf{K}$-image, then each n-xel in $\mathbb{I}^{-1}[\{1\}]$ is called a 1 of $\mathbb{I}$, and each n-xel in $\mathbb{I}^{-1}[\{0\}]$ is called a 0 of $\mathbb{I}$. If $\mathcal{D}$ is any subset of the set of 1's of a $\mathbf{K}$-image $\mathbb{I}$, then we write $\mathbb{I} - \mathcal{D}$ to denote the $\mathbf{K}$-image whose set of 1's is $\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{D}$. Changing $\mathbb{I}$ to $\mathbb{I} - \mathcal{D}$ is referred to as *deletion* of the set $\mathcal{D}$ from $\mathbb{I}$.

We write $\mathbb{I}^{\mathbf{c}}$ to denote the $\mathbf{K}$-image defined by $\mathbb{I}^{\mathbf{c}}(X) = 1 - \mathbb{I}(X)$ for all $X \in \mathcal{G}$. Thus the set of 1's of $\mathbb{I}^{\mathbf{c}}$ is the set of 0's of $\mathbb{I}$.

Each weakly (strongly) connected component of $\mathbb{I}^{-1}[\{1\}]$ will be called a *weak foreground component* (*strong foreground component*) of $\mathbb{I}$. Each weakly (strongly) connected component of $\mathbb{I}^{-1}[\{0\}]$ will be called a *weak background component* (*strong background component*) of $\mathbb{I}$.

If $D \in \mathbb{I}^{-1}[\{1\}]$, then D is said to be *simple* in $\mathbb{I}$ if (loosely speaking) "the deletion of $\{D\}$ from $\mathbb{I}$ preserves topology". A precise definition of this concept is as follows:

Definition 1. *Let* $\mathbf{K}$ *be a xel-complex, and let* D *be a* 1 *of a* $\mathbf{K}$*-image* $\mathbb{I}$*. Then we say* D *is* simple *in* $\mathbb{I}$ *if* $\bigcup(\mathbb{I}^{-1}[\{1\}] - \{D\})$ *is a deformation retract of* $\bigcup \mathbb{I}^{-1}[\{1\}]$.

In other words, D is simple in $\mathbb{I}$ if and only if the union of all the 1's of $\mathbb{I}$ can be continuously deformed over itself onto the union of all the 1's of $\mathbb{I}$ other than D, in such a way that all points in the latter union remain fixed throughout the deformation process.

The idea of defining simpleness in terms of continuous deformation is an old one that dates back to the 1960's: In the case of the 2D cubical complex, the above definition is very similar to an informally stated connectivity preservation condition given by Hilditch in an early paper on thinning [8, p. 411, condition 5].

A complementary concept to that of a simple 1 is that of a *cosimple* 1:

Definition 2. *Let* $\mathbf{K}$ *be a xel-complex, and let* D *be a* 1 *of a* $\mathbf{K}$*-image* $\mathbb{I}$*. Then we say* D *is* cosimple *in* $\mathbb{I}$ *if* D *is simple in* $(\mathbb{I} - \{D\})^{\mathbf{c}}$*. Equivalently,* D *is* cosimple *in* $\mathbb{I}$ *if and only if* $\bigcup \mathbb{I}^{-1}[\{0\}]$ *is a deformation retract of* $\bigcup(\mathbb{I}^{-1}[\{0\}] \cup \{D\})$*.*

Let $\mathbb{I}$ be a $\mathbf{K}$-image for some xel-complex $\mathbf{K}$, and let D be any 1 of $\mathbb{I}$. Then we define two sets $\mathbf{Attach}(D, \mathbb{I})$ and $\mathbf{Coattach}(D, \mathbb{I})$ of xels in ∂D as follows:

$$\mathbf{Attach}(D, \mathbb{I}) = \{X \in \mathbf{K} \mid X \subsetneq \partial D \quad \text{and} \quad \exists Q \in \mathbb{I}^{-1}[\{1\}] \setminus \{D\} \ (X \subsetneq \partial Q)\}$$
$$\mathbf{Coattach}(D, \mathbb{I}) = \{X \in \mathbf{K} \mid X \subsetneq \partial D \quad \text{and} \quad \exists Q \in \mathbb{I}^{-1}[\{0\}] \ (X \subsetneq \partial Q)\}$$

If a xel X is in $\mathbf{Attach}(D, \mathbb{I})$ or in $\mathbf{Coattach}(D, \mathbb{I})$, then so is every proper face of X. Note also that $\mathbf{Coattach}(D, \mathbb{I}) = \mathbf{Attach}(D, (\mathbb{I} - \{D\})^{\mathbf{c}})$. Conditions 3 and 4 in the definition of a xel-complex and Property 2 imply that $\bigcup \mathbf{Attach}(D, \mathbb{I}) = D \cap \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \{D\})$ and $\bigcup \mathbf{Coattach}(D, \mathbb{I}) = D \cap \bigcup \mathbb{I}^{-1}[\{0\}]$.

We can now state essentially discrete characterizations of simple and cosimple 1's in binary images on xel-complexes of dimension ≤ 4:

Theorem 1. *Let* $\mathbf{K}$ *be an nD xel-complex, where* $n \leq 4$*, and let* D *be a* 1 *of a* $\mathbf{K}$*-image* $\mathbb{I}$*. Then:*

1. *D is simple in $\mathbb{I}$ if and only if $\bigcup \mathbf{Attach}(D, \mathbb{I})$ is contractible.*
2. *D is cosimple in $\mathbb{I}$ if and only if $\bigcup \mathbf{Coattach}(D, \mathbb{I})$ is contractible.* □

Note that, since D is cosimple in $\mathbb{I}$ if and only if D is simple in $(\mathbb{I} - \{D\})^{\mathbf{c}}$, and since $\mathbf{Coattach}(D, \mathbb{I}) = \mathbf{Attach}(D, (\mathbb{I} - \{D\})^{\mathbf{c}})$, the two assertions of this theorem are really equivalent. The "if" parts of the theorem can be deduced from the fact that if A and B are contractible polyhedra such that $B \subseteq A$, then B is a deformation retract of A.[1] The "only if" parts of the theorem can be proved using methods of algebraic topology.[2]

[1] A self-contained proof of this fact is given in [13, Sect. 4].

[2] More specifically, it follows from the excision theorem and the exact homology sequence of a pair [16, Ch. 4] that if D is simple in $\mathbb{I}$ then the polyhedron $\bigcup \mathbf{Attach}(D, \mathbb{I})$ is nonempty and its reduced homology groups are all trivial. A polyhedron in $\mathbb{R}^3$ or in the boundary of a polyhedral 4-cell is contractible if (and only if) it has these properties. This is a consequence of (1) the theorems of Whitehead and Hurewicz [16, Chs. 7, 8] and (2) the fact that a polyhedron P in $\mathbb{R}^3$ or in the boundary of a polyhedral 4-cell is simply connected if its first homology group $H_1(P)$ is trivial. In the case where P is in $\mathbb{R}^3$, a proof of (2) is given in [12]. The truth of (2) for a polyhedron P in $\mathbb{R}^3$ implies its truth for a polyhedron P in the boundary of a polyhedral 4-cell X, because if $P \subsetneq \partial X$ then, by Thm. 2 of [17, Ch. 36], there is a homeomorphism $h : \partial X \to \mathbb{R}^3 \cup \{\infty\}$ such that $h[P]$ is a polyhedron in $\mathbb{R}^3$.

For $n \leq 4$, if D is a polyhedral n-cell then a polyhedron $P \subseteq \partial D$ is contractible if and only if P is connected, $(\partial D) \setminus P$ is connected, and the Euler characteristic of P is 1.[3] An important consequence of this and Theorem 1 is that it is computationally straightforward to determine whether or not a given 1 of a binary image on a xel-complex of dimension ≤ 4 is simple or cosimple.

The concepts of simple and cosimple 1's are extended to finite sets of 1's as follows:

Definition 3. *Let* $\mathbf{K}$ *be a xel-complex, and let* $\mathcal{D}$ *be a set of* 1*'s of a* $\mathbf{K}$*-image* $\mathbb{I}$*. Then we say* $\mathcal{D}$ *is* simple (cosimple) *in* $\mathbb{I}$ *if* $\mathcal{D}$ *is a finite set and there is an enumeration* $D_1, \ldots, D_k$ *of all the elements of* $\mathcal{D}$ *such that, for* $1 \leq i \leq k$, D_i *is a simple* (*cosimple*) 1 *in the* $\mathbf{K}$*-image* $\mathbb{I} - \{D_j | 1 \leq j < i\}$.

Note that the empty set is both simple and cosimple in every $\mathbf{K}$-image. Also note that, if D is a 1 of $\mathbb{I}$, then the singleton set $\{D\}$ is simple (cosimple) in $\mathbb{I}$ if and only if D is simple (cosimple) in $\mathbb{I}$.

