Construction of Switching Components

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Abstract. Switching components play an important role investigating uniqueness of problems in discrete tomography. General projections and additive projections as well as switching components w.r.t. these projections are defined. Switching components are derived by combining other switching components.

The composition of switching components into minimal ones in case of additive projections is proved. We also prove, that the product of minimal switching components is also minimal.

1 Introduction

In many scenarios of discrete tomography the so-called switching components play an important role. In the current paper we consider switching components from a more general point of view. The paper is intended to serve as a starting point for further description or even construction of the set of all switching components w.r.t. a given projection. This may be done via composing switching components into bigger ones. For projections derived from lower dimensional projections – let them be additive, as in the unabsorbed or absorbed case, or even more general – we can decide whether a switching component is minimal. This will possibly give the opportunity to describe or construct the set of all the switching components of this kind of projections.

2 Switching Components

2.1 Generalized Projections

Consider a finite or countably infinite set $\mathcal{L}$ (for example an integer lattice) and a set $\mathcal{R}$ containing the so-called rays of $\mathcal{L}$. This set may, but need not be a set of certain subsets of $\mathcal{L}$. A ray of $\mathcal{L}$ will be denoted by R, the *set of the rays* is $\mathcal{R}(\mathcal{L})$.

Given a (finite or countably infinite) set $\mathcal{L}$ and a set of rays $\mathcal{R} = \mathcal{R}(\mathcal{L})$ on $\mathcal{L}$. Furthermore, let $\mathbb{F}$ be a (commutative) ring, for example the ring of the integer, real or complex numbers.

Definition 1. *A function $\mathcal{P}$ is called a generalized ($\mathbb{F}$-valued) projection of the pair $(\mathcal{L}, \mathcal{R}(\mathcal{L}))$, if for every (finite) subset $G \subset \mathcal{L}$ and for every ray $R \in \mathcal{R}(\mathcal{L})$ $\mathcal{P}$*

A. Kuba, L.G. Nyúl, and K. Palágyi (Eds.): DGCI 2006, LNCS 4245, pp. 157–168, 2006.

returns a value $\mathcal{P}(G, R) \in \mathbb{F}$*. In order to emphasize the projection as a function on the subsets of the set* $\mathcal{L}$ *we will also write*

$$\mathcal{P}^{(R)}(G) \;:=\; \mathcal{P}(G, R)\ .$$

Definition 2. *A generalized projection* $\mathcal{P}$ *is called an additive projection, if*

$$\mathcal{P}^{(R)}(G_1 \cup G_2) \;=\; \mathcal{P}^{(R)}(G_1) \;+\; \mathcal{P}^{(R)}(G_2) \tag{1}$$

$$\begin{array}{lll} \text{for all rays} & R \in \mathcal{R}(\mathcal{L}) & \\ \text{for all (finite) sets} & G_1, G_2 \subset \mathcal{L} & \text{with } G_1 \cap G_2 = \emptyset\ . \end{array}$$

Let $\mathcal{P}$ an additive projection. Considering the projection on an arbitrary ray R as above, we then get for a (finite) subset G of $\mathcal{L}$

$$\mathcal{P}^{(R)}(G) \;=\; \sum_{g \in G} \mathcal{P}^{(R)}(\{g\}) \;=\; \sum_{g \in \mathcal{L}} \chi_G(g) \cdot \omega_g^{(R)}\ . \tag{2}$$

with the weights $\omega_g^{(R)} := \mathcal{P}^{(R)}(\{g\}) \in \mathbb{F}$.

2.2 Product of Projections

Let $(\mathcal{L}_1, \mathcal{R}_1(\mathcal{L}_1), \mathcal{P}_1)$ and $(\mathcal{L}_2, \mathcal{R}_2(\mathcal{L}_2), \mathcal{P}_2)$ be two projections. $\mathcal{L} := \mathcal{L}_1 \times \mathcal{L}_2$.

Let a ray on the base set $\mathcal{L}$ be defined either as a pair (g_1, r_2) with $g_1 \in \mathcal{L}_1$ and $r_2 \in \mathcal{R}_2(\mathcal{L}_2)$ or as a pair (r_1, g_2) with $r_1 \in \mathcal{R}_1(\mathcal{L}_1)$ and $g_2 \in \mathcal{L}_2$. Hence, the rays on $\mathcal{L}$ form the set

$$\mathcal{R}(\mathcal{L}) \;=\; (\,\mathcal{L}_1 \times \mathcal{R}_2(\mathcal{L}_2)\,)\ \dot{\cup}\ (\,\mathcal{R}_1(\mathcal{L}_1) \times \mathcal{L}_2\,)\ .$$

