Upward Spirality and Upward Planarity Testing*

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Abstract. The upward planarity testing problem is known to be NP-hard. We describe an $O(n^4)$-time upward planarity testing and embedding algorithm for the class of digraphs that do not contain rigid triconnected components. We also present a new FPT algorithm that solves the upward planarity testing and embedding problem for general digraphs.

1 Introduction

An *upward planar drawing* of a planar digraph G is a crossing-free drawing of G such that the vertices of G are mapped to points of the plane and the edges of G are drawn as simple curves that are monotone in the upward direction. A digraph that admits an upward planar drawing is an *upward planar digraph*. Unfortunately, not all planar digraphs are upward planar. The digraph of Figure 1(a) is not upward planar independent of the choice of its planar embedding. The upward planarity testing problem asks whether a planar digraph G has an upward planar drawing.

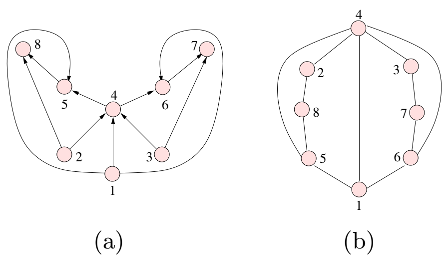

Fig. 1. (a) A digraph G that is not upward planar. (b) The underlying undirected graph of G is a series-parallel graph, i.e., it does not have rigid components.

The upward planarity testing problem is a classical subject of investigation in the graph drawing literature, and many papers have been devoted to this subject during the last decade. Bertolazzi et al. [1] present an $O(n^2)$-time algorithm that tests whether a digraph with a given planar embedding is upward planar. Garg and Tamassia [9] show that the problem in the variable embedding setting is NP-complete. Papakostas [14] presents an $O(n^2)$-time algorithm for testing the

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upward planarity of outerplanar digraphs. Hutton and Lubiw [13] describe an $O(n^2)$-time testing algorithm for digraphs that have a single source. Bertolazzi et al. [3] improve this last result by showing an optimal $O(n)$ testing algorithm for the same class of digraphs studied by Hutton and Lubiw. Bertolazzi et al. [2] describe a branch-and-bound testing algorithm for biconnected planar digraphs. Recently, fixed parameter tractable (FPT) algorithms have also been designed: Chan [4] presents an $O(t! \cdot 8^t \cdot n^3 + (2 \cdot t)^{3 \cdot 2^c} t! \cdot 8^t \cdot n)$-time algorithm where c and t are the number of cut-vertices and the number of triconnected components of G, respectively. Healy and Lynch [12] improve Chan's result by giving an $O(2^t \cdot t! \cdot n^2)$-time algorithm; in the same paper, Healy and Lynch describe a second upward planarity testing algorithm whose time complexity is $O(n^2 + k^4(2k+1)!)$, with $k = |E| - |V|$.

In this paper we describe a polynomial time algorithm and a new FPT algorithm for the upward planarity testing problem in the variable embedding setting. More precisely:

- We introduce and study the concept of *upward spirality* (Section 3), which is a measure of how much a component of a digraph is "rolled-up" in an upward planar drawing. A similar concept was introduced in the literature in the context of orthogonal drawings [6].
- We describe an $O(n^4)$-time upward planarity testing and embedding algorithm for the class of series-parallel digraphs, i.e. biconnected digraphs whose $SPQR$-tree does not have any R-node (Section 4). Our algorithm still runs in polynomial time even if the digraph is not biconnected and any block is a series-parallel digraph.
- Using the above results, we design a new FPT algorithm for upward planarity testing of general digraphs whose time complexity is $O(d^t \cdot n^3 + d \cdot t^2 \cdot n + d^2 \cdot n^2)$, where d is the maximum diameter of any split component of G and t is the number of (non-trivial) triconnected components of G (Section 5).

For reasons of space, all proofs are omitted and some sections are sketched. Details can be found in [8].

2 Preliminaries

We assume familiarity with basic concepts of graph drawing and graph planarity [5]. Let G be a planar digraph with a given planar embedding. A vertex of G is *bimodal* if the circular list of its incident edges can be partitioned into two (possibly empty) lists, one consisting of incoming edges and the other consisting of outgoing edges. If all vertices of G are bimodal then G and its embedding are called *bimodal*. Acyclicity and bimodality are necessary conditions for the upward planar drawability of an embedded planar digraph [1]. However, they are not sufficient conditions.

Let f be a face of an embedded planar bimodal digraph G and suppose that the boundary of f is visited clockwise if f is internal, and counterclockwise if f is external. Let $a = (e_1, v, e_2)$ be a triplet such that v is a vertex of the boundary of f and e_1, e_2 are incident edges of v that are consecutive on the boundary of f.

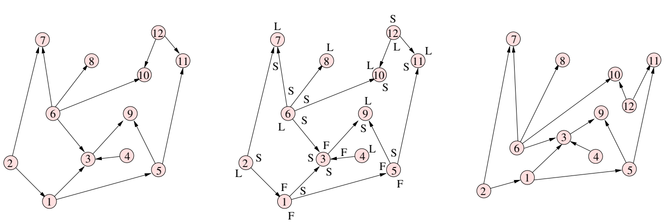
Triplet a is called an *angle of* f. Also, a is a *switch angle of* f if the direction of e_1 is opposite to the direction of e_2 (note that e_1 and e_2 may coincide if G is not biconnected). If e_1 and e_2 are both incoming in v, then a is a *sink-switch of* f; if they are both outgoing, a is a *source-switch of* f. A source or a sink of G is called a *switch vertex* of G; a vertex that is not a switch vertex is called an *internal vertex* of G.