Important properties[4] of simple sets of 1's are that the deletion of such a set can never split a weak foreground component, can never completely eliminate a weak foreground component, can never merge different strong background components, and can never create a new strong background component. More precisely, if $\mathcal{D}$ is a set of 1's that is simple in $\mathbb{I}$, then each weak foreground component of $\mathbb{I}$ contains exactly one weak foreground component of $\mathbb{I} - \mathcal{D}$, and each strong background component of $\mathbb{I} - \mathcal{D}$ contains exactly one strong background component of $\mathbb{I}$.

Analogously, deletion of a cosimple set can never split a strong foreground component, can never completely eliminate a strong foreground component, can never merge different weak background components, and can never create a new weak background component: If $\mathcal{D}$ is a set of 1's that is cosimple in $\mathbb{I}$, then each strong foreground component of $\mathbb{I}$ contains exactly one strong foreground component of $\mathbb{I} - \mathcal{D}$, and each weak background component of $\mathbb{I} - \mathcal{D}$ contains exactly one weak background component of $\mathbb{I}$.

We are now ready to define the principal concepts of this paper, namely MNS and MNCS sets:

Definition 4. *Let* $\mathbf{K}$ *be a xel-complex, and let* $\mathcal{D}$ *be a set of* 1*'s of a* $\mathbf{K}$*-image* $\mathbb{I}$*. Then we say* $\mathcal{D}$ *is* minimal non-simple*, or* MNS (minimal non-cosimple*, or* MNCS) *in the* $\mathbf{K}$*-image* $\mathbb{I}$ *if* $\mathcal{D}$ *is non-simple* (*non-cosimple*) *in* $\mathbb{I}$*, but every proper subset of* $\mathcal{D}$ *is simple* (*cosimple*) *in* $\mathbb{I}$*.*

Note that, if D is any 1 of $\mathbb{I}$, then the singleton set $\{D\}$ is MNS (MNCS) in $\mathbb{I}$ if and only if D is non-simple (non-cosimple) in $\mathbb{I}$. Note also that all MNS and MNCS sets are finite, because simple and cosimple sets are, by definition, finite.

[3] This can be deduced from the fact stated in the second sentence of footnote 2 and the Alexander duality theorem [16, Ch. 4].

[4] These properties can be deduced from the first sentence of footnote 2 and the Alexander duality theorem, which imply that if D is a 1 of $\mathbb{I}$ that is simple in $\mathbb{I}$ then $\bigcup \mathbf{Attach}(D, \mathbb{I})$ is a nonempty connected proper subset of ∂D whose complement in ∂D is also connected.

If a finite set $\mathcal{Q}$ of 1's of a $\mathbf{K}$-image $\mathbb{I}$ is non-simple (non-cosimple) in $\mathbb{I}$, then $\mathcal{Q}$ must contain a subset that is MNS (MNCS) in $\mathbb{I}$. Thus if $\mathcal{P}$ is a set of 1's of $\mathbb{I}$ such that no subset of $\mathcal{P}$ is MNS (MNCS) in $\mathbb{I}$, then *every* subset of $\mathcal{P}$ is simple (cosimple) in $\mathbb{I}$. We say that a set $\mathcal{P}$ of 1's of $\mathbb{I}$ is *hereditarily simple* (*hereditarily cosimple*) in $\mathbb{I}$ if $\mathcal{P}$ has this property. It can be shown that, if $\mathbb{I}$ is a binary image on the 3D cubical complex, then $\mathcal{P}$ is hereditarily simple (hereditarily cosimple) in $\mathbb{I}$ if and only if $\mathcal{P}$ is $\mathcal{P}_{26}$-simple ($\mathcal{P}_6$-simple) in the sense of Bertrand [1].

The arguments in this paper will be based on the characterizations of MNS and MNCS sets that are stated in the following theorem:

Theorem 2. *Let $\mathbf{K}$ be an nD xel-complex, where $n \leq 4$, and let $\mathcal{D}$ be a set of 1's of a $\mathbf{K}$-image $\mathbb{I}$. Then:*

1. *$\mathcal{D}$ is MNS in $\mathbb{I}$ if and only if the following conditions hold for all $D \in \mathcal{D}$:*
   MNS0 *$\mathcal{D}$ is nonempty and finite.*
   MNS1 *D is non-simple in $\mathbb{I} - (\mathcal{D} \setminus \{D\})$.*
   MNS2 *D is simple in $\mathbb{I} - \mathcal{D}'$ for every $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{D\}$.*
2. *$\mathcal{D}$ is MNCS in $\mathbb{I}$ if and only if the following conditions hold for all $D \in \mathcal{D}$:*
   MNCS0 *$\mathcal{D}$ is nonempty and finite.*
   MNCS1 *D is non-cosimple in $\mathbb{I} - (\mathcal{D} \setminus \{D\})$.*
   MNCS2 *D is cosimple in $\mathbb{I} - \mathcal{D}'$ for every $\mathcal{D}' \subsetneq \mathcal{D} \setminus \{D\}$.* □

Both assertions of this theorem are special cases of Prop. 6 in [9]. Explanations of why the hypotheses of that proposition are satisfied are given in [5, p. 123] (for the MNS case) and in [11, p. 326] (for the MNCS case).

We say that a set $\mathcal{S}$ of n-xels of an nD xel-complex $\mathbf{K}$ *can be MNS* (*can be MNCS*) if there exists a $\mathbf{K}$-image $\mathbb{I}$ in which $\mathcal{S}$ is an MNS (MNCS) set of 1's. We say that $\mathcal{S}$ *can be MNS* (*MNCS*) *without being a weak* (*strong*) *foreground component* if there exists a $\mathbf{K}$-image $\mathbb{I}$ in which $\mathcal{S}$ is an MNS (MNCS) set of 1's and $\mathcal{S}$ is not a weak (strong) foreground component of $\mathbb{I}$. The main goals of this paper are to determine, for every xel-complex $\mathbf{K}$ of dimension ≤ 4, exactly which sets of xels can be MNS, exactly which sets can be MNCS, and exactly which sets can be MNS (MNCS) without being a weak (strong) foreground component.

5 Properties of Contractible Polyhedra in $\mathbb{R}^3$ or in the Boundary of a Polyhedral 4-Cell

The proofs of our main theorems will depend on the following fact:

Property 7. Let A and B be polyhedra in $\mathbb{R}^3$ or in the boundary of a polyhedral 4-cell, such that at least two of the following three statements are true:

1. Each of A and B is contractible.
2. $A \cup B$ is contractible.
3. $A \cap B$ is contractible.

Then all three of these statements are true. □

The hypotheses of Property 7 evidently imply that none of the polyhedra A, B, and $A \cap B$ is empty. Indeed, if any of these sets is empty then $A \cap B = \emptyset$ (so that statement 3 is false), and either A and B are disjoint nonempty closed sets (in which case $A \cup B$ is disconnected and statement 2 is false) or one of A and B is empty (in which case statement 1 is false).

Property 7 is a consequence of the reduced Mayer-Vietoris sequence (see, e.g., [16, pp. 128–129]) and the fact, mentioned in footnote 2, that a polyhedron in $\mathbb{R}^3$ or in the boundary of a polyhedral 4-cell is contractible if and only if it is nonempty and its reduced homology groups are all trivial.

The following lemma and its corollary state some consequences of Property 7. Note that the hypotheses of assertions 1 and 2 of the lemma imply that each member of the collection $\mathcal{S}$ is contractible, since "every nonempty subcollection" includes subcollections that consist of just one member.

Lemma 1. *Let $\mathcal{S}$ be a nonempty finite collection of polyhedra in $\mathbb{R}^3$ or in the boundary of a polyhedral 4-cell. Then:*

1. *$\bigcap \mathcal{S}$ is contractible if every nonempty subcollection of $\mathcal{S}$ has a contractible union.*
2. *$\bigcup \mathcal{S}$ is contractible if every nonempty subcollection of $\mathcal{S}$ has a contractible intersection.*

Proof. First, we prove assertion 1. Assertion 1 is evidently true if $|\mathcal{S}| = 1$. Now assume as an induction hypothesis that, for some integer $l > 1$, assertion 1 is true whenever $|\mathcal{S}| < l$. Suppose $|\mathcal{S}| = l$, and every nonempty subcollection of $\mathcal{S}$ has a contractible union. We need to show that $\bigcap \mathcal{S}$ is contractible. Let $\mathcal{S} = \{A_i \mid 1 \leq i \leq l\}$, $\mathcal{S}' = \mathcal{S} \setminus \{A_l\}$, and $\mathcal{S}'' = \{A_l \cup A_i \mid 1 \leq i \leq l-1\}$. Since every nonempty subcollection of $\mathcal{S}$ has a contractible union, we have that:

(a) A_l is contractible.
(b) Every nonempty subcollection of $\mathcal{S}'$ has a contractible union.
(c) Every nonempty subcollection of $\mathcal{S}''$ has a contractible union.