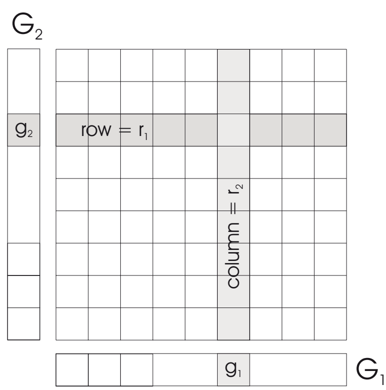

Fig. 1. Rays on the Cartesian Product

Let the transections of a subset $G \subset \mathcal{L}$ be defined by

$$T_2(G\,|\,g_1) \;:=\; \{g \in \mathcal{L}_2 \,|\, (g_1, g) \in G\}$$

and

$$T_1(G\,|\,g_2) \;:=\; \{g \in \mathcal{L}_1 \,|\, (g, g_2) \in G\}\ .$$

Definition 3. *Let*

$$\begin{aligned} \forall\ (g_1, r_2) \in \mathcal{L}_1 \times \mathcal{R}_2(\mathcal{L}_2) \quad \mathcal{P}^{(g_1,r_2)}(G) \ &:= \ \mathcal{P}_2^{(r_2)}(T_2(G\,|\,g_1)) \\ \forall\ (r_1, g_2) \in \mathcal{R}_1(\mathcal{L}_1) \times \mathcal{L}_2 \quad \mathcal{P}^{(r_1,g_2)}(G) \ &:= \ \mathcal{P}_1^{(r_1)}(T_1(G\,|\,g_2))\ . \end{aligned} \tag{3}$$

We call the triple $(\mathcal{L}, \mathcal{R}(\mathcal{L}), \mathcal{P})$ *the product of the projections* $(\mathcal{L}_1, \mathcal{R}_1(\mathcal{L}_1), \mathcal{P}_1)$ *and* $(\mathcal{L}_2, \mathcal{R}_2(\mathcal{L}_2), \mathcal{P}_2)$ *and denote it by* $\mathcal{P}_1 \times \mathcal{P}_2$.

2.3 Generalized Switching Components

Let $\mathcal{L}$ be a given set, $\mathcal{R}(\mathcal{L})$ a set of rays in $\mathcal{L}$ and $\mathcal{P}$ a given projection. Let G be a (finite) subset of $\mathcal{L}$ and c and c^S two functions on G with values in $\{0\,;1\}$.

Definition 4. *We say a function* ε *on* $\mathcal{L}$ *switchable w.r.t. the pair* (G, c) *if*

$$\forall\ g \in G\ :\ \varepsilon(g) = c(g)\ .$$

Definition 5. *If* ε *is switchable w.r.t.* (G, c)*, then we say the function* $\varepsilon^S = \varepsilon^S_{(G,\,c,\,c^S)}$ *that we get by replacing the values of* ε *by those of* c^S *on the set* G*, the switching result of* ε *with respect to the triplet* (G, c, c^S).

$$\varepsilon^S(g) = \varepsilon^S_{(G,\,c,\,c^S)}(g) := \begin{cases} c^S(g) \text{ if } g \in G \\ \varepsilon(g) \quad \text{otherwise} \end{cases} . \tag{4}$$

Definition 6. *The triplet* (G, c, c^S) *is said a switching component with respect to the projection function* $\mathcal{P}$*, if whenever* ε *is switchable w.r.t.* (G, c) *the projection values* $\mathcal{P}\varepsilon$ *and* $\mathcal{P}\varepsilon^S_{(G,\,c,\,c^S)}$ *are identical.*

$$\forall\ R \in \mathcal{R}(\mathcal{L})\ :\ \mathcal{P}^{(R)}(\varepsilon^S_{(G,\,c,\,c^S)})\ =\ \mathcal{P}^{(R)}(\varepsilon)\ . \tag{5}$$

For a switching component $S = (G, c, c^S)$ we call the set G the *domain* of S and denote it by $G = dom(S)$.

Lemma 1

(i) *For every set* $G \subseteq \mathcal{L}$ *and every function* $c : G \to \{0; 1\}$ *the triplet* (G, c, c) *is a switching component of each projection* $\mathcal{P}$.

(ii) *If* (G, c, c^S) *is a switching component with respect to the projection function* $\mathcal{P}$*, then* (G, c^S, c) *is also a switching component.*

(iii) *If* $(G, c, c^{(1)})$ *and* $(G, c^{(1)}, c^{(2)})$ *are switching components with respect to the projection function* $\mathcal{P}$ *then* $(G, c, c^{(2)})$ *is also a switching component.*

Proof

The proof is evident and can be omitted. □

For a switching component $S = (G, c, c^S)$ we say the switching component (G, c^S, c) the *switched switching component* and denote it by $S^{sw} = (G, c^S, c)$.

The empty switching component is denoted by $E = (\emptyset, \emptyset, \emptyset)$.