Let Γ be an upward planar drawing of G and let a be an angle of G. Label a with a letter L (resp. a letter S) if it is a switch angle and has in Γ a value greater (resp. less) than π. Label a with a letter F if it is not a switch angle. The labeled embedded digraph U_G so obtained is called an *upward planar representation* of G, and can be viewed as the equivalence class of all (embedding preserving) upward planar drawings of G that induce the same angle labeling on G. Drawing Γ is also said to be an upward planar drawing that *preserves* U_G.

Now, consider an embedded planar digraph G and a labeling of its angles with labels L, S, and F. If v is a vertex of G, we denote by $L(v)$, $S(v)$, and $F(v)$ the number of angles at v that are labeled L, S, and F, respectively. The *degree* of v is defined as the number of angles at v, and is denoted as $deg(v)$. Also, if f is a face of G, $L(f)$, $S(f)$, and $F(f)$ denote the number of angles of f that are labeled L, S, and F, respectively. The following result is a restatement of the results in [1].

Lemma 1. *Let G be an acyclic planar bimodal embedded digraph with angle labels L, S, F. G and its labeling define an upward planar representation if and only if the following properties hold:* `(UP1)` *If v is a switch vertex of G then: $L(v) = 1$, $S(v) = deg(v) - 1$, $F(v) = 0$;* `(UP2)` *If v is not a switch vertex of G then: $L(v) = 0$, $S(v) = deg(v) - 2$, $F(v) = 2$;* `(UP3)` *If f is a face of G then: $L(f) = S(f) - 2$ if f is internal and $L(f) = S(f) + 2$ if f is external.*

From an upward planar representation U_G it is always possible to construct in linear time an upward planar drawing of G that preserves U_G, where each edge is drawn as a straight-line segment or as a polyline. Figure 2 shows an embedded planar digraph G, an upward planar representation U_G of G, and an upward planar drawing of G within U_G. Given an upward planar representation U_G, the angles labeled L, S, and F are called *large*, *small*, and *flat* angles, respectively. If



(a) (b) (c)

Fig. 2. (a) A planar embedded bimodal digraph G. (b) An upward planar representation U_G of G. (c) An upward planar drawing of G within U_G.

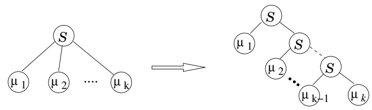


Fig. 3. Transformation of an *SPQR*-tree into its canonical form

G' is a subgraph of G, then G' has an *upward planar representation* $U_{G'}$ *induced by* U_G, which is defined as follows. Let $a = (e_1, v, e_2)$ be an angle of G', and let A be the counterclockwise sequence of angles of U_G between e_1 and e_2. Angle a in $U_{G'}$ is labeled: L if A either contains one large angle or two flat angles; F if A contains only one flat angle; S otherwise.

Let G be a biconnected graph and let $e = \{s, t\}$ be any edge of G, called *reference edge*. The *SPQR-tree* of G with respect to e describes a decomposition of G in terms of its triconnected components, and implicitly represents all planar embeddings of G with e on the external face. We assume familiarity with all formal definitions about *SPQR*-trees [7]. Suppose that G is given with an *st*-numbering of its vertices, such that the source and the sink of this numbering are the end-vertices s, t of the reference edge of G. If T is the *SPQR*-tree of G with respect to e, given any node μ of T, let u and v be the two poles of μ, so that u precedes v in the *st*-numbering. We call u and v the *first pole* and the *second pole* of the pertinent graph G_μ of μ. If G has a fixed planar embedding with reference edge e on the external face, the *right face* of G_μ is the face to the right of G_μ in G, while moving from u to v. The *left face* of G_μ is the face to the left of G_μ in G, while moving from u to v. The path on the right face of G_μ, going from u to v, is called the *right path* of G_μ. The path on the left face of G_μ, going from u to v, is called the *left path* of G_μ.

In the remainder of the paper, we consider *SPQR*-trees of directed graphs (digraphs) G. In this case, the computation of the decomposition tree is done exactly as for undirected graphs, by ignoring the orientation of the edges of G. Notice that, there is no connection between the orientation of the edges of G and the definition of first and second poles of the pertinent digraphs. In order to simplify the description of our upward planarity testing algorithm, we use *canonical SPQR-trees*, i.e., *SPQR*-trees where each S-node has always two children. A canonical *SPQR*-tree T of G can be constructed from an *SPQR*-tree of G by applying on every S-node the transformation illustrated in Figure 3. A canonical *SPQR*-tree of G has a number of nodes that is still linear in the number of vertices of G.

We say that a biconnected digraph G is a *series-parallel digraph* if its *SPQR*-tree only consists of Q-, S-, and P-nodes.

3 Upward Spirality

In the following, we assume that G is a biconnected digraph, T an *SPQR*-tree of G, U_G an upward representation of G, and G_μ the pertinent digraph of a node μ of T, with first pole u and second pole v.