It follows from (b), (c), and the induction hypothesis that each of the two sets $\bigcap \mathcal{S}' = \bigcap_{i=1}^{l-1} A_i$ and $\bigcap \mathcal{S}'' = A_l \cup \bigcap_{i=1}^{l-1} A_i$ is contractible. This, (a), and Property 7 imply that $A_l \cap \bigcap_{i=1}^{l-1} A_i = \bigcap \mathcal{S}$ is contractible, as required. This proves assertion 1. By a symmetrical argument, with unions in place of intersections, and vice versa, assertion 2 is also true. □

Corollary 1. *Let $\mathcal{S}$ be a nonempty finite collection of polyhedra, in $\mathbb{R}^3$ or in the boundary of a polyhedral 4-cell, that satisfies one of the following conditions:*

1. *Every nonempty proper subcollection of $\mathcal{S}$ has a contractible union.*
2. *Every nonempty proper subcollection of $\mathcal{S}$ has a contractible intersection.*

Then $\mathcal{S}$ satisfies both of these conditions. Moreover, $\bigcup \mathcal{S}$ is contractible if and only if $\bigcap \mathcal{S}$ is contractible.

Proof. If condition 1 holds, and $\mathcal{S}'$ is any nonempty proper subcollection of $\mathcal{S}$, then every nonempty subcollection of $\mathcal{S}'$ has a contractible union, and so $\bigcap \mathcal{S}'$ is contractible by assertion 1 of Lemma 1. Hence condition 2 holds if condition 1 holds. Since condition 2 holds, if $\bigcap \mathcal{S}$ is contractible then every nonempty subcollection of $\mathcal{S}$ has a contractible intersection, and so $\bigcup \mathcal{S}$ is contractible by assertion 2 of Lemma 1. By symmetrical arguments, condition 1 holds if condition 2 holds, and $\bigcap \mathcal{S}$ is contractible if $\bigcup \mathcal{S}$ is contractible. □

Another property of contractible polyhedra that we will need is:

Property 8. Let X be a polyhedral n-cell, and let $\mathcal{T}$ be a nonempty finite collection of polyhedra in ∂X such that:

1. $\bigcap \mathcal{T} = \emptyset$.
2. $\bigcap \mathcal{T}'$ is a contractible set whenever $\emptyset \neq \mathcal{T}' \subsetneq \mathcal{T}$.

Then $|\mathcal{T}| - 1 \leq n$, and $\bigcup \mathcal{T} = \partial X$ if and only if $|\mathcal{T}| - 1 = n$. □

The hypotheses of Property 8 imply that the polyhedron of the nerve complex of $\mathcal{T}$ is the boundary of a $(|\mathcal{T}|-1)$-simplex. So it follows from the nerve theorem [3, Thm. 10.6(i)] that the $|\mathcal{T}| - 2^{\text{nd}}$ Betti number of $\bigcup \mathcal{T}$ is 1 if $|\mathcal{T}| \geq 3$. Property 8 can be deduced from this and the fact that $\bigcup \mathcal{T}$ is a polyhedron in ∂X.

6 The Fundamental Lemma

We now use the results of Sect. 5 to establish some key facts (stated in the following lemma) on which the proofs of our main theorems will be based.

Lemma 2 (Fundamental Lemma). *Let X be an n-xel of a xel-complex $\mathbf{K}$, where $n \leq 4$, let $(X_i)_{1 \leq i \leq k}$ be a nonempty finite family of xels of $\mathbf{K}$ in ∂X, and let $P \subseteq \partial X$ be a union of xels of $\mathbf{K}$ for which*

$$P \cup \bigcup \{X_i \mid i \in \mathcal{M}\} \text{ is contractible whenever } \emptyset \neq \mathcal{M} \subsetneq \{1, \ldots, k\} \qquad (*)$$

Then:

1. *For all $\mathcal{S}$ such that $\emptyset \neq \mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}$, $P \cap \bigcap \mathcal{S}$ is contractible if and only if P is contractible and $\bigcap \mathcal{S} \neq \emptyset$.*
2. *If $\bigcap_{i=1}^k X_i = \emptyset$, then P is contractible if and only if $P \cup \bigcup_{i=1}^k X_i$ is contractible.*
3. *If $P \cap \bigcap_{i=1}^k X_i$ is contractible, then P is contractible if and only if $P \cup \bigcup_{i=1}^k X_i$ is contractible.*
4. *If P is contractible, and there is some $\mathcal{S}$ such that $\emptyset \neq \mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}$ and $\bigcap \mathcal{S} = \bigcap_{i=1}^k X_i$, then $P \cup \bigcup_{i=1}^k X_i$ is contractible.*
5. *If $\bigcap_{i=1}^k X_i \neq \emptyset$ but $P \cap \bigcap_{i=1}^k X_i = \emptyset$ and $P \cup \bigcup_{i=1}^k X_i$ is contractible, then $P = \emptyset$.*
6. *If $\bigcap_{i=1}^k X_i \neq \emptyset$ but $P \cap \bigcap_{i=1}^k X_i = \emptyset$ and P is contractible, then $k \leq n$, and $P \cup \bigcup_{i=1}^k X_i = \partial X$ if and only if $k = n$.*

Proof. We claim that it is enough to prove this lemma in the case where no two of the X_i's are equal. For assertions 1 – 5 this is because, if $(X'_i)_{1 \leq i \leq k'}$ is a family of *distinct* sets such that $\{X'_i \mid 1 \leq i \leq k'\} = \{X_i \mid 1 \leq i \leq k\}$, then when we replace X_i with X'_i and k with k' it is evident that (*) still holds and none of assertions 1 – 5 changes in meaning. In the case of assertion 6, if $X_j = X_l$ for some $j \neq l$, then the case $\mathcal{M} = \{1, \ldots, k\} \setminus \{j\}$ of (*) implies that $P \cup \bigcup_{i=1}^{k} X_i$ is contractible, and so the hypotheses of assertion 6 are inconsistent with assertion 5. In other words, if assertion 5 is true then assertion 6 is vacuously true if $X_j = X_l$ for some $j \neq l$. This justifies our claim, and in the rest of this proof we shall assume that the X_i's are all distinct.

To prove assertion 1, let $\mathcal{S}$ satisfy $\emptyset \neq \mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}$ and let $\mathcal{S}^* = \{P \cup Y \mid Y \in \mathcal{S}\}$. By (*), every nonempty subcollection of $\mathcal{S}^*$ has a contractible union. Hence, by assertion 1 of Lemma 1, $P \cup \bigcap \mathcal{S} = \bigcap \mathcal{S}^*$ is contractible. If $\bigcap \mathcal{S} = \emptyset$ then $P \cap \bigcap \mathcal{S} = \emptyset$ is not contractible, which is consistent with assertion 1. Now suppose $\bigcap \mathcal{S} \neq \emptyset$. Then $\bigcap \mathcal{S}$ is a xel of $\mathbf{K}$ (by condition 4 of the definition of a xel-complex) and is therefore contractible. Since each of $P \cup \bigcap \mathcal{S}$ and $\bigcap \mathcal{S}$ is contractible, it follows from Property 7 that P is contractible if and only if $P \cap \bigcap \mathcal{S}$ is contractible. This proves assertion 1.

Next, we prove assertion 2. Suppose $\bigcap_{i=1}^{k} X_i = \emptyset$ (which implies $k \geq 2$). By (*), every nonempty proper subcollection of $\{P \cup X_1 \cup X_i \mid 2 \leq i \leq k\}$ has a contractible union, and so it follows from Corollary 1 of Lemma 1 that $P \cup \bigcup_{i=1}^{k} X_i = \bigcup_{i=2}^{k} (P \cup X_1 \cup X_i)$ is contractible if and only if $(P \cup X_1) \cup \bigcap_{i=2}^{k} X_i = \bigcap_{i=2}^{k} (P \cup X_1 \cup X_i)$ is contractible.

There are now two cases: $\bigcap_{i=2}^{k} X_i = \emptyset$, and $\bigcap_{i=2}^{k} X_i \neq \emptyset$. In the first case, the set $(P \cup X_1) \cup \bigcap_{i=2}^{k} X_i = P \cup X_1$ is contractible (by (*)), so it follows from the equivalence established in previous paragraph that $P \cup \bigcup_{i=1}^{k} X_i$ is also contractible. Moreover, in this case it follows from assertion 1 of Lemma 1 that $P = P \cup \bigcap_{i=2}^{k} X_i = \bigcap_{i=2}^{k} (P \cup X_i)$ is contractible as well, because every nonempty subcollection of $\{P \cup X_i \mid 2 \leq i \leq k\}$ has a contractible union (by (*)). Thus the sets P and $P \cup \bigcup_{i=1}^{k} X_i$ are both contractible, which is consistent with assertion 2.