Definition 7. *Let $\mathcal{P}$ be a projection on $\mathcal{L}$ and (G, c, c^S) a switching component w.r.t. this projection. The switching component is called a minimal switching component if whenever (G', c', c'^S) is also a switching component w.r.t. $\mathcal{P}$ and G' is a subset of G and $c|_{G'} = c'$ and $c^S|_{G'} = c'^S$ then $G' = G$ or $G' = \emptyset$.*

Lemma 2. *If $\mathcal{P}$ is an additive projection on $\mathcal{L}$ and $S = (G, c, c^S)$ is a minimal switching component w.r.t. $\mathcal{P}$, then*

$$\forall\ g \in G\ :\ c(g) \neq c^S(g)\ .$$

Proof
Let $G_e = \{g \in G \,|\, c(g) = c^S(g)\}$. Consider the set $G - G_e$ and the restrictions of c and c^S to this set. Let R be an arbitrary ray of $\mathcal{R}(\mathcal{L})$ and $\varepsilon : \mathcal{L} \to \{0\,;1\}$ a function on $\mathcal{L}$ with values in $\{0\,;1\}$. Suppose that ε is switchable w.r.t. $(G-G_e, c)$.

Since $\mathcal{P}$ is additive we can calculate the projections value $\mathcal{P}^{(R)}$ of ε on a ray $R \in \mathcal{R}$ as follows

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{G^c}(\varepsilon) + \mathcal{P}_{G-G_e}(\varepsilon) + \mathcal{P}_{G_e}(\varepsilon)\ ,$$

where $\mathcal{P}_G(f) := \mathcal{P}^{(R)}(f \cdot \chi_G)$ for a set G and a function f, and $G^c := \mathcal{L} - G$.

ε equals c on $G - G_e$, i.e.

$$\begin{aligned}\mathcal{P}^{(R)}(\varepsilon) &= \mathcal{P}_{G^c}(\varepsilon) + \mathcal{P}_{G-G_e}(c) + \mathcal{P}_{G_e}(c) - \mathcal{P}_{G_e}(c) + \mathcal{P}_{G_e}(\varepsilon) \\ &= \mathcal{P}_{G^c}(\varepsilon) + \mathcal{P}_{G}(c) - \mathcal{P}_{G_e}(c) + \mathcal{P}_{G_e}(\varepsilon)\ .\end{aligned}$$

Since (G, c, c^S) is a switching component we have $\mathcal{P}_G(c) = \mathcal{P}_G(c^S)$. Additionally, c and c^S are identical on G_e, thus $\mathcal{P}_{G_e}(c) = \mathcal{P}_{G_e}(c^S)$ also holds. Replacing these expressions we get

$$\begin{aligned}\mathcal{P}^{(R)}(\varepsilon) &= \mathcal{P}_{G^c}(\varepsilon) + \mathcal{P}_{G}(c^S) - \mathcal{P}_{G_e}(c^S) + \mathcal{P}_{G_e}(\varepsilon) \\ &= \mathcal{P}_{G^c}(\varepsilon) + \mathcal{P}_{G-G_e}(c^S) + \mathcal{P}_{G_e}(\varepsilon)\ ,\end{aligned}$$

which means that replacing the values of c on $G - G_e$ by those of c^S already results in the same projection value, i.e. $(G - G_e, c|_{G-G_e}, c^S|_{G-G_e})$ is also a switching component w.r.t. $\mathcal{P}$. Since (G, c, c^S) is minimal, $G_e = \emptyset$. This is what we wanted to show. □

3 Deriving Switching Components

In the following we are going to derive switching components from other, known switching components.

3.1 Composition of Switching Components

Let $\mathcal{P}$ be a given projection, $S_1 = (G_1, c_1, c_1^S)$ and $S_2 = (G_2, c_2, c_2^S)$ two switching components with respect to $\mathcal{P}$.

Definition 8. *If the functions c_1^S and c_2 are identical on the set $G_1 \cap G_2$, i.e.*

$$\forall\ g \in G_1 \cap G_2 \ :\ c_1^S(g) \ = \ c_2(g) \ ,$$

then the two switching components are called composible.

Note, that the composible relation betwen switching components is not symmetric.

Lemma 3. *Suppose that S_1 and S_2 are two composible switching components. Defining c and c^S as*

$$c(g) \ = \ \begin{cases} c_1(g) & \text{if } g \in G_1 \\ c_2(g) & \text{if } g \in G_2 - G_1 \end{cases}$$

and

$$c^S(g) \ = \ \begin{cases} c_1^S(g) & \text{if } g \in G_1 - G_2 \\ c_2^S(g) & \text{if } g \in G_2 \end{cases} \ ,$$

the triplet $(G_1 \cup G_2, c, c^S)$ is a switching component with respect to the projection $\mathcal{P}$.