Let $P =< v_1, e_1, v_2, \ldots, v_i, e_i, \ldots, e_{k-1}, v_k >$ be any simple (undirected) path (possibly a simple cycle) in G, and let U_P be the upward planar representation of P induced by U_G. Consider a vertex v_i ($i \in \{2, \ldots, k-1\}$) that is a switch of P, and denote by $a = (e_{i-1}, v_i, e_i), a' = (e_i, v_i, e_{i-1})$ the two angles at v_i in U_P. Walking on P from v_1 to v_k, we say that v_i is a *left turn* (resp. *right turn*) of U_P if a (resp. a') is large. We denote by $n(U_P)$ the number of right turns minus the number of left turns of U_P, and we call $n(U_P)$ the *turn number* of P in U_G, or simpler, the *turn number* of U_P. Similarly, if P is a simple cycle, i.e. $v_1 = v_k$, and we walk clockwise on P, we say that we encounter a left turn (resp. right turn) of U_P on any switches of P that has a large angle (resp. small angle) inside the cycle. Because of Lemma 1, if P is a simple cycle of U_G, then its turn number is $n(U_P) = 2$.

Denoted by $w \in \{u, v\}$ any of the two poles of G_μ, we want to classify w on the basis of the labeling of the angles at w in U_G. The label of the angle at w in the right face (resp. in the left face) of G_μ is called the *right inter-label* (resp. the *left inter-label*) of w. An *intra-label* of w is any label of an angle at w internal at G_μ. We assign to each angle label an integer weight, in such a way that labels S, F, and L have weight 0, 1, and 2, respectively. The *intra-labeling weight* of w is the sum of the weights of all intra-labels of w. From properties UP1 and UP2 of Lemma 1, the intra-labeling weight of w ranges from 0 to 2.

In U_G, we describe the angles labeling of the pole w of G_μ, by using a string $t_w = XY\lambda$, such that X is the left inter-label of w, Y is the right inter-label of w, and λ is the intra-labeling weight of w. We say that t_w is the *pole category* of w. We remark that, since U_G is an upward planar representation, not all categories $XY\lambda$ ($X, Y \in \{S, F, L\}, \lambda \in \{0, 1, 2\}$) are possible for a pole w of a pertinent digraph of G. Indeed, as also observed above, the sum of all angle labels at w must verify UP1 and UP2, and w must be bimodal. Hence, the following lemma immediately follows (see also Figure 4):

Lemma 2. *The possible pole categories of any pole of G_μ in U_G are: $SS0$, $SS1$, $SS2$, $SF0$, $SF1$, $FS0$, $FS1$, $FF0$, $SL0$, $LS0$.*

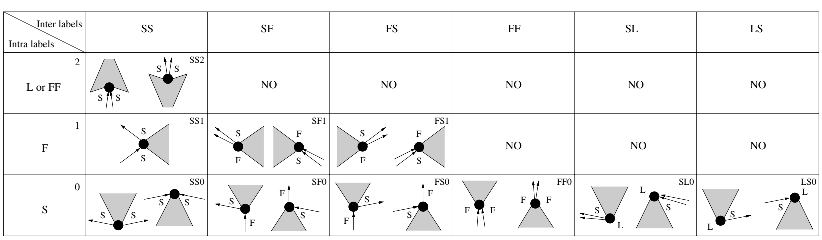


Fig. 4. Illustration of the pole categories for the first pole of a pertinent digraph within an upward planar representation. Grey portions are the internal parts of the pertinent digraph. The two labels around the pole are the inter-labels of the pole. The illustration for the second pole is symmetric.

In order to introduce the notion of upward spirality we need to identify two suitable vertices that we call the *left external vertex of* w, denoted as w_l, and the *right external vertex of* w, denoted as w_r, where w is still any of the two poles of G_μ. The right and the left external vertices of w are defined based on the pole category t_w of w, with respect to G_μ in U_G. More precisely, let e_l be the edge incident on w, that is on the left path of G_μ and that does not belong to G_μ; let e_r be the edge incident on w, that is on the right path of G_μ and that does not belong to G_μ. Also, let x be the end-vertex of e_l other than w and let y be the end-vertex of e_r other than w. The external vertices w_l and w_r of w are defined as follows: **(Case 1)** One of the following three subcases is verified: (i) $t_w \in \{SS0, SF0, FS0, FF0\}$; (ii) $t_w = SL0$ and w is the first pole of G_μ; (iii) $t_w = LS0$ and w is the second pole of G_μ. In this case $w_l = w_r = w$. **(Case 2)** One of the following two subcases is verified: (i) $t_w \in \{FS1, SF1, SS1, SS2\}$; (ii) $t_w = SL0$ and w is the second pole of G_μ; (iii) $t_w = LS0$ and w is the first pole of G_μ. In this case $w_l = x$ and $w_r = y$.

Let u_l, u_r be the left and the right external vertices of the first pole u of G_μ and let v_l, v_r be the left and the right external vertices of the second pole v of G_μ. Let P_{uv} be an (undirected) path from u to v in G_μ. The undirected path $P_l = (u_l, u) \cup P_{uv} \cup (v, v_l)$ is called a *left spine of* G_μ. The path $P_r = (u_r, u) \cup P_{uv} \cup (v, v_r)$ is called a *right spine of* G_μ. For example, the left spine and the right spine of a pertinent digraph are highlighted in Figure 5.