In the second case, where $\bigcap_{i=2}^{k} X_i \neq \emptyset$, the set $(P \cup X_1) \cup \bigcap_{i=2}^{k} X_i$ is the union of the set $\bigcap_{i=2}^{k} X_i$ (which is a xel of $\mathbf{K}$, by condition 4 in the definition of a xel-complex, and is therefore contractible) with the set $P \cup X_1$ (which is contractible because of (*)). Hence, by Property 7, we have that $(P \cup X_1) \cup \bigcap_{i=2}^{k} X_i$ is contractible if and only if $(P \cup X_1) \cap \bigcap_{i=2}^{k} X_i = P \cap \bigcap_{i=2}^{k} X_i$ is contractible. But, by assertion 1, $P \cap \bigcap_{i=2}^{k} X_i$ is contractible if and only if P is contractible. We conclude that $(P \cup X_1) \cup \bigcap_{i=2}^{k} X_i$ is contractible if and only if P is contractible. As we showed earlier that $(P \cup X_1) \cup \bigcap_{i=2}^{k} X_i$ is contractible if and only if $P \cup \bigcup_{i=1}^{k} X_i$ is contractible, assertion 2 is proved.

To prove assertion 3, suppose $P \cap \bigcap_{i=1}^{k} X_i$ is contractible. This implies that $\bigcap_{i=1}^{k} X_i \neq \emptyset$, and so $\bigcap_{i=1}^{k} X_i$ is a xel of $\mathbf{K}$ (by condition 4 in the definition of a xel-complex) and is therefore contractible. Since $P \cap \bigcap_{i=1}^{k} X_i$ and $\bigcap_{i=1}^{k} X_i$ are both contractible, it follows from Property 7 that P is contractible if and only if $P \cup \bigcap_{i=1}^{k} X_i$ is contractible. But, since every nonempty proper subcollection of $\{P \cup X_i \mid 1 \leq i \leq k\}$ has a contractible union (by (*)), $P \cup \bigcap_{i=1}^{k} X_i = \bigcap_{i=1}^{k}(P \cup X_i)$ is contractible if and only if $P \cup \bigcup_{i=1}^{k} X_i = \bigcup_{i=1}^{k}(P \cup X_i)$ is contractible, by Corollary 1 of Lemma 1. This proves assertion 3.

To prove assertion 4, suppose that P is contractible, and that there is some $\mathcal{S}$ such that $\emptyset \neq \mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}$ and $\bigcap \mathcal{S} = \bigcap_{i=1}^{k} X_i$. Now if $\bigcap \mathcal{S} = \bigcap_{i=1}^{k} X_i = \emptyset$ then assertion 4 is true, by assertion 2. If, on the other hand, $\bigcap \mathcal{S} = \bigcap_{i=1}^{k} X_i \neq \emptyset$, then assertion 1 implies that $P \cap \bigcap_{i=1}^{k} X_i = P \cap \bigcap \mathcal{S}$ is contractible, and so assertion 4 is true, by assertion 3. This proves assertion 4.

To prove assertion 5, suppose $\bigcap_{i=1}^{k} X_i \neq \emptyset$ but $P \cap \bigcap_{i=1}^{k} X_i = \emptyset$. Let $P' = P \cup \bigcap_{i=1}^{k} X_i$. Then the hypotheses of the lemma still hold when we replace P with P', and $P' \cap \bigcap_{i=1}^{k} X_i = \bigcap_{i=1}^{k} X_i$ is a xel of $\mathbf{K}$ (by condition 4 in the definition of a xel-complex) and is therefore contractible. Hence assertion 3 of the lemma (with P' in place of P) implies that if $P \cup \bigcup_{i=1}^{k} X_i = P' \cup \bigcup_{i=1}^{k} X_i$ is contractible then P' is contractible. However, P' is contractible only if $P = \emptyset$ (for if $P \neq \emptyset$ then $P' = P \cup \bigcap_{i=1}^{k} X_i$ is disconnected, as P and $\bigcap_{i=1}^{k} X_i$ are disjoint nonempty closed sets).

To prove assertion 6, suppose $\bigcap_{i=1}^{k} X_i \neq \emptyset$ but $P \cap \bigcap_{i=1}^{k} X_i = \emptyset$ and P is contractible. Let $\mathcal{T} = \{X_i \mid 1 \leq i \leq k\} \cup \{P\}$. Then it follows from assertion 1 that every nonempty proper subcollection of $\mathcal{T}$ has a contractible intersection. Moreover, $\bigcap \mathcal{T} = P \cap \bigcap_{i=1}^{k} X_i = \emptyset$. So it follows from Property 8 that $k \leq n$, and that $P \cup \bigcup_{i=1}^{k} X_i = \bigcup \mathcal{T} = \partial X$ if and only if $k = n$. $\square$

7 The Main Theorems

Theorem 3 (First Main Theorem). *Let* $\mathbf{K}$ *be an* n*D xel-complex, where* $1 \leq n \leq 4$, *and let* $\mathcal{T}$ *be a nonempty finite collection of* n*-xels of* $\mathbf{K}$*. Then:*

1. *If* $\bigcap \mathcal{T} = \emptyset$, *then there is no* $\mathbf{K}$*-image* $\mathbb{I}$ *such that* $\mathcal{T}$ *is MNS in* $\mathbb{I}$*.*
2. *If* $\bigcap \mathcal{T} \neq \emptyset$, *and* $\mathcal{T}$ *is a weak foreground component of a* $\mathbf{K}$*-image* $\mathbb{I}$, *then* $\mathcal{T}$ *is MNS in* $\mathbb{I}$*.*
3. *If* $\bigcap \mathcal{T}$ *is a* 0*-xel of* $\mathbf{K}$, *and* $\mathcal{T}$ *is MNS in a* $\mathbf{K}$*-image* $\mathbb{I}$, *then* $\mathcal{T}$ *is a weak foreground component of* $\mathbb{I}$*.*
4. *If* $\bigcap \mathcal{T}$ *is an* m*-xel of* $\mathbf{K}$ *for some* $m \geq 1$, *then there is a* $\mathbf{K}$*-image* $\mathbb{I}$ *such that* $\mathcal{T}$ *is MNS in* $\mathbb{I}$ *and* $\mathcal{T}$ *is not a weak foreground component of* $\mathbb{I}$*.*

Proof. Let $k = |\mathcal{T}| - 1$, let $\mathcal{T} = \{X, T_1, \ldots, T_k\}$ and, for $1 \leq i \leq k$, write X_i for $X \cap T_i$.

We first prove assertions 1 and 3. For this purpose we may assume $k \neq 0$, as this is implied by the hypotheses of assertions 1 and 3. Suppose there is a $\mathbf{K}$-image $\mathbb{I}$ such that $\mathcal{T}$ is an MNS set of 1's of $\mathbb{I}$. We will deduce that $\bigcap \mathcal{T} \neq \emptyset$ (which will prove assertion 1). We will further deduce that if $\bigcap \mathcal{T}$ is a 0-xel of $\mathbf{K}$ then $\mathcal{T}$ is a weak foreground component of $\mathbb{I}$ (which will prove assertion 3).

Let $P = X \cap \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{T})$. Thus $P = \bigcup \mathbf{Attach}(X, \mathbb{I} - \{T_i \mid 1 \leq i \leq k\})$. Then $\bigcup \mathbf{Attach}(X, \mathbb{I} - (\{T_i \mid 1 \leq i \leq k\} \setminus \mathcal{W})) = P \cup \bigcup \{X_i \mid T_i \in \mathcal{W}\}$ for any subcollection $\mathcal{W}$ of $\{T_i \mid 1 \leq i \leq k\}$.

Since $\mathcal{T} = \{X, T_1, \ldots, T_k\}$ is MNS in $\mathbb{I}$, it follows from Theorem 2 that X is simple in $\mathbb{I} - (\{T_i \mid 1 \leq i \leq k\} \setminus \mathcal{W})$ for every nonempty subcollection $\mathcal{W}$ of $\{T_i \mid 1 \leq i \leq k\}$, and that X is non-simple in $\mathbb{I} - \{T_i \mid 1 \leq i \leq k\}$. Hence, by Theorem 1, $P = \bigcup \mathbf{Attach}(X, \mathbb{I} - \{T_i \mid 1 \leq i \leq k\})$ is not contractible, but $P \cup \bigcup\{X_i \mid T_i \in \mathcal{W}\} = \bigcup \mathbf{Attach}(X, \mathbb{I} - (\{T_i \mid 1 \leq i \leq k\} \setminus \mathcal{W}))$ is contractible whenever $\emptyset \neq \mathcal{W} \subseteq \{T_i \mid 1 \leq i \leq k\}$.