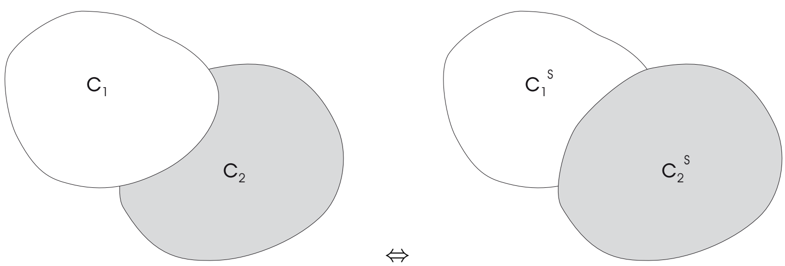

Fig. 2. The composition of two switching components

Proof
Let $R \in \mathcal{R}(\mathcal{L})$, and ε a function $\mathcal{L} \rightarrow \{0\,;1\}$ switchable w.r.t to $(G_1 \cup G_2, c)$.

Because of the definition of c, ε is identical to c_1 on G_1. Thus, the switching component (G_1, c_1, c_1^S) can be applied, and as the result we get the following

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}^{(R)}(\varepsilon_1) \ ,$$

where

$$\varepsilon_1(g) \ = \ \begin{cases} c_1^S(g) & \text{if } g \in G_1 \\ \varepsilon(g) & \text{otherwise} \end{cases} \ .$$

Now, (G_2, c_2, c_2^S) can be applied, since c_1^S and c_2 are identical on the set $G_1 \cap G_2$ and ε and also ε_1 are equal to c_2 on $G_2 - G_1$ and we have

$$\mathcal{P}^{(R)}(\varepsilon_1) \ = \ \mathcal{P}^{(R)}(\varepsilon_2) \ ,$$

where

$$\varepsilon_2(g) \;=\; \begin{cases} c_2^S(g) & \text{if } g \in G_2 \\ \varepsilon_1(g) & \text{otherwise} \end{cases} \;=\; \begin{cases} c_2^S(g) & \text{if } g \in G_2 \\ c_1^S(g) & \text{if } g \in G_1 - G_2 \\ \varepsilon(g) & \text{otherwise} \end{cases} .$$

This is the function after applying $(G_1 \cup G_2, c, c^S)$ and, hence, the projection value is the same. □

Definition 9. *Given two composible switching components S_1 and S_2. The switching component constructed in Lemma 3 is called the composition of the switching components, and we write*

$$(G_1 \cup G_2, c, c^S) \;=\; (G_1, c_1, c_1^S) \;\circ\; (G_2, c_2, c_2^S)$$

Lemma 4. *Let $\mathcal{L}$ be a set, $\mathcal{R}(\mathcal{L})$ a set of rays of $\mathcal{L}$ and $\mathcal{P}$ a projection on $(\mathcal{L}, \mathcal{R}(\mathcal{L}))$. Let $S = (G, c, c^S)$, $S_1 = (G_1, c_1, c_1^S)$, $S_2 = (G_2, c_2, c_2^S)$ and $S_3 = (G_3, c_3, c_3^S)$ be switching components w.r.t. $(\mathcal{L}, \mathcal{R}(\mathcal{L}), \mathcal{P})$. For the composition of switching components w.r.t. $(\mathcal{L}, \mathcal{R}(\mathcal{L}), \mathcal{P})$ the following properties hold.*

(i) S and the empty switching component E are composible and

$$S \circ E \;=\; E \circ S \;=\; S \;.$$

(ii) The switching components S and S^{sw} as well as S^{sw} and S are composible and

$$S \circ S^{sw} \;=\; (G, c, c) \quad \text{and} \quad S^{sw} \circ S \;=\; (G, c^S, c^S)$$

(iii) If S_1 and S_2 are composible, then S_2^{sw} and S_1^{sw} are also composible and

$$S_2^{sw} \circ S_1^{sw} \;=\; (S_1 \circ S_2)^{sw} \;.$$

(iv) If S_1 and S_2 are composible and $S_1 \circ S_2$ and S_3 are also composible, then S_2 and S_3 are composible as well as S_1 and $S_2 \circ S_3$ and

$$(S_1 \circ S_2) \circ S_3 \;=\; S_1 \circ (S_2 \circ S_3) \;.$$

Proof
The proof of these properties follow immediately from the definition. □

3.2 Symmetric Composition of Switching Components

In the following, let $\mathcal{P}$ be an additive projection. $S_1 = (G_1, c_1, c_1^S)$ and $S_2 = (G_2, c_2, c_2^S)$ two switching components w.r.t. $\mathcal{P}$.

Definition 10. *The switching components S_1 and S_2 are called symmetric composible, if*

$$\forall\; g \in G_1 \cap G_2 \;:\; c_1^S(g) = c_2(g) \quad \text{and} \quad c_2^S(g) = c_1(g) \;.$$

Note, that the symmetric composible relation betwen switching components is a symmetric one.