The following lemma shows that the turn number of a spine of a pertinent digraph of an upward representation is an invariant property of the upward representation itself.

Lemma 3. *Let P'_r, P''_r be two distinct right spines of G_μ and let P'_l, P''_l be two distinct left spines of G_μ. Then $n(U_{P'_r}) = n(U_{P''_r})$ and $n(U_{P'_l}) = n(U_{P''_l})$.*

For example, in Figure 5, $G_{\mu'}$ has only two left spines, that also concide with the right spines. The turn number of these spines is -1. Based on Lemma 3, we can denote by $n_l(U_{G_\mu})$ the turn number of any left spine of G_μ in U_G, without

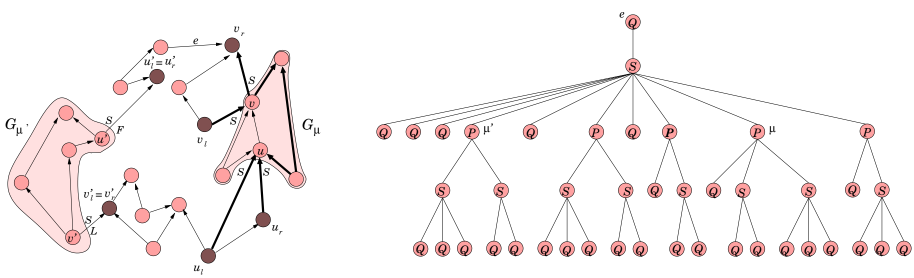


Fig. 5. An upward planar representation of a series-parallel digraph G, and an $SPQR$-tree T of G rooted at edge e. G_μ and $G_{\mu'}$ are the pertinent digraphs of nodes μ and μ' of T, with poles u, v, u', v', respectively. The pole categories of u and v are $SS2$ and $SS1$, respectively. The ones of u' and v' are $FS1$ and $SL0$, respectively. The left and the right spines of G_μ constructed on the right path of G_μ are highlighted.

ambiguity; similarly, $n_r(U_{G_\mu})$ denotes the turn number of any right spine of G_μ. The *upward spirality* of G_μ within U_G (or simpler, the *upward spirality* of U_{G_μ}), is denoted as $\sigma(U_{G_\mu})$ and is defined as follows: $\sigma(U_{G_\mu}) = \frac{n_l(U_{G_\mu})+n_r(U_{G_\mu})}{2}$.

For example, in Figure 5, $\sigma(U_{G_\mu}) = -1/2$, and $\sigma(U_{G_{\mu'}}) = -1$. Suppose now that P_l and P_r are a left spine and a right spine of G_μ, constructed using the same path $P_{uv} =< u, w_1, w_2, \ldots, w_k, v >$ between the poles u, v of G_μ. We can rewrite the turn number of the spines as follows: $n(U_{P_l}) = n(U_{P_{uv}}) + a_{u_l} + a_{v_l}$, $n(U_{P_r}) = n(U_{P_{uv}}) + a_{u_r} + a_{v_r}$, where $a_{u_l} = n(U_{P_{u_l}})$, $a_{u_r} = n(U_{P_{u_r}})$, $a_{v_l} = n(U_{P_{v_l}})$, $a_{v_r} = n(U_{P_{v_r}})$, and $P_{u_l} =< u_l, u, w_1 >$, $P_{u_r} =< u_r, u, w_1 >$, $P_{v_l} =< w_k, v, v_l >$, $P_{v_r} =< w_k, v, v_r >$. Of course, $a_{u_l}, a_{u_r}, a_{v_l}, a_{v_r} \in \{-1, 0, 1\}$. From the invariant property of Lemma 3, the upward spirality of U_{G_μ}, can be rewritten as follows:

$$\sigma(U_{G_\mu}) = n(U_{P_{uv}}) + \frac{(a_{u_l} + a_{u_r})}{2} + \frac{(a_{v_l} + a_{v_r})}{2} \qquad (1)$$

In order to uniquely refer to the values $a_{u_l}, a_{u_r}, a_{v_l}, a_{v_r}$ for the upward spirality of U_{G_μ}, we aim at rewriting $\sigma(U_{G_\mu})$ in a kind of canonical form, choosing always a "special" path P_{uv}. We define the following equivalence relationship between any two paths P'_{uv}, P''_{uv} of G_μ, within a given upward representation U_G of G. We say that P'_{uv}, P''_{uv} are *turn equivalent* if $n(U_{P'_{uv}}) = n(U_{P''_{uv}})$, i.e, if they have the same turn number. Since $\sigma(U_{G_\mu})$ assumes the same value if we use P'_{uv} or P''_{uv} in Formula (1), and since $a_{u_l}, a_{u_r}, a_{v_l}, a_{v_r} \in \{-1, 0, 1\}$, then the turn-equivalence relationship partitions the set of the undirected paths of U_{G_μ}, from the first to the second pole, into a finite set of equivalence classes. The following lemma gives a useful property of the paths of G_μ.

