INTRINSIC MODELING OF LINEAR THERMO-DYNAMIC THIN SHELLS

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Abstract We consider the problem of modeling dynamic thin shells with thermal effects based on the intrinsic geometry methods of Michel Delfour and Jean-Paul Zolésio. This model relies on the oriented distance function which describes the geometry. Here we further develop the Kirchhoff-based shell model introduced in our previous work by subjecting the elastically and thermally isotropic shell to an unknown temperature distribution. This yields a fully-coupled system of four linear equations whose variables are the displacement of the shell mid-surface and the thermal stress resultants.

keywords: Intrinsic shell model, dynamic thermoelasticity

1. Introduction

In this paper we continue the development of a Kirchhoff-based shell model using the intrinsic-geometric methods introduced by Michel Delfour and Jean-Paul Zolésio [6, 5]. The aim of this method is to produce a coordinate-free version of the shell equations, in contrast to the classical equations which require explicit representation of the nonconstant coefficients. With the intrinsic approach, one can exploit the underlying geometry of the shell to derive equations in which the nonconstant coefficients are written in the form of tangential operators. This enables us to better modify and apply known techniques that were developed for use in the constant-coefficient case (flat plate models).

In our previous work [2–4] we have developed a linear dynamic model of the thin shell and shown several interesting stability/controllability results. However, as thermal effects are very important in many applications of engineering, we wish to include them in our shell model. We proceed in the development of a (linear) thermoelastic shell model based essentially on similar assumptions to those which are used in the derivation of classical linear thermoelastic plate models (see, *e.g.* [7]). As such, we subject the elastically and thermally

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isotropic shell to an unknown temperature distribution. Eventually this yields a fully-coupled system of four linear equations whose variables are the displacement of the shell mid-surface and the thermal stress resultants. The form of these equations is familiar – in fact it looks very similar to a 'linear' version of the well-known Von Kármán system [7]. However, it must be noted that all the operators are tangential operators and thus the curvature of the shell is very much in evidence.

2. Preliminary Considerations

In this section we present a brief overview of the oriented distance function and the intrinsic tangential calculus that forms the basis of our shell model. In addition, we introduce the set of hypotheses on the shell that will be in force for the rest of this paper.

2.1 Overview of the intrinsic geometry

In order to improve readability we here include a brief discussion of the oriented distance function and the intrinsic geometric methods of Delfour and Zolésio. Since by necessity this overview will lack detail, the reader is referred to [5, 6] for a definitive exposition on this topic.

Consider a domain $\mathcal{O} \subset R^3$ whose nonempty boundary $\partial\mathcal{O}$ is a C^1 two-dimensional submanifold of R^3. Define the oriented (or signed) distance function to $\mathcal{O}$ as $b(x) = d_{\mathcal{O}}(x) - d_{R^3\backslash\mathcal{O}}(x)$ where d is the Euclidean distance from the point x to the domain $\mathcal{O}$. In other words, $b(x)$ is simply the positive or negative distance to the boundary $\partial\mathcal{O}$. It can be shown that for every $x \in \partial\mathcal{O}$, there exists a neighborhood where the function $\nabla b = \nu$, the unit outward external normal to $\partial\mathcal{O}$ [6].

Consider a subset $\Gamma \subseteq \partial\mathcal{O}$ which will eventually become the mid-surface of our shell. We define the projection $p(x)$ of a point x onto Γ as $p(x) = x - b(x)\nabla b(x)$. Then, we define a shell S_h of thickness h as

$$S_h(\Gamma) \equiv \left\{x \in R^3 : p(x) \in \Gamma,\ |b(x)| < h/2\right\} \quad . \tag{1}$$

When $\Gamma \neq \partial\mathcal{O}$, the shell S_h has a lateral boundary $\Sigma_h(\Gamma) \equiv \{x \in R^3 : p(x) \in \Upsilon,\ |b(x)| < h/2\}$ where $\Upsilon \equiv \partial\Gamma$ denotes the boundary of Γ. A natural curvilinear coordinate system (X, z) is thus induced on the shell S_h, where the coordinate vector X gives the position of a point on the mid-surface Γ, and $z \in \left(-\frac{h}{2}, \frac{h}{2}\right)$ gives the vertical (normal) distance from the mid-surface. We also define the "flow mapping" $T_z(X)$ as $T_z(X) = X + z\nabla b(X)$ for all X and z in S_h. The curvatures of the shell will be denoted H and K. These can be reconstructed from the boundary distance function $b(x)$ by noting that at any point (X, z), the matrix D^2b has eigenvalues $0,\ \lambda_1,\ \lambda_2$. The curvatures are then given by $\mathrm{tr}(D^2b) = 2H = \lambda_1 + \lambda_2$ and $K = \lambda_1\lambda_2$.