Lemma 5. *Assume, that S_1 and S_2 are symmetric composible. Define c and c^S on $G_1 \triangle G_2$, the symmetric difference of G_1 and G_2, as*

$$c(g) \ := \ \begin{cases} c_1(g) & \text{if } g \in G_1 - G_2 \\ c_2(g) & \text{if } g \in G_2 - G_1 \end{cases}$$

and

$$c^S(g) \ := \ \begin{cases} c_1^S(g) & \text{if } g \in G_1 - G_2 \\ c_2^S(g) & \text{if } g \in G_2 - G_1 \end{cases}$$

Then the triplet $(G_1 \triangle G_2, c, c^S)$ is a switching component with respect to the projection $\mathcal{P}$.

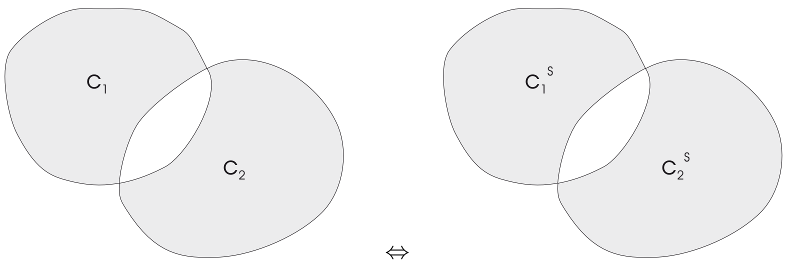

Fig. 3. The symmetric composition of two switching components

Proof
Let $R \in \mathcal{R}(\mathcal{L})$ be a ray, and $\varepsilon : \mathcal{L} \to \{0\,;1\}$ a function switchable w.r.t. $(G_1 \triangle G_2, c)$.

For a set G and a function f, we denote

$$\mathcal{P}_G(f) := P^{(R)}(f \cdot \chi_G) \ .$$

G^c denotes the complement of the set G w.r.t. to the set $\mathcal{L}$, i.e. $G^c = \mathcal{L} - G$.

Since the projection $\mathcal{P}$ is additive and ε equals c_1 and c_2 on $G_1 - G_2$ and $G_2 - G_1$ respectively, for the projection of ε we then have

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{(G_1 \cup G_2)^c}(\varepsilon) + \mathcal{P}_{G_1 - G_2}(c_1) + \mathcal{P}_{G_1 \cap G_2}(\varepsilon) + \mathcal{P}_{G_2 - G_1}(c_2) \ .$$

From the additivity of $\mathcal{P}$ we have

$$\mathcal{P}_{G_1 - G_2}(c_1) = \mathcal{P}_{G_1}(c_1) - \mathcal{P}_{G_1 \cap G_2}(c_1)$$

and, hence,

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{(G_1 \cup G_2)^c}(\varepsilon) + \mathcal{P}_{G_1}(c_1) - \mathcal{P}_{G_1 \cap G_2}(c_1) + \mathcal{P}_{G_1 \cap G_2}(\varepsilon) + \mathcal{P}_{G_2 - G_1}(c_2) \ .$$

(G_1, c_1, c_1^S) is a switching component w.r.t. $\mathcal{R}$, that's why we can replace c_1 by c_1^S

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{(G_1 \cup G_2)^c}(\varepsilon) + \mathcal{P}_{G_1}(c_1^S) - \mathcal{P}_{G_1 \cap G_2}(c_1) + \mathcal{P}_{G_1 \cap G_2}(\varepsilon) + \mathcal{P}_{G_2 - G_1}(c_2) \ .$$

Since $\mathcal{P}$ is additive and $c_1^S = c_2$ on the set $G_1 \cap G_2$

$$\mathcal{P}_{G_1}(c_1^S) = \mathcal{P}_{G_1-G_2}(c_1^S) + \mathcal{P}_{G_1\cap G_2}(c_1^S) = \mathcal{P}_{G_1-G_2}(c_1^S) + \mathcal{P}_{G_1\cap G_2}(c_2)$$

and

$$\mathcal{P}_{G_2-G_1}(c_2) + \mathcal{P}_{G_1\cap G_2}(c_2) = \mathcal{P}_{G_2}(c_2)$$

and, thus,

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{(G_1\cup G_2)^c}(\varepsilon) + \mathcal{P}_{G_1-G_2}(c_1^S) - \mathcal{P}_{G_1\cap G_2}(c_1) + \mathcal{P}_{G_1\cap G_2}(\varepsilon) + \mathcal{P}_{G_2}(c_2) \ .$$

From (G_2, c_2, c_2^S) also being a switching component w.r.t. $\mathcal{P}$ we get

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{(G_1\cup G_2)^c}(\varepsilon) + \mathcal{P}_{G_1-G_2}(c_1^S) - \mathcal{P}_{G_1\cap G_2}(c_1) + \mathcal{P}_{G_1\cap G_2}(\varepsilon) + \mathcal{P}_{G_2}(c_2^S) \ .$$