The collection of sets $\{P \cup \bigcup\{X_i \mid T_i \in \mathcal{W}\} \mid \emptyset \neq \mathcal{W} \subseteq \{T_i \mid 1 \leq i \leq k\}\}$ is the same as the collection of sets $\{P \cup \bigcup \mathcal{S} \mid \emptyset \neq \mathcal{S} \subseteq \{X_i \mid 1 \leq i \leq k\}\}$. Hence $P \cup \bigcup \mathcal{S}$ is contractible whenever $\emptyset \neq \mathcal{S} \subseteq \{X_i \mid 1 \leq i \leq k\}$. Since this implies $P \cup X_i$ is contractible for $1 \leq i \leq k$, and since we saw above that P is not contractible, none of the sets X_i is empty, and so each set $X_i = X \cap T_i$ is a xel of $\mathbf{K}$. Thus we have established the following:

(a) P, X, and the family $(X_i)_{1 \leq i \leq k}$ satisfy the hypotheses of the Fundamental Lemma.
(b) P is not contractible.
(c) $P \cup \bigcup_{i=1}^{k} X_i$ is contractible.

Assertion 2 of the Fundamental Lemma now implies:

$$\bigcap \mathcal{T} = \bigcap_{i=1}^{k} X_i \neq \emptyset \qquad (\dagger)$$

This proves assertion 1.

Now suppose $\bigcap \mathcal{T} = \bigcap_{i=1}^{k} X_i$ is a 0-xel of $\mathbf{K}$. If $P \cap \bigcap_{i=1}^{k} X_i \neq \emptyset$ then $P \cap \bigcap_{i=1}^{k} X_i$ is the 0-xel $\bigcap_{i=1}^{k} X_i$ (as a 0-xel has no nonempty proper subset), and so $P \cap \bigcap_{i=1}^{k} X_i$ is contractible, which contradicts assertion 3 of the Fundamental Lemma (in view of (a), (b), and (c) above). Hence $P \cap \bigcap_{i=1}^{k} X_i = \emptyset$. In view of this, (a), (c), ($\dagger$), and assertion 5 of the Fundamental Lemma, we have that $X \cap \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{T}) = P = \emptyset$.

As X is an arbitrary element of $\mathcal{T}$, it follows that $T \cap \bigcup(\mathbb{I}^{-1}[\{1\}] \setminus \mathcal{T}) = \emptyset$ for every $T \in \mathcal{T}$. Moreover, every two elements of $\mathcal{T}$ are weakly adjacent (since $\bigcap \mathcal{T} \neq \emptyset$), and so $\mathcal{T}$ is weakly connected. Hence $\mathcal{T}$ is a weak foreground component of $\mathbb{I}$. This proves assertion 3.

To prove assertion 4, suppose $\bigcap \mathcal{T}$ is an m-xel of $\mathbf{K}$ for some $m \geq 1$. Then, by Property 3 of a xel-complex, there exist two distinct 0-xels $\{q_1\}$ and $\{q_2\}$ of $\mathbf{K}$ in $\bigcap \mathcal{T}$. By condition 6 of the definition of a xel-complex, there exist n-xels Q_1 and Q_2 of $\mathbf{K}$ such that $q_1 \in Q_1$, $q_2 \in Q_2$, and $Q_1 \cap Q_2 = \emptyset$. Let $\mathbb{I}^*$ be the $\mathbf{K}$-image whose set of 1's is $\mathcal{T} \cup \{Q_1, Q_2\}$.

We claim that $\mathcal{T}$ is MNS in $\mathbb{I}^*$. To justify this claim, let $P^* = X \cap (Q_1 \cup Q_2)$, so $\bigcup \mathbf{Attach}(X, \mathbb{I}^* - \{T_i \mid 1 \leq i \leq k\}) = P^*$. Then, for any $\mathcal{W} \subseteq \{T_i \mid 1 \leq i \leq k\}$, $\bigcup \mathbf{Attach}(X, \mathbb{I}^* - (\{T_i \mid 1 \leq i \leq k\} \setminus \mathcal{W})) = P^* \cup \bigcup\{X_i \mid T_i \in \mathcal{W}\}$. So (since X is an arbitrary element of $\mathcal{T}$) our claim that $\mathcal{T}$ is MNS in $\mathbb{I}^*$ will follow from Theorems 1 and 2 if we can show that:

(a) P^* is not contractible.
(b) $P^* \cup \bigcup \mathcal{S}$ is contractible whenever $\emptyset \neq \mathcal{S} \subseteq \{X_i \mid 1 \leq i \leq k\}$.

Here (a) is true since $P^* = X \cap (Q_1 \cup Q_2)$ is disconnected (as $Q_1 \cap Q_2 = \emptyset$). To prove (b), let $\emptyset \neq \mathcal{S} \subseteq \{X_i \mid 1 \leq i \leq k\}$, let $\mathcal{S}_1 = \mathcal{S} \cup \{X \cap Q_1\}$, and let $\mathcal{S}_2 = \mathcal{S} \cup \{X \cap Q_2\}$. Then $(\bigcup \mathcal{S}_1) \cup (\bigcup \mathcal{S}_2) = P^* \cup \bigcup \mathcal{S}$. Note that if $\mathcal{X}$ is $\mathcal{S}$, $\mathcal{S}_1$, or $\mathcal{S}_2$, then the intersection of any nonempty subcollection of $\mathcal{X}$ is nonempty, and is therefore a xel of $\mathbf{K}$. So if $\mathcal{X}$ is $\mathcal{S}$, $\mathcal{S}_1$, or $\mathcal{S}_2$ then every nonempty subcollection of $\mathcal{X}$ has a contractible intersection, which implies (by assertion 2 of Lemma 1) that $\bigcup \mathcal{X}$ is contractible. Thus each of the sets $\bigcup \mathcal{S}$, $\bigcup \mathcal{S}_1$, and $\bigcup \mathcal{S}_2$ is contractible. Since $(\bigcup \mathcal{S}_1) \cap (\bigcup \mathcal{S}_2) = (\bigcup \mathcal{S}) \cup (X \cap Q_1 \cap Q_2) = \bigcup \mathcal{S}$ is also contractible, we see from Property 7 that $P^* \cup \bigcup \mathcal{S} = (\bigcup \mathcal{S}_1) \cup (\bigcup \mathcal{S}_2)$ is contractible. This proves (b) and completes the proof of assertion 4.

To prove assertion 2, suppose $\bigcap \mathcal{T} \neq \emptyset$, and let $\mathbb{I}'$ be any $\mathbf{K}$-image of which $\mathcal{T}$ is a weak foreground component. We will show that $\mathcal{T}$ is MNS in $\mathbb{I}'$.

Now $\mathbf{Attach}(X, \mathbb{I}' - \{T_i \mid 1 \leq i \leq k\}) = \emptyset$, as $\mathcal{T}$ is a weak foreground component of $\mathbb{I}'$. We also have that $\bigcup \mathbf{Attach}(X, \mathbb{I}' - (\{T_i \mid 1 \leq i \leq k\} \setminus \mathcal{W}))$ $= \bigcup \{X_i \mid T_i \in \mathcal{W}\}$ for any subcollection $\mathcal{W}$ of $\{T_i \mid 1 \leq i \leq k\}$. So (since X is an arbitrary element of $\mathcal{T}$) our claim that $\mathcal{T}$ is MNS in $\mathbb{I}'$ will follow from Theorems 1 and 2 if we can just show that $\bigcup \mathcal{S}$ is contractible whenever $\emptyset \neq \mathcal{S} \subseteq \{X_i \mid 1 \leq i \leq k\}$. Now the intersection of any nonempty subcollection of $\{X_i \mid 1 \leq i \leq k\}$ is nonempty (as $\bigcap \mathcal{T} \neq \emptyset$) and is therefore a xel of $\mathbf{K}$. Thus if $\emptyset \neq \mathcal{S} \subseteq \{X_i \mid 1 \leq i \leq k\}$ then every nonempty subcollection of $\mathcal{S}$ has a contractible intersection, and so $\bigcup \mathcal{S}$ is contractible (by assertion 2 of Lemma 1), as required. □

The proof of our second main theorem will depend on two more lemmas, which we now establish. Note that if $\mathcal{S}$ is a finite set of 1's of a binary image $\mathbb{I}$ on a xel-complex of dimension ≤ 4, then it follows from assertion 2 of the first lemma below that $\mathcal{S}$ is cosimple if (and only if) the intersection of $\mathcal{S}$ with each strong foreground component of $\mathbb{I}$ is cosimple, and so $\mathcal{S}$ cannot be MNCS in $\mathbb{I}$ if $\mathcal{S}$ intersects more than one strong foreground component of $\mathbb{I}$.