Lemma 4. *Let P^r_{uv} be a path of G_μ that is turn-equivalent to the right path of G_μ, and let P^l_{uv} be a path of G_μ that is turn-equivalent to the left path of G_μ. If P_{uv} is any path of G_μ between u and v, then $n(U_{P^l_{uv}}) \geq n(U_{P_{uv}}) \geq n(U_{P^r_{uv}})$.*

In Formula (1) we now choose as path P_{uv} any path P^r_{uv} that is turn-equivalent to the right path of G_μ, and we consider the corresponding values $(a_{u_l} + a_{u_r})/2$ and $(a_{u_l} + a_{u_r})/2$. Denote $n(U_{P^r_{uv}})$ by $\alpha(U_{G_\mu})$, and denote $(a_{u_l} + a_{u_r})/2$, $(a_{u_l} + a_{u_r})/2$ by $\alpha_u(U_{G_\mu})$ and $\alpha_v(U_{G_\mu})$, respectively.

The upward spirality of U_{G_μ} can be rewritten in the following canonical form: $\sigma(U_{G_\mu}) = \alpha(U_{G_\mu}) + \alpha_u(U_{G_\mu}) + \alpha_v(U_{G_\mu})$. We call $\alpha(U_{G_\mu})$ the *internal spirality* of U_{G_μ}, and $\alpha_u(U_{G_\mu})$, $\alpha_v(U_{G_\mu})$ the *first-pole spirality* and the *second-pole spirality*, respectively. From Lemma 4, each of the terms $a_{u_l}, a_{u_r}, a_{v_l}, a_{v_r}$ in Formula (1) takes the maximum possible value when $P_{uv} = P^r_{uv}$. This also implies that, for any choice of P_{uv}, $(a_{u_l} + a_{u_r})/2 \leq \alpha_u(U_{G_\mu})$ and $(a_{v_l} + a_{v_r})/2 \leq \alpha_v(U_{G_\mu})$. Therefore, for each pole category, it is possible to determine the exact value of the two pole spiralities, since we know that they take the maximum possible value and since we know what are the two external vertices. The next results prove that the upward spirality can only take a linear number of values.

Lemma 5. *Let $\overline{n}$ be the minimum number of switches on any path between the poles u and v of G_μ. Then, $-\overline{n}-2 \leq \sigma(U_{G_\mu}) \leq \overline{n}+2$. Also, $\alpha_u(U_{G_\mu}) + \alpha_v(U_{G_\mu}) \in \{-1, -1/2, 0, 1/2, 1, 3/2, 2\}$.*

Theorem 1. *Let G be a digraph with n vertices, T an SPQR-tree of G, and G_μ the pertinent digraph of a node μ of T. There are at most $O(n)$ values for the upward spirality of G_μ within any upward planar representation of G.*

The following lemmas describe the relationships between the upward spiralities of series and parallel compositions, and the ones of their components.

Lemma 6. *Let μ be an S-node of T with children μ_1 and μ_2. Let G_μ be the pertinent digraph of μ, with poles u and v, and let G_{μ_1}, G_{μ_2} be the pertinent digraphs of μ_1, μ_2, with poles $u_1 = u, v_1$, and $u_2 = v_1, v_2 = v$, respectively. The following relationship holds: $\sigma(U_{G_\mu}) = \sigma(U_{G_{\mu_1}}) + \sigma(U_{G_{\mu_2}})$.*

Lemma 7. *Let μ be a P node of T with children $\mu_1, \ldots, \mu_k$, ordered from left to right. Let G_μ be the pertinent digraph of μ and let $G_{\mu_1}, \ldots, G_{\mu_k}$ be the pertinent digraphs of $\mu_1, \ldots, \mu_k$, respectively. For each $i = 1, \ldots, k$, the following relationships hold:* **(1)** $\alpha(U_{G_\mu}) = \alpha(U_{G_{\mu_i}}) + \delta^{(i)}(U_{G_\mu})$, $\delta^{(i)}(U_{G_\mu}) \in \{0, 1, 2, 3, 4\}$; **(2)** $\alpha(U_{G_{\mu_1}}) \geq \alpha(U_{G_{\mu_2}}) \geq \cdots \geq \alpha(U_{G_{\mu_k}}) = \alpha(U_{G_\mu})$.

Consider now the subgraph G' of G consisting of G_μ plus the edges incident on u and v that are external to G_μ, and let $U'_{G'}$ be any upward planar representation of G' such that the planar embedding of the external edges of G_μ and the angle labels between these edges in $U'_{G'}$ are the same as in U_G. Notice that, the planar embedding of G_μ in $U'_{G'}$ can be different from the one in U_G. Denote by $t'_u = X'_u Y'_u \lambda_u$ and $t'_v = X'_v Y'_v \lambda_v$ the pole categories of u and v for U'_{G_μ}. The operation of *substitution* of U_{G_μ} with U'_{G_μ} in U_G defines a new planar embedded digraph $S(U'_{G_\mu}, U_G)$ with angle labels S, F, and L such that: (i) The planar embedding and the labels of the angles of subgraph $G - G_\mu$ are the same as in U_G; (ii) The planar embedding and the labels of the angles of subgraph G_μ are the same as in U'_{G_μ}; (iii) The inter-labels of G_μ at u and at v are X'_u, Y'_u, X'_v, Y'_v, respectively. We say that U_{G_μ} is *substitutable* with U'_{G_μ} in U_G if $S(U'_{G_\mu}, U_G)$ is still an upward planar representation of G. The following theorem is the main result of this section.