Next, we mention briefly some useful aspects of the tangential differential calculus. Given $f \in C^1(\Gamma)$, we define the tangential gradient ∇_Γ of the scalar function f by means of the projection as

$$\nabla_\Gamma f \equiv \nabla(f \circ p)(x)|_\Gamma \quad . \tag{2}$$

This notion of the tangential gradient is equivalent to the classical definition using an extension F of f in the neighborhood of Γ, *i.e.* $\nabla_\Gamma f = \nabla F|_\Gamma - \frac{\partial F}{\partial \nu}\nu$ [6]. Following the same idea we can define the tangential Jacobian matrix of a vector function $v \in C^1(\Gamma)^3$ as $D_\Gamma v \equiv D(v \circ p)|_\Gamma$ or $(D_\Gamma v)_{ij} = (\nabla_\Gamma v_i)_j$, the tangential divergence as $\mathrm{div}_\Gamma v \equiv \mathrm{div}(v \circ p)|_\Gamma$, the Hessian $D^2_\Gamma f$ of $f \in C^2(\Gamma)$ as $D^2_\Gamma f = D_\Gamma(\nabla_\Gamma f)$, the Laplace-Beltrami operator of $f \in C^2(\Gamma)$ as $\Delta_\Gamma f \equiv \mathrm{div}_\Gamma(\nabla_\Gamma f) = \Delta(f \circ p)|_\Gamma$, the tangential linear strain tensor of elasticity as $\varepsilon_\Gamma(v) \equiv \frac{1}{2}(D_\Gamma v + {}^*D_\Gamma v) = \varepsilon(v \circ p)|_\Gamma$, and the tangential vectorial divergence of a second-order tensor A as $\mathrm{div}_\Gamma A \equiv \mathrm{div}(A \circ p)|_\Gamma = \mathrm{div}_\Gamma A_i$. Using these definitions one can derive Green's formula in the tangential calculus [6]:

$$\int_\Gamma f \mathrm{div}_\Gamma v\, d\Gamma + \int_\Gamma \langle \nabla_\Gamma f, v\rangle\, d\Gamma = \int_\Upsilon \langle fv, \nu\rangle\, d\Upsilon + 2\int_\Gamma f\, H\langle v, \nabla b\rangle\, d\Gamma \tag{3}$$

where ν is the outward unit normal to the curve Υ. From [6, 5] we have that $\langle \nabla_\Gamma w, \nabla b\rangle = 0$ and $D_\Gamma v \nabla b = 0$ by definition for any scalar w and vector v. In addition, if we consider a purely tangent vector $v = v_\Gamma$, *i.e.* $\langle v_\Gamma, \nabla b\rangle = 0$, we can take the tangential gradient of both sides of this expression and derive that $D^2b\, v_\Gamma + {}^*D_\Gamma v_\Gamma \nabla b = 0$. Finally, throughout this paper we will use $\langle\cdot,\cdot\rangle$ to denote the scalar product of two vectors and $A..B$ to denote the double contraction of two matrices – *i.e.* $A..B = tr(AB)$.

2.2 Model hypotheses

ASSUMPTION 1 *We impose the following assumptions on the shell.*

(i) The shell is assumed to be made of an isotropic and homogeneous material, so that the Lamé coefficients $\lambda > 0$ and $\mu > 0$ are constant.

(ii) The thickness h of the shell is small enough to accommodate the curvatures H and K, i.e. the product of the thickness by the curvatures is small as compared to 1. As a consequence we shall drop terms of order equal or greater than 2 in the series expansions.

(iii) (Kirchhoff Hypothesis) Let T be a transformation of the shell S_h, and let $\mathrm{e} = (e_\Gamma, w)$ be the corresponding transformation of the mid-surface. In the classical thin plate theory named after Kirchhoff, the displacement vectors T and $\mathrm{e} \circ p$ are related by the hypothesis that the filaments of the plate initially perpendicular to the middle surface remain straight and perpendicular

to the deformed surface, and undergo neither contraction nor extension. In the intrinsic geometry we have $T = \mathbf{e} \circ p - b(\,^*D_{\Gamma_0}\mathbf{e}\,\nabla b) \circ p$.