Lemma 3. *Let $\mathbf{K}$ be an nD xel-complex, where $n \leq 4$, and let $\mathcal{T}$ be any set of n-xels of $\mathbf{K}$. Let $\mathbb{I}_1$ and $\mathbb{I}_2$ be $\mathbf{K}$-images such that $\mathbb{I}_1(X) = \mathbb{I}_2(X) = 1$ for every $X \in \mathcal{T}$, and $\mathbb{I}_1(X) = \mathbb{I}_2(X) = 0$ for every n-xel X that is not in $\mathcal{T}$ but is strongly adjacent to an n-xel in $\mathcal{T}$ (i.e., $\mathcal{T}$ is a union of strong foreground components both of $\mathbb{I}_1$ and of $\mathbb{I}_2$). Then:*

1. *For every $T \in \mathcal{T}$, $\mathbf{Coattach}(T, \mathbb{I}_1) = \mathbf{Coattach}(T, \mathbb{I}_2)$.*
2. *For every $T \in \mathcal{T}$, T is cosimple in $\mathbb{I}_1$ if and only if T is cosimple in $\mathbb{I}_2$.*
3. *For every $\mathcal{T}' \subseteq \mathcal{T}$, $\mathcal{T}'$ is MNCS in $\mathbb{I}_1$ if and only if $\mathcal{T}'$ is MNCS in $\mathbb{I}_2$.*

Proof. To prove assertion 1, let $T \in \mathcal{T}$. We now show that $\mathbf{Coattach}(T, \mathbb{I}_1) \subseteq \mathbf{Coattach}(T, \mathbb{I}_2)$. Let $Y \in \mathbf{Coattach}(T, \mathbb{I}_1)$. Then $Y \subsetneq \partial T$ and there is an n-xel $Q \in \mathbb{I}_1^{-1}[\{0\}]$ such that $Y \subsetneq \partial Q$. Thus $Y \subseteq T \cap Q$ and so, by Property 5, there exists a sequence $Q_0, Q_1, \ldots, Q_k$ of n-xels of $\mathbf{K}$ such that $Q_0 = T$, $Q_k = Q$, and, for $1 \leq i \leq k$, $Q_{i-1} \cap Q_i$ is an $(n-1)$-xel of $\mathbf{K}$ that contains Y. Now $Q_0 = T \in \mathcal{T}$ and $Q_k = Q \notin \mathcal{T}$ (since $Q \in \mathbb{I}_1^{-1}[\{0\}]$). Let Q_j be the first element of the sequence $Q_0, Q_1, \ldots, Q_k$ that does not belong to $\mathcal{T}$. Then, $Q_{j-1} \in \mathcal{T}$. Since $Q_{j-1} \cap Q_j$ is an

$(n-1)$-xel of $\mathbf{K}$, the n-xel Q_j is strongly adjacent to an n-xel in $\mathcal{T}$, and therefore $\mathbb{I}_1(Q_j) = \mathbb{I}_2(Q_j) = 0$. As $Q_j \in \mathbb{I}_2^{-1}[\{0\}]$ and $Y \subseteq Q_{j-1} \cap Q_j \subsetneq Q_j$ (which implies that $Y \subsetneq \partial Q_j$), it follows that $Y \in \mathbf{Coattach}(\mathcal{T}, \mathbb{I}_2)$. As Y is an arbitrary xel in $\mathbf{Coattach}(\mathcal{T}, \mathbb{I}_1)$, this shows that $\mathbf{Coattach}(\mathcal{T}, \mathbb{I}_1) \subseteq \mathbf{Coattach}(\mathcal{T}, \mathbb{I}_2)$. By a symmetrical argument, $\mathbf{Coattach}(\mathcal{T}, \mathbb{I}_2) \subseteq \mathbf{Coattach}(\mathcal{T}, \mathbb{I}_1)$. This proves assertion 1. Assertion 2 follows from assertion 1 and Theorem 1. Assertion 3 follows from assertion 2 and Theorem 2, because if $\mathcal{W}$ is any subset of $\mathcal{T}$ then the hypotheses of the lemma must still hold when $\mathcal{T}$, $\mathbb{I}_1$, and $\mathbb{I}_2$ are respectively replaced by $\mathcal{T} \setminus \mathcal{W}$, $\mathbb{I}_1 - \mathcal{W}$, and $\mathbb{I}_2 - \mathcal{W}$. □

Lemma 4. *Let* $\mathbf{K}$ *be an nD xel-complex, and let* $\mathcal{T}$ *be a nonempty finite collection of n-xels of* $\mathbf{K}$ *such that* $\bigcap \mathcal{T} \neq \emptyset$ *and there is no* $\mathcal{T}' \subsetneq \mathcal{T}$ *such that* $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$*. Then* $|\mathcal{T}| \leq n+1$*. Moreover, if* $|\mathcal{T}| = n+1$ *then* $\bigcap \mathcal{T}^*$ *is an* $(n+1-|\mathcal{T}^*|)$*-xel of* $\mathbf{K}$ *whenever* $\emptyset \neq \mathcal{T}^* \subseteq \mathcal{T}$*.*

Proof. Let $k = |\mathcal{T}| - 1$ and let $T^0, T^1, \ldots, T^k$ be an enumeration of the elements of $\mathcal{T}$. Since $\bigcap \mathcal{T} \neq \emptyset$, $\bigcap_{i=0}^{l} T^i$ is a xel of $\mathbf{K}$ for $0 \leq l \leq k$. Hence:

$$\dim(T^0) - \dim(\bigcap_{i=0}^{k} T^i) = \sum_{l=0}^{k-1} (\dim(\bigcap_{i=0}^{l} T^i) - \dim(\bigcap_{i=0}^{l+1} T^i)) \qquad (\star)$$

But we must have $\bigcap_{i=0}^{l} T^i \supsetneq \bigcap_{i=0}^{l+1} T^i$ for $0 \leq l \leq k-1$ (for if $\bigcap_{i=0}^{l} T^i = \bigcap_{i=0}^{l+1} T^i$ then $\bigcap(\mathcal{T} \setminus \{T^{l+1}\}) = \bigcap \mathcal{T}$), and so it follows from Property 2 that $\dim(\bigcap_{i=0}^{l} T^i) - \dim(\bigcap_{i=0}^{l+1} T^i) \geq 1$ for $0 \leq l \leq k-1$. Hence the right side of $(\star)$ is $\geq k$. Since the left side of $(\star)$ is $\leq \dim(T^0) = n$, we have that $n \geq k$, and so $|\mathcal{T}| = k+1 \leq n+1$.

Now suppose $|\mathcal{T}| = n+1$. Then $k = n$ and the left side of $(\star)$ is $\leq k$, so no term on the right exceeds 1 and we have that $\dim(\bigcap_{i=0}^{l} T^i) - \dim(\bigcap_{i=0}^{l+1} T^i) = 1$ for $0 \leq l \leq k-1$. Hence $\dim(\bigcap_{i=0}^{l} T^i) = n - l = (n+1) - |\{T^i \mid 0 \leq i \leq l\}|$ for $0 \leq l \leq k$, since $\dim(\bigcap_{i=0}^{0} T^i) = \dim(T^0) = n$. As this holds for any enumeration $T^0, T^1, \ldots, T^k$ of $\mathcal{T}$, the lemma is proved. □

Theorem 4 (Second Main Theorem). *Let* $\mathbf{K}$ *be an nD xel-complex, where* $1 \leq n \leq 4$*, and let* $\mathcal{T}$ *be a nonempty finite collection of n-xels of* $\mathbf{K}$*. Then:*

1. *If* $\bigcap \mathcal{T} = \emptyset$*, then there is no* $\mathbf{K}$*-image* $\mathbb{I}$ *such that* $\mathcal{T}$ *is MNCS in* $\mathbb{I}$*.*
2. *If there is some* $\mathcal{T}' \subsetneq \mathcal{T}$ *such that* $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$*, then there is no* $\mathbf{K}$*-image* $\mathbb{I}$ *such that* $\mathcal{T}$ *is MNCS in* $\mathbb{I}$*.*
3. *If* $\bigcap \mathcal{T} \neq \emptyset$ *and there is no* $\mathcal{T}' \subsetneq \mathcal{T}$ *such that* $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$*, and* $|\mathcal{T}| = n+1$*, then* $\mathcal{T}$ *is MNCS in a* $\mathbf{K}$*-image if and only if* $\mathcal{T}$ *is a strong foreground component of that* $\mathbf{K}$*-image.*
4. *If* $\bigcap \mathcal{T} \neq \emptyset$ *and there is no* $\mathcal{T}' \subsetneq \mathcal{T}$ *such that* $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$*, and* $|\mathcal{T}| \leq n$*, then there is a* $\mathbf{K}$*-image* $\mathbb{I}$ *such that* $\mathcal{T}$ *is MNCS in* $\mathbb{I}$ *and* $\mathcal{T}$ *is not a strong foreground component of* $\mathbb{I}$*.*

Proof. Let $k = |\mathcal{T}| - 1$, let $\mathcal{T} = \{X, T_1, \ldots, T_k\}$ and, for $1 \leq i \leq k$, write X_i for $X \cap T_i$.