From the additivity of $\mathcal{P}$ again and from $c_2^S = c_1$ on the set $G_1 \cap G_2$ it follows

$$\mathcal{P}_{G_2}(c_2^S) = \mathcal{P}_{G_2-G_1}(c_2^S) + \mathcal{P}_{G_1\cap G_2}(c_2^S) = \mathcal{P}_{G_2-G_1}(c_2^S) + \mathcal{P}_{G_1\cap G_2}(c_1)$$

and replacing this expression we get

$$\mathcal{P}^{(R)}(\varepsilon) \ = \ \mathcal{P}_{(G_1\cup G_2)^c}(\varepsilon) + \mathcal{P}_{G_1-G_2}(c_1^S) + \mathcal{P}_{G_1\cap G_2}(\varepsilon) + \mathcal{P}_{G_2-G_1}(c_2^S) \ .$$

The expression on the right hand side is an expression for $\mathcal{P}^{(R)}(\varepsilon^S)$ and, hence, as the final result we have

$$\mathcal{P}^{(R)}(\varepsilon) = \mathcal{P}^{(R)}(\varepsilon^S) \ .$$

This is what we wanted to prove. □

Definition 11. *For two symmetric composible switching components S_1 and S_2, the switching component constructed in Lemma 5 is called the symmetric composition of the two switching components, and we write*

$$(G_1 \triangle G_2, c, c^S) \ = \ (G_1, c_1, c_1^S) \ \odot \ (G_2, c_2, c_2^S) \ .$$

The following properties of the symmetric composition hold.

Lemma 6. *Let $\mathcal{L}$ be a set, $\mathcal{R}(\mathcal{L})$ be a set of rays on $\mathcal{L}$ and $\mathcal{P}$ be an additive projection on these rays. Furthermore, let $S = (G, c, c^S)$, $S_1 = (G_1, c_1, c_1^S)$, $S_2 = (G_2, c_2, c_2^S)$, and $S_3 = (G_3, c_3, c_3^S)$ switching components w.r.t. $(\mathcal{L}, \mathcal{R}(\mathcal{L}), \mathcal{P})$.*

(i) S and E are always symmetric composible and

$$S \odot E \ = \ E \odot S \ = \ S \ .$$

(ii) S and S^{sw} are symmetric composible and

$$S \odot S^{sw} \ = \ S^{sw} \odot S \ = \ E \ .$$

(iii) If $S_1 \odot S_2$ and S_3 are symmetric composible then S_1 and $S_2 \odot S_3$ are also symmetric composible and

$$(S_1 \odot S_2) \odot S_3 \;=\; S_1 \odot (S_2 \odot S_3) \;.$$

(iv) If S_1 and S_2 are symmetric composible, then

$$S_2 \odot S_1 \;=\; S_1 \odot S_2 \;.$$

Proof
The proof of these properties follows immediately from the definition. □

Proposition 1. *Let S be an arbitrary switching component w.r.t. an additive projection $\mathcal{P}$. There exists a constant switching component S_0 and a sequence of **minimal** switching components $\{S_i\}_{i=1}^N$ with pairwise disjunct domains with the symmetric composition being S*

$$S \;=\; S_0 \odot S_1 \odot \cdots \odot S_N$$

with

$$c\,|_{S_0} \;\equiv\; c^S\,|_{S_0} \quad \text{and} \quad dom(S_i) \cap dom(S_j) \;=\; \emptyset \quad \text{for } i \neq j \;.$$

Proof
This easily follows from the definitions. □

3.3 Product of Switching Components

Let $\mathcal{P}_1$ be a generalized projection w.r.t. $(\mathcal{L}_1, \mathcal{R}_1(\mathcal{L}_1))$ and $\mathcal{P}_2$ a generalized projection w.r.t. $(\mathcal{L}_2, \mathcal{R}_2(\mathcal{L}_2))$ and $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ their product as defined in Section 2.2.

Furthermore, let $S_1 = (G_1, c_1, c_1^S)$ and $S_2 = (G_2, c_2, c_2^S)$ be a switching component w.r.t. $\mathcal{P}_1$ and $\mathcal{P}_2$, resp.

The following figure gives an idea how to create the switching component on the set $G_1 \times G_2$ or on a subset of $G_1 \times G_2$ based on the two given switching components.

Let

$$G_1 \times_{(c_1,\,c_2)} G_2 \;:=\; G_1 \times G_2 - \{(g_1, g_2)\,|\, c_1(g_1) = c_1^S(g_1) \wedge c_2(g_2) = c_2^S(g_2)\} \;.$$

For $c_1(g_1) \neq c_1^S(g_1) \;\wedge\; c_2(g_2) \neq c_2^S(g_2)$ we define

$$c(g_1, g_2) = \begin{cases} 0 \;\text{ if } \; c_1(g_1) = c_2(g_2) \;\; \wedge \;\; c_1^S(g_1) = c_2^S(g_2) \\ 1 \;\text{ if } \; c_1(g_1) = c_2^S(g_2) \;\; \wedge \;\; c_1^S(g_1) = c_2(g_2) \end{cases}$$

and

$$c^S(g_1, g_2) = \begin{cases} 1 \;\text{ if } \; c_1(g_1) = c_2(g_2) \;\; \wedge \;\; c_1^S(g_1) = c_2^S(g_2) \\ 0 \;\text{ if } \; c_1(g_1) = c_2^S(g_2) \;\; \wedge \;\; c_1^S(g_1) = c_2(g_2) \end{cases} \;.$$