Theorem 2. *If U'_{G_μ} and U_{G_μ} have the same upward spirality and the same pole categories (i.e. $t'_u = t_u$, $t'_v = t_v$), then U_{G_μ} is substitutable with U'_{G_μ} in U_G.*

4 Upward Planarity Testing of Series-Parallel Digraphs

The outline of our upward planarity testing and embedding algorithm for series-parallel digraphs is as follows. For each possible choice of an edge e of G, the algorithm computes the *SPQR*-tree T of G with reference edge e. Then, the algorithm visits T from bottom to top, in post-order. Each time a node μ of T is visited, μ is equipped with a set of upward planar representations of G_μ (which we call *feasible set of* μ), such that each upward planar representation is constrained to have assigned pole categories and an assigned value of upward spirality. Using the result of Theorem 2, for each possible combination of pole categories and upward spirality value, the algorithm stores only one constrained upward planar representation, if there exists one. The feasible set of each S-node and P-node of T is computed by considering the feasible sets of its children. In this way, the algorithm incrementally tries to construct an upward planar representation of G with edge e on the external face, from the leaves to the root, while exploring a subset of upward planar representations that is "representative" of the whole set of upward planar representations of G. The algorithm ends if the feasible set of a node is empty or if the feasible sets of all nodes have been successfully computed. In the following we formalize the definition of feasible set and then describe how the feasible sets of the different types of nodes can be computed.

A *feasible tuple* of μ is defined as follows: $\tau_\mu =< U_{G_\mu}, \sigma(U_{G_\mu}), t_u, t_v >$, where U_{G_μ} is an upward planar representation of G_μ with pole categories t_u, t_v and upward spirality $\sigma(U_{G_\mu})$. Let $\tau'_\mu =< U'_{G_\mu}, \sigma(U'_{G_\mu}), t'_u, t'_v >$ and $\tau''_\mu =< U''_{G_\mu}, \sigma(U''_{G_\mu}), t''_u, t''_v >$ be two feasible tuples of μ. We say that U'_{G_μ} and U''_{G_μ} are *spirality equivalent* if $\sigma(U'_{G_\mu}) = \sigma(U''_{G_\mu})$, $t'_u = t''_u$, and $t'_v = t''_v$. In this case, we also say that τ'_μ and τ''_μ are *spirality equivalent*. A *feasible set* $\mathcal{F}_\mu$ of μ is a set of feasible tuples of μ such that there is exactly one representative tuple for each class of spirality equivalent feasible tuples of μ. The next lemma guarantees that our algorithm is able to find an upward planar representation of G with e on the external face, if there exists one.

Lemma 8. *Let G be an upward planar digraph with edge e on the external face, and let T be the SPQR-tree of G with respect to e. There exists an upward planar representation U_G of G such that: (i) e is on the external face of U_G; (ii) for each node μ of T, there exists a feasible tuple $\tau_\mu =< U_{G_\mu}, \sigma(U_{G_\mu}), t_u, t_v >$ in the feasible set of μ, where U_{G_μ} is the upward representation of G_μ induced by U_G.*

All the Q-nodes have the same feasible set, which can be computed with a preprocessing step in $O(1)$ time. Namely, if μ is a Q-node, both the internal spirality and the internal-labeling weight of any upward planar representation U_{G_μ} of G_μ are equal to 0. We can only have three upward spirality values for U_{G_μ}: 0, 1, and -1. More precisely, if (u, v) is the (undirected) edge represented by μ, the algorithm inserts in $\mathcal{F}_\mu$ a tuple for each of the following combinations of upward spirality and pole categories: **(1)** $\sigma(U_{G_\mu}) = 0$, $t_u \in \{SS0, SF0, FS0, FF0, SL0\}$, $t_v \in \{SS0, SF0, FS0, FF0, LS0\}$. **(2)** $\sigma(U_{G_\mu}) = 0$, $t_u = LS0$ and $t_v = SL0$. **(3)** $\sigma(U_{G_\mu}) = 1$, $t_u = LS0$, $t_v \in \{SS0, SF0, FS0, FF0, LS0\}$. **(4)** $\sigma(U_{G_\mu}) = -1$, $t_u \in \{SS0, SF0, FS0, FF0, SL0\}$, $t_v = SL0$. In all these tuples, U_{G_μ} is the edge (u, v) oriented upward.

Let μ be an S-node of T, and let u and v be the first pole and the second pole of G_μ, respectively. Let μ_1, μ_2 be the two children of μ; denote by $u_1 = u$, v_1 the first pole and the second pole of G_{μ_1}; also denote by $u_2 = v_1$, $v_2 = v$ the first pole and the second pole of G_{μ_2}. The feasible set of μ is computed using the relationship of Lemma 6. For each pair of tuples $\tau_1 =< U_{G_{\mu_1}}, \sigma(U_{G_{\mu_1}}), t_{u_1}, t_{v_1} > \in \mathcal{F}_{\mu_1}$, $\tau_2 =< U_{G_{\mu_2}}, \sigma(U_{G_{\mu_2}}), t_{u_2}, t_{v_2} > \in \mathcal{F}_{\mu_2}$, the algorithm checks if the inter-labels of t_{v_1} and t_{u_2} are the same, and if the orientations of the edges incident on