We first prove assertions 1 and 2, and the "only if" part of assertion 3. For this purpose we may assume $k \neq 0$, as this is implied by the hypotheses of assertions 1, 2, and 3. Suppose there is a **K**-image $\mathbb{I}$ such that $\mathcal{T}$ is an MNCS set of 1's of $\mathbb{I}$. We will deduce that $\bigcap \mathcal{T} \neq \emptyset$ (which will prove assertion 1). We will further deduce that there is no set $\mathcal{T}' \subsetneq \mathcal{T}$ such that $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$ (which will prove assertion 2). Then we will prove the "only if" part of assertion 3 by showing that $\mathcal{T}$ must be a strong foreground component of $\mathbb{I}$ if $|\mathcal{T}| = n + 1$.

Let $P = X \cap \bigcup \mathbb{I}^{-1}[\{0\}] = \bigcup \mathbf{Coattach}(X, \mathbb{I})$. Then $\bigcup \mathbf{Coattach}(X, \mathbb{I} - \mathcal{W}) = P \cup \bigcup \{X_i \mid T_i \in \mathcal{W}\}$ for any subcollection $\mathcal{W}$ of $\{T_i \mid 1 \leq i \leq k\}$.

Since $\mathcal{T} = \{X, T_1, \ldots, T_k\}$ is MNCS in $\mathbb{I}$, it follows from Theorem 2 that X is cosimple in $\mathbb{I} - \mathcal{W}$ for every proper subcollection $\mathcal{W}$ of $\{T_i \mid 1 \leq i \leq k\}$, and that X is non-cosimple in $\mathbb{I} - \{T_i \mid 1 \leq i \leq k\}$. Hence, by Theorem 1, $P \cup \bigcup_{i=1}^{k} X_i = \bigcup \mathbf{Coattach}(X, \mathbb{I} - \{T_i \mid 1 \leq i \leq k\})$ is not contractible, but $P \cup \bigcup \{X_i \mid T_i \in \mathcal{W}\} = \bigcup \mathbf{Coattach}(X, \mathbb{I} - \mathcal{W})$ is contractible for every collection $\mathcal{W} \subsetneq \{T_i \mid 1 \leq i \leq k\}$. As a special case of the latter fact, P is contractible.

The collection of sets $\{P \cup \bigcup \{X_i \mid T_i \in \mathcal{W}\} \mid \mathcal{W} \subsetneq \{T_i \mid 1 \leq i \leq k\}\}$ includes the collection of sets $\{P \cup \bigcup \mathcal{S} \mid \mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}\}$. Hence $P \cup \bigcup \mathcal{S}$ is contractible whenever $\mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}$. Since this implies $P \cup \bigcup (\{X_j \mid 1 \leq j \leq k\} \setminus \{X_i\})$ is contractible for $1 \leq i \leq k$, and since we saw above that $P \cup \bigcup_{i=1}^{k} X_i$ is not contractible, none of the sets X_i is empty, and so each set $X_i = X \cap T_i$ is a xel of **K**. Thus we have established the following:

(a) P, X, and the family $(X_i)_{1 \leq i \leq k}$ satisfy the hypotheses of the Fundamental Lemma.
(b) P is contractible.
(c) $P \cup \bigcup_{i=1}^{k} X_i$ is not contractible.

Assertion 2 of the Fundamental Lemma now implies that $\bigcap \mathcal{T} = \bigcap_{i=1}^{k} X_i \neq \emptyset$. This proves assertion 1.

To prove assertion 2, we suppose there is a set $\mathcal{T}' \subsetneq \mathcal{T}$ such that $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$, and deduce a contradiction. We may assume without loss of generality that $T_1 \in \mathcal{T} \setminus \mathcal{T}'$. Then $\bigcap_{i=2}^{k} X_i = X \cap \bigcap_{i=2}^{k} T_i \subseteq \bigcap \mathcal{T}' = \bigcap \mathcal{T} = X \cap \bigcap_{i=1}^{k} T_i = \bigcap_{i=1}^{k} X_i$, which implies $\bigcap_{i=2}^{k} X_i = \bigcap_{i=1}^{k} X_i$. This and (a) – (c) above contradict assertion 4 of the Fundamental Lemma, and so we have established assertion 2.

To prove the "only if" part of assertion 3, we continue to suppose that $\mathcal{T}$ is MNCS in the **K**-image $\mathbb{I}$, but now also suppose that $|\mathcal{T}| = n + 1$ (so that $k = n$). We need to deduce that $\mathcal{T}$ is a strong foreground component of $\mathbb{I}$.

By assertions 1 and 2, $\bigcap \mathcal{T} \neq \emptyset$ and there is no set $\mathcal{T}' \subsetneq \mathcal{T}$ such that $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$. So Lemma 4 implies that, for any two distinct elements T and T' of $\mathcal{T}$, the intersection $T \cap T'$ is an $(n-1)$-xel of **K**. Hence $\mathcal{T}$ is strongly connected. It also follows from Lemma 4 that $\bigcap_{i=1}^{k} X_i = \bigcap \mathcal{T}$ is a 0-xel of **K**.

Now if $P \cap \bigcap_{i=1}^{k} X_i \neq \emptyset$ then $P \cap \bigcap_{i=1}^{k} X_i$ is the 0-xel $\bigcap_{i=1}^{k} X_i$, and so $P \cap \bigcap_{i=1}^{k} X_i$ is contractible, which contradicts assertion 3 of the Fundamental Lemma (in view of (a) – (c) above).

Hence we may assume $P \cap \bigcap_{i=1}^{k} X_i = \emptyset$. Then assertion 6 of the Fundamental Lemma implies that $\bigcup \mathbf{Coattach}(X, \mathbb{I} - \{T_i \mid 1 \leq i \leq k\}) = P \cup \bigcup_{i=1}^{k} X_i = \partial X$.

It follows that there is no n-xel Y of $\mathbf{K}$ such that Y is a 1 of $\mathbb{I} - \{T_i \mid 1 \leq i \leq k\}$ and $X \cap Y$ is an $(n-1)$-xel. (For if such an n-xel Y exists, and $Z = X \cap Y$, then (by Property 4) no point in $Z \setminus \partial Z$ lies on a 0 of $\mathbb{I} - \{T_i \mid 1 \leq i \leq k\}$ and all points of $Z \setminus \partial Z$ must lie in $\partial X \setminus \bigcup \mathbf{Coattach}(X, \mathbb{I} - \{T_i \mid 1 \leq i \leq k\})$.) Hence X is not strongly adjacent to any 1 of $\mathbb{I} - \mathcal{T}$. As X is an *arbitrary* element of $\mathcal{T}$ (and we already know $\mathcal{T}$ is strongly connected) it follows that $\mathcal{T}$ is a strong foreground component of $\mathbb{I}$. This proves the "only if" part of assertion 3.

It remains to establish the "if" part of assertion 3, and assertion 4. For this purpose we suppose that $\bigcap \mathcal{T} \neq \emptyset$, and that there is no set $\mathcal{T}' \subsetneq \mathcal{T}$ for which $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$. (To begin with, we do not suppose that $|\mathcal{T}| = n+1$.) We will define a $\mathbf{K}$-image $\mathbb{I}^*$, and show that $\mathcal{T}$ is MNCS in $\mathbb{I}^*$.

Let $\mathcal{H}$ be the set of all n-xels of $\mathbf{K}$ that intersect the xel $\bigcap \mathcal{T}$, and let $\overline{\mathcal{H}}$ be the set of all n-xels of $\mathbf{K}$ that do not intersect the xel $\bigcap \mathcal{T}$. Let $\mathbb{I}^*$ be the $\mathbf{K}$-image whose set of 1's is $\mathcal{H}$ (and whose set of 0's is $\overline{\mathcal{H}}$). We will show that $\mathcal{T}$ is MNCS in $\mathbb{I}^*$.

Now $\overline{\mathcal{H}}$ consists of the xels of dimension n in the set $\{C \in \mathbf{K} \mid C \cap \bigcap \mathcal{T} = \emptyset\}$. Moreover, condition 6 in the definition of a xel-complex implies that each xel of dimension $< n$ in $\{C \in \mathbf{K} \mid C \cap \bigcap \mathcal{T} = \emptyset\}$ is contained in an n-xel in $\overline{\mathcal{H}}$. Hence $\bigcup \overline{\mathcal{H}} = \bigcup\{C \in \mathbf{K} \mid C \cap \bigcap \mathcal{T} = \emptyset\}$. Therefore $\bigcup \mathbf{Coattach}(X, \mathbb{I}^*) = X \cap \bigcup \overline{\mathcal{H}} = \bigcup\{X \cap C \mid C \in \mathbf{K} \text{ and } C \cap \bigcap \mathcal{T} = \emptyset\} \bigcup\{D \in \mathbf{K} \mid D \subsetneq X \text{ and } D \cap \bigcap \mathcal{T} = \emptyset\}$.