For the other cases we define

$$c(g_1, g_2) = \begin{cases} 0 & \text{if } c_1(g_1) = c_1^S(g_1) = 0 \ \vee \ c_2(g_2) = c_2^S(g_2) = 0 \\ 1 & \text{if } c_1(g_1) = c_1^S(g_1) = 1 \ \vee \ c_2(g_2) = c_2^S(g_2) = 1 \end{cases}$$

and

$$c^S(g_1, g_2) = \begin{cases} 0 & \text{if } c_1(g_1) = c_1^S(g_1) = 0 \ \vee \ c_2(g_2) = c_2^S(g_2) = 0 \\ 1 & \text{if } c_1(g_1) = c_1^S(g_1) = 1 \ \vee \ c_2(g_2) = c_2^S(g_2) = 1 \end{cases} .$$

Lemma 7. *The triple $(G_1 \times_{(c_1,\,c_2)} G_2, c, c^S)$ with the two functions c and c^S as defined above is a switching component w.r.t. the product projection $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$.*

Proof

Let $\varepsilon : A \to \{0\,;1\}$ an arbitry function for which

$$\forall\ (g_1, g_2) \in G_1 \times_{(c_1,\,c_2)} G_2 \ : \ \varepsilon(g_1, g_2) \ = \ c(g_1, g_2)$$

and we investigate the function ε^S with

$$\varepsilon^S(g_1, g_2) \ := \begin{cases} c^S(g_1, g_2) & \text{if } (g_1, g_2) \in G_1 \times_{(c_1,\,c_2)} G_2 \\ \varepsilon(g_1, g_2) & \text{otherwise} \end{cases}$$

We define

$$\forall\ g \in G_1 \ : \ \varepsilon_1^{(g_2)}(g) \ := \ \varepsilon(g, g_2)$$

and

$$\forall\ g \in G_2 \ : \ \varepsilon_2^{(g_1)}(g) \ := \ \varepsilon(g_1, g)$$

The appropriate projections $\mathcal{P}_1$ and $\mathcal{P}_2$ can be applied to the functions $\varepsilon_1^{(g_2)}$ and $\varepsilon_2^{(g_1)}$.

First, let $(g_1, r_2) \in \mathcal{L}_1 \times \mathcal{R}_2(\mathcal{L}_2)$. For $g_1 \notin G_1$ the following equality is trivial.

$$\mathcal{P}^{(g_1,r_2)}(\varepsilon^S) \ = \ \mathcal{P}_2^{(r_2)}([\varepsilon_2^S]^{(g_1)}) \ = \ \mathcal{P}_2^{(r_2)}(\varepsilon_2^{(g_1)}) \ = \ \mathcal{P}^{(g_1,r_2)}(\varepsilon)$$

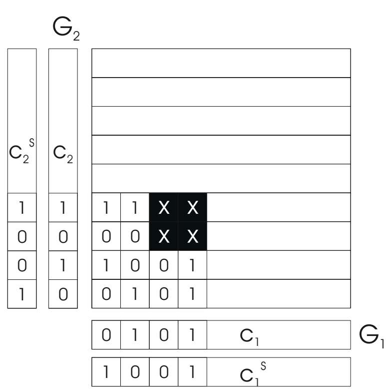

Fig. 4. Product of Switching Components

Let now $g_1 \in G_1$ and $c_1(g_1) \neq c_1^S(g_1)$. As we can see from the definition of c, in this case it holds that

$$c(g_1, g) \equiv c_2(g) \quad \text{and} \quad c^S(g_1, g) \equiv c_2^S(g) \quad \text{for } g \in G_2 \ ,$$

or

$$c(g_1, g) \equiv c_2^S(g) \quad \text{and} \quad c^S(g_1, g) \equiv c_2(g) \quad \text{for } g \in G_2 \ .$$

Since (G_2, c_2, c_2^S) is a switching component for the projection $\mathcal{P}_2$, again we have

$$\mathcal{P}^{(g_1,r_2)}(\varepsilon^S) \ = \ \mathcal{P}^{(g_1,r_2)}(\varepsilon)$$

If $g_1 \in G_1$ and $c_1(g_1) = c_1^S(g_1)$ we can see from the definition of c that $c^S(g_1, g) \equiv c(g_1, g)$ for $g \in G_2$, if $c_2(g_2) \neq c_2^S(g_2)$. Hence, $\varepsilon(g_1, g) \equiv \varepsilon^S(g_1, g)$ for these $g \in G_2$ and, again,

$$\mathcal{P}^{(g_1,r_2)}(\varepsilon^S) \ = \ \mathcal{P}^{(g_1,r_2)}(\varepsilon) \ .$$