$u_2 = v_1$ in $U_{G_{\mu_1}}$ and $U_{G_{\mu_2}}$ are compatible. In the affirmative case, it constructs a new tuple $\tau =< U_{G_\mu}, \sigma(U_{G_\mu}), t_u, t_v >$, which will be inserted in $\mathcal{F}_\mu$, only if $\mathcal{F}_\mu$ does not already contain a spirality equivalent tuple; τ is defined as follows: $\sigma(U_{G_\mu}) = \sigma(U_{G_{\mu_1}}) + \sigma(U_{G_{\mu_2}})$; $t_u = t_{u_1}$, $t_v = t_{v_2}$; U_{G_μ} is the series composition of $U_{G_{\mu_1}}$ and $U_{G_{\mu_2}}$ on the common vertex $u_2 = v_1$. Since each feasible set has $O(n)$ tuples, the feasible set of an S-node can be computed in $O(n^2)$ time.

The computation of the feasible set of a P-node is a more complicated task, since the skeleton of a P-node with k children has $O(k!)$ possible planar embeddings, and we want to keep the computation polynomial in the number of vertices of the graph. Let μ be a P-node of T, with first pole u and second pole v. Let $\mu_1, \ldots, \mu_k$ be the children of μ. We remark that each G_{μ_i} $(i = 1, \ldots, k)$ has $u_i = u$ and $v_i = v$ as the first pole and the second pole, respectively. In order to construct the feasible set of μ, we evaluate the possibility of constructing an upward planar representation U_{G_μ} for each possible way of fixing $\sigma(U_{G_\mu}), t_u$, and t_v. Namely, for each choice of $\sigma(U_{G_\mu}), t_u, t_v$, the algorithm must verify if it is possible to select from the feasible sets of $\mu_1, \ldots, \mu_k$, a subset of upward planar representations $U_{G_{\mu_1}}, \ldots, U_{G_{\mu_k}}$ that can assume a "parallel configuration" compatible with $\sigma(U_{G_\mu}), t_u, t_v$. The conditions of Lemma 7 allow us to limit the number of these configurations, so that it is not needed to consider all permutations of the children of μ in the $skel(\mu)$. Actually, it can be proved that the total number of configurations is constant with respect to the number of vertices of G. The set of possible configurations is defined on the basis of t_u and t_v; each configuration consists of a sequence of groups, such that each group can host a certain number of upward planar representations, all having the same pole categories and the same internal spirality (which also implies the same upward spirality). The groups in the sequence are ordered according to their values of internal spirality. In this way, on the basis of $\sigma(U_{G_\mu})$ and for each configuration above defined, the algorithm tries to select a set of upward representations $U_{G_{\mu_1}}, \ldots, U_{G_{\mu_k}}$ from the feasible sets of $\mu_1, \ldots, \mu_k$ and to assign each of them to a group in the configuration. This assignment problem is solved by searching a feasible flow in a suitable network constructed from the configuration. The formal description of the configurations and the construction of the feasible set using a sequence of flow-based algorithms can be found in [8]. The construction of the feasible sets of all P-nodes can be done in $O(n^3)$ time.

Once all feasible sets have been computed for the nodes of T, the algorithm performs a final step to verify if it is possible to construct an upward planar representation from the feasible set of the root of T (which is a Q-node) and the one of its child. Namely, let μ be the root and let ν be its child. The following lemma holds.

Lemma 9. *G has an upward planar representation U_G if and only if there exist two tuples $\tau_\mu =< U_{G_\mu}, \sigma(U_{G_\mu}), t_{u_\mu}, t_{v_\mu} >\in \mathcal{F}_\mu$, $\tau_\nu =< U_{G_\nu}, \sigma(U_{G_\nu}), t_{u_\nu}, t_{v_\nu} >\in \mathcal{F}_\nu$ such that: (1) $\sigma(U_{G_\mu}) - \sigma(U_{G_\nu}) = 2$; (2) $Y_{u_\mu} = X_{u_\nu}$, $Y_{v_\mu} = X_{v_\nu}$, where $t_w = X_w Y_w \lambda_w$ and $w \in \{u_\mu, u_\nu, v_\mu, v_\nu\}$.*

According to Lemma 9, the algorithm looks for two tuples that verify the conditions (1) and (2) in the statement. If these tuples are found, the final upward planar representation is returned, otherwise the upward planarity testing fails. The next theorem summarizes the main result of this section. The final time complexity of the testing algorithm follows from the above discussion, iterating over all $SPQR$-trees of G (one for each choice of the reference edge).

Theorem 3. *Let G be a biconnected series-parallel digraph with n vertices. There exists an $O(n^4)$-time algorithm that tests if G is upward planar and, if so, that constructs an upward planar drawing of G.*