Let $P = \bigcup \mathbf{Coattach}(X, \mathbb{I}^*)$. Then, for any $\mathcal{W} \subseteq \{T_i \mid 1 \leq i \leq k\}$, we have that $\bigcup \mathbf{Coattach}(X, \mathbb{I}^* - \mathcal{W}) = P \cup \bigcup\{X_i \mid T_i \in \mathcal{W}\}$. Now we observe that $\{X_i \mid T_i \in \mathcal{W}\}$ is a proper subset of $\{X_i \mid 1 \leq i \leq k\}$ whenever $\mathcal{W}$ is a proper subset of $\{T_i \mid 1 \leq i \leq k\}$. (This is because there cannot exist $j \neq j'$ for which $X_j = X_{j'}$. For if such j and j' existed then $X \cap T_j \cap T_{j'} = X_j \cap X_{j'} = X_j = X \cap T_j$, which would imply that $\bigcap \mathcal{T} = \bigcap(\mathcal{T} \setminus \{T_{j'}\})$, contrary to our hypothesis that there is no set $\mathcal{T}' \subsetneq \mathcal{T}$ for which $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$.) In view of this, and since X is an arbitrary element of $\mathcal{T}$, if we can show that the following statements (i) and (ii) are both true, then Theorems 1 and 2 will imply that $\mathcal{T}$ is indeed MNCS in $\mathbb{I}^*$:

(i) $P \cup \bigcup \mathcal{S}$ is contractible whenever $\mathcal{S} \subsetneq \{X_i \mid 1 \leq i \leq k\}$.
(ii) $P \cup \bigcup_{i=1}^{k} X_i$ is not contractible.

Recall that $P = \bigcup\{D \in \mathbf{K} \mid D \subsetneq X \text{ and } D \cap \bigcap \mathcal{T} = \emptyset\}$. If $k = 0$, then $\bigcap \mathcal{T} = X$, $P = \emptyset$, (i) is vacuously true, and (ii) is true.

Now suppose $k \neq 0$. Then condition 5 of the definition of a xel-complex implies that P is contractible, since $\bigcap \mathcal{T}$ is a nonempty proper subset of X. The intersection of any nonempty subcollection of $\{X_i \mid 1 \leq i \leq k\}$ is contractible, as it is nonempty (since $\bigcap \mathcal{T} \neq \emptyset$) and is therefore a xel of $\mathbf{K}$. Now let $\mathcal{S}'$ be any nonempty proper subcollection of $\{X_i \mid 1 \leq i \leq k\}$. Then $\bigcap \mathcal{T}$ is a nonempty proper subset of $\bigcap \mathcal{S}'$, since there is no set $\mathcal{T}' \subsetneq \mathcal{T}$ such that $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$. Hence $P \cap \bigcap \mathcal{S}' \bigcup\{E \in \mathbf{K} \mid E \subsetneq \bigcap \mathcal{S}' \text{ and } E \cap \bigcap \mathcal{T} = \emptyset\}$ is contractible, by condition 5 of the definition of a xel-complex.

The observations in the preceding paragraph imply that, if $k \neq 0$, then the intersection of any nonempty proper subcollection of $\{X_i \mid 1 \leq i \leq k\} \cup \{P\}$ is contractible. It follows, by Corollary 1 of Lemma 1, that the union of any nonempty proper subcollection of $\{X_i \mid 1 \leq i \leq k\} \cup \{P\}$ is contractible. This proves (i). Corollary 1 also tells us that $P \cup \bigcup_{i=1}^{k} X_i = \bigcup(\{X_i \mid 1 \le i \le k\} \cup \{P\})$ is contractible if and only if $\bigcap(\{X_i \mid 1 \le i \le k\} \cup \{P\}) = P \cap \bigcap_{i=1}^{k} X_i$ is contractible. But $P \cap \bigcap_{i=1}^{k} X_i = P \cap \bigcap \mathcal{T} = \emptyset$ is not contractible, and so we have proved (ii). Thus we have shown that $\mathcal{T}$ is MNCS in $\mathbb{I}^*$.

Now suppose, again, that $|\mathcal{T}| = n+1$. Since $\mathcal{T}$ is MNCS in $\mathbb{I}^*$, the "only if" part of assertion 3 implies that $\mathcal{T}$ must be a strong foreground component of $\mathbb{I}^*$, and so it follows from assertion 3 of Lemma 3 that $\mathcal{T}$ is MNCS in any **K**-image of which $\mathcal{T}$ is a strong foreground component. This establishes the "if" part of assertion 3.

Finally, we suppose, instead, that $|\mathcal{T}| \le n$ (so that $k \le n-1$), and complete the proof of assertion 4 by deducing that $\mathcal{T}$ is not a strong foreground component of $\mathbb{I}^*$. First of all, we claim that $\bigcup \mathbf{Coattach}(X, \mathbb{I}^* - \{T_i \mid 1 \le i \le k\}) = P \cup \bigcup_{i=1}^{k} X_i \subsetneq \partial X$. Recall that $P = \bigcup\{D \in \mathbf{K} \mid D \subsetneq X \text{ and } D \cap \bigcap \mathcal{T} = \emptyset\}$. If $k = 0$ then $\bigcap \mathcal{T} = X$ and $P \cup \bigcup_{i=1}^{k} X_i = P = \emptyset$, so that our claim is valid. If $k \neq 0$ then, since P is contractible (as we observed earlier), and since $\bigcap \mathcal{T} \neq \emptyset$, $P \cap \bigcap \mathcal{T} = \emptyset$, and $k \le n-1$, the validity of our claim follows from assertion 6 of the Fundamental Lemma.

Let p be any point in $\partial X \setminus \bigcup \mathbf{Coattach}(X, \mathbb{I}^* - \{T_i \mid 1 \le i \le k\})$. Then (by Properties 1 and 6) there must exist an $(n-1)$-xel $Z \subsetneq \partial X$ such that $p \in Z$. Since $p \notin \bigcup \mathbf{Coattach}(X, \mathbb{I}^* - \{T_i \mid 1 \le i \le k\})$, we also have that $Z \not\subseteq \bigcup \mathbf{Coattach}(X, \mathbb{I}^* - \{T_i \mid 1 \le i \le k\})$, and so the n-xel Y of **K** such that $X \cap Y = \partial X \cap \partial Y = Z$ (which must exist, by Property 4) is a 1 of $\mathbb{I}^* - \{T_i \mid 1 \le i \le k\}$. Hence X is strongly adjacent to a 1 of $\mathbb{I}^* - \{T_i \mid 1 \le i \le k\}$ and $\mathcal{T}$ is not a strong foreground component of $\mathbb{I}^*$. This completes the proof. $\square$

Note that, in view of Lemma 4, every nonempty finite collection $\mathcal{T}$ of n-xels of **K** must satisfy the hypotheses of one of the four assertions of Theorem 4.

8 Concluding Remarks

We say that a set $\mathcal{T}$ of n-dimensional xels of an n-dimensional xel-complex *can be minimal non-simple* (*can be minimal non-cosimple*) if there exists a binary image in which $\mathcal{T}$ is a minimal non-simple (minimal non-cosimple) set of 1's. We say that $\mathcal{T}$ *can be minimal non-simple* (*minimal non-cosimple*) *without being a weak* (*strong*) *foreground component* if there exists a binary image in which $\mathcal{T}$ is a proper subset of a weak (strong) foreground component and $\mathcal{T}$ is a minimal non-simple (minimal non-cosimple) set.

This paper has determined just which sets of xels can be minimal non-simple, just which sets can be minimal non-cosimple, and just which sets can be minimal non-simple (minimal non-cosimple) without being a weak (strong) foreground component, in arbitrary xel-complexes of dimension ≤ 4. A number of earlier papers [4,5,6,10,11,15,19] have solved these problems for particular xel-complexes—specifically, the 2D, 3D, and 4D cubical, 2D hexagonal, and 3D face-centered-cubical complexes. This paper generalizes that earlier work.

We have established that, for $n \le 4$, a nonempty finite collection $\mathcal{T}$ of n-dimensional xels of an n-dimensional xel-complex can be minimal non-simple if and only if $\bigcap \mathcal{T} \neq \emptyset$. We have shown, too, that $\mathcal{T}$ can be minimal non-simple without being a weak foreground component if and only if $\bigcap \mathcal{T}$ is an m-dimensional xel for some $m \geq 1$.

We have further established that $\mathcal{T}$ can be minimal non-cosimple if and only if $\bigcap \mathcal{T} \neq \emptyset$ and there is no nonempty proper subcollection $\mathcal{T}'$ of $\mathcal{T}$ such that $\bigcap \mathcal{T}' = \bigcap \mathcal{T}$, and we have shown that $\mathcal{T}$ can be minimal non-cosimple without being a strong foreground component if and only if, in addition, $|\mathcal{T}| \leq n$.

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