The second case, namely if $(r_1, g_2) \in \mathcal{R}_1(\mathcal{L}_1) \times \mathcal{L}_2$, can be shown similarly. □

Definition 12. *We call the switching component $S = (G_1 \times_{(c_1,\, c_2)} G_2, c, c^S)$ the product of the switching components $S_1 = (G_1, c_1, c_1^S)$ and $S_2 = (G_2, c_2, c_2^S)$ and denote it by $S = S_1 \times S_2 = (G_1, c_1, c_1^S) \times (G_2, c_2, c_2^S)$.*

Proposition 2. *If $S_1 = (G_1, c_1, c_1^S)$ and $S_2 = (G_2, c_2, c_2^S)$ are two minimal (non-empty) switching components w.r.t. the generalized projections $\mathcal{P}_1$ and $\mathcal{P}_2$ then their product $S = S_1 \times S_2$ is also a minimal switching component w.r.t. to the product projection $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$.*

Proof
Suppose that $S = S_1 \times S_2$ is not a minimal switching component, i.e. there exists a (G', c', c'^S) switching component w.r.t. $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$ for which $\emptyset \subsetneq G' \subsetneq G = G_1 \times_{(c_1,\, c_2)} G_2$ and $c(S)|_{G'} = c'$ and $c(S)^S|_{G'} = c'^S$.

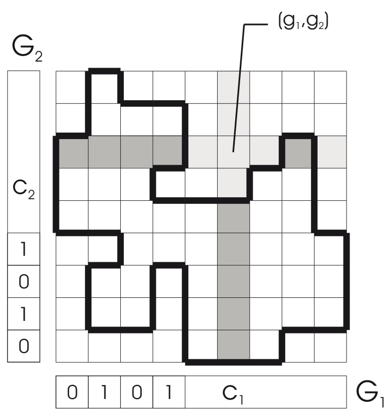

Fig. 5. Product of minimal switching Components

Let $(g_1, g_2) \in G_1 \times_{(c_1, c_2)} G_2 - G'$ and consider the sets $T_2(G' \mid g_1)$ and $T_1(G' \mid g_2)$.

Since $(g_1, g_2) \in G_1 \times_{(c_1, c_2)} G_2$ we have $c_1(g_1) \neq c_1^S(g_1)$ or $c_2(g_2) \neq c_2^S(g_2)$.

If $c_1(g_1) \neq c_1^S(g_1)$ then

$$c \mid T_2(G \mid g_1) \equiv c_2 \quad \text{or} \quad c \mid T_2(G \mid g_1) \equiv c_2^S$$

and, hence, $T_2(G' \mid g_1) = T_2(G \mid g_1)$ or $T_2(G' \mid g_1) = \emptyset$ because c_2 is minimal.

Similarly, if $c_2(g_2) \neq c_2^S(g_2)$ then

$$c \mid T_1(G \mid g_2) \equiv c_1 \quad \text{or} \quad c \mid T_1(G \mid g_2) \equiv c_1^S$$

and, hence, $T_1(G' \mid g_2) = T_1(G \mid g_2)$ or $T_1(G' \mid g_2) = \emptyset$ because c_1 is minimal.

Equality cannot hold because we supposed $(g_1, g_2) \in G_1 \times_{(c_1, c_2)} G_2 - G'$. Hence, the appropriate transection sets are all empty.

As one of the possibilties, let's suppose now, that $c_1(g_1) \neq c_1^S(g_1)$. In this case $T_2(G' \mid g_1) = \emptyset$. This implies that $\forall j \in G_2 : (g_1, j) \notin G'$ and so, whenever $j \in G_2 \wedge c_2(j) \neq c_2^S(j)$ we have $T_1(G' \mid j) = \emptyset$ also.

From this, on the other hand, it follows that whenever $i \in G_1 \wedge c_1(i) \neq c_1^S(i)$ we have $T_2(G' \mid i) = \emptyset$, and as the final consequence $G' = \emptyset$. This completes the proof. □

References

1. Balogh, E., Kuba, A., Del Lungo, A., and Nivat, M. : Reconstruction of binary matrices from absorbed projections. In: Braquelaire, A., Lachaud, J.-O., Vialard, A., eds.: Discrete Geometry for Computer Imagery, LNCS 2301. Springer-Verlag, Berlin Heidelberg New York (2002) 392–403
2. Kuba, A., Nivat, M. : Reconstruction of Discrete Sets with Absorption. Linear Algebra and Its Applications **339** (2000) 171–194
3. Kuba, A., Nivat, M. : A sufficient condition for non-uniqueness in binary tomography with absorption. Technical Report Vol. 1953. University of Szeged (2001)
4. Ryser, H. : Combinatorical properties of matrices of zeros and ones. Canad. J. Math. **9** (1957) 371–377