5 An FPT Algorithm for General Digraphs

To extend the upward planarity testing algorithm above described to general biconnected digraphs, we need to describe how to compute the feasible sets of R-nodes. Unfortunately, to compute the feasible set of an R-node μ, we cannot rely on any relationship between the upward spirality of U_{G_μ} and the upward spirality of its children. Therefore, we simply consider all possible combinations of tuples for each virtual edge of $skel(\mu)$ in constructing U_{G_μ}. Namely, let e_i be a virtual edge of $skel(\mu)$ and let μ_i be the child of μ corresponding to e_i. We substitute to e_i the upward planar representation $U_{G_{\mu_i}}$ of a tuple in the feasible set of μ_i. We repeat this process for each virtual edge, until a "partial candidate" upward planar representation U'_{G_μ} of G_μ is constructed. We then apply on this partial representation the flow-based upward planarity testing algorithm proposed by Bertolazzi et al. [1], where the assignment of the switches to the faces is constrained for the part of the representation that is already fixed. In order to construct the feasible set of μ, we need to run the testing algorithm over all possible combinations of upward spirality and pole categories of U_{G_μ}. For each given value of upward spirality σ and for each choice of pole categories t_u, t_v, we enrich the partial upward representation U'_{G_μ} with a suitable external gadget, that forces U_{G_μ} to have upward spirality σ and pole categories t_u, t_v. This gadget will have a fixed upward planar representation, which is still translated into a set of constraints on the flow network. See [8] for a detailed construction of the external gadgets.

The feasible set of an R-node μ, computed with the above procedure, requires to consider all possible combinations of tuples in the feasible set of the children of μ, and, for each of these combinations, we need to consider all possible values of upward spirality and pole categories. The procedure must be also applied to the two possible planar embeddings of $skel(\mu)$. Denote by t the number of non-trivial triconnected components of G and denote by d the maximum diameter of a split component of G. The feasible set of an R-node of μ can be then computed in $O(d^{t_\mu} \cdot n^2)$ time, where n is the number of vertices of G, and $t_\mu \leq t$ is the number of virtual edges (distinct from the reference edge) of μ. Indeed, the minimum number of switches in any path between the poles of G_μ is at most d, and therefore, from Lemma 5, the upward spirality of U_{G_μ} can take $O(d)$ possible values and the feasible set of any node of T has $O(d)$ tuples. Also, $O(n^2)$ is the complexity of the upward planarity testing of Bertolazzi et al. Hence, the feasible set of all R-nodes can be computed in $O(d^t \cdot n^2)$ time.

Our FPT algorithm can be eventually extended to general planar digraphs, using a recent result of Healy and Lynch [10, 11] about the upward planarity testing of simply connected graphs (refer to [8]). The following theorem holds, by observing that the feasible sets of P- and S-nodes of each $SPQR$-tree T can be computed in $O(d \cdot t^2)$-time and $O(d^2 n)$-time, respectively, and by iterating over all decomposition trees of G.

Theorem 4. *Let G be a connected planar digraph with n vertices. Suppose that each block of G has at most t (non-trivial) triconnected components, and that each split component of a block has a diameter at most d. There exists an $O(d^t \cdot n^3 + d \cdot t^2 \cdot n + d^2 \cdot n^2)$-time algorithm that tests if G is upward planar and, if so, that constructs an upward planar drawing of G.*

Theorem 5. *Let G be a connected planar digraph with n vertices and such that each block is a series-parallel digraph. There exists an $O(n^4)$-time algorithm that tests if G is upward planar and, if so, that constructs an upward planar drawing of G.*

References

1. P. Bertolazzi, G. D. Battista, G. Liotta, and C. Mannino. Upward drawings of triconnected digraphs. *Algorithmica*, 6(12):476–497, 1994.
2. P. Bertolazzi, G. Di Battista, and W. Didimo. Quasi-upward planarity. *Algorithmica*, 32(3):474–506, 2002.
3. P. Bertolazzi, G. Di Battista, C. Mannino, and R. Tamassia. Optimal upward planarity testing of single-source digraphs. *SIAM J. Comput.*, 27:132–169, 1998.
4. H. Chan. A parameterized algorithm for upward planarity testing. In *Proc. ESA '04*, volume 3221 of *LNCS*, pages 157–168, 2004.
5. G. Di Battista, P. Eades, R. Tamassia, and I. G. Tollis. *Graph Drawing*. Prentice Hall, Upper Saddle River, NJ, 1999.
6. G. Di Battista, G. Liotta, and F. Vargiu. Spirality and optimal orthogonal drawings. *SIAM J. Comput.*, 27(6):275–298, 1998.
7. G. Di Battista and R. Tamassia. On-line planarity testing. *SIAM J. Comput.*, 25:956–997, 1996.
8. W. Didimo, F. Giordano, and G. Liotta. Upward spirality and upward planarity testing. Technical report, 2005. RT-006-05, DIEI - Universitá di Perugia, Italy.
9. A. Garg and R. Tamassia. On the computational complexity of upward and rectilinear planarity testing. *SIAM J. Comput.*, 31(2):601–625, 2001.
10. P. Healy and K. Lynch. Building blocks of upward planar digraphs. In *Proc. GD '04*, volume 3383 of *LNCS*, pages 296–306, 2004.
11. P. Healy and K. Lynch. Building blocks of upward planar digraphs. Technical report, 2005. TR UL-CSIS-05-2.
12. P. Healy and K. Lynch. Fixed-parameter tractable algorithms for testing upward planarity. In *Proc. SOFSEM '05*, volume 3381 of *LNCS*, pages 199–208, 2005.
13. M. D. Hutton and A. Lubiw. Upward planarity testing of single-source acyclic digraphs. *SIAM J. Comput.*, 25(2):291–311, 1996.
14. A. Papakostas. Upward planarity testing of outerplanar dags. In *Proc. GD '95*, volume 894 of *LNCS*, pages 7298–306, 1995.