(iv) We will assume the boundary Υ *consists of two open connected regions* Υ_0 *and* Υ_1, *with* $\Upsilon = \overline{\Upsilon_0 \cup \Upsilon_1}$ *and* $\emptyset = \Upsilon_0 \cap \Upsilon_1$. *We will clamp the shell on* Υ_0, *and allow* Υ_1 *to be free.*

(v) The shell is assumed to be subject to an unknown temperature distribution $\tau(\mathbf{x}, t)$ *which is measured from a reference temperature. The shell is assumed to be thermally isotropic, the change in* τ *is small compared to the reference temperature* τ_0 *of the shell, and the thermal strain is assumed to be linear. Thus the thermal strains of the shell are given by* $\varepsilon^\tau(T) = \bar{\alpha}\tau I$, *where* $\bar{\alpha}$ *is the coefficient of thermal expansion.*

We denote by $\mathbf{e}$ the transformation of the shell mid-surface and by e_Γ and e_n the tangential and normal components of $\mathbf{e}$ in local coordinates. We define w to be the magnitude of the normal displacement. As such, we have that

$$w = \langle \mathbf{e}, \nabla b \rangle, \qquad e_n = w\,\nabla b, \qquad e_\Gamma = \mathbf{e} - e_n\,. \tag{4}$$

The variable τ denotes the temperature in the shell body, as measured from a reference temperature τ_0, taken to be the absolute temperature of the body. Because of the assumptions of thinness of the shell and linearity of the thermal strains (Hypothesis 1 (ii) and (v)), it is reasonable to suppose that the temperature varies linearly with respect the thickness of the shell,

$$\tau = \tau_1 \circ p + b\,\tau_2 \circ p\,, \tag{5}$$

with τ_1, τ_2 variables defined on the mid-surface of the shell Γ. Note that τ_1 corresponds physically to the thermal energy of stretching (membrane energy), whereas τ_2 corresponds to the thermal effect of shell bending. The final form of the equations of the shell will not involve τ_i, but instead will naturally involve the thermal stress resultants θ and φ, defined as

$$\varphi = \bar{\alpha}\,\tau_1, \qquad \theta = \bar{\alpha}\,\tau_2\,, \tag{6}$$

where $\bar{\alpha}$ is the coefficient of thermal expansion.

Here we list the following definitions and properties derived in [2]:

LEMMA 2 *The following strain-displacement relation holds for a shell modeled in the intrinsic geometry under Hypothesis 1 (i)-(iii).*

$$\begin{aligned} \varepsilon(T) = {} & (\varepsilon_\Gamma(e_\Gamma) + wD^2b + V_\Gamma e_\Gamma) \circ p \\ & -b\,(-\varepsilon_\Gamma(D^2b\,e_\Gamma) + C_\Gamma e_\Gamma + S_\Gamma w + G_\Gamma w + w(D^2b)^2) \circ p\,, \end{aligned} \tag{7}$$

where ε_Γ is the tangential linear strain tensor of elasticity and

$$C_\Gamma u \;=\; \frac{1}{2}(D^2b\ ^*D_\Gamma u + D_\Gamma u\, D^2b)$$

$$
\begin{aligned}
V_\Gamma u &= \frac{1}{2}((D^2b\,u)\otimes\nabla b+\nabla b\otimes(D^2b\,u)) \\
G_\Gamma w &= \frac{1}{2}((\nabla b\otimes\nabla_\Gamma w)D^2b+D^2b(\nabla_\Gamma w\otimes\nabla b)) \\
S_\Gamma w &= \frac{1}{2}(D^2_\Gamma w+{}^*D^2_\Gamma w)\,. \qquad (8)
\end{aligned}
$$

C_Γ and V_Γ are 1st-order and 0-order operators, respectively, that in practice operate on a tangential vector u. G_Γ is a 1st-order operator, and S_Γ is the symmetrization of the Hessian matrix of a scalar function w (the Hessian matrix is not symmetric in the tangential calculus [6]). Define the space V

$$
V=\left\{\mathbf{e}\in\left[H^1(\Gamma)\right]^2\times H^2(\Gamma)\,\middle|\,e_\Gamma=w=\frac{\partial}{\partial\nu}w=0\text{ on }\Upsilon_0\right\}. \qquad (9)
$$

3. Thermoelastic shell model

THEOREM 3 *Define the following operator $\mathcal{C}$ acting on a matrix A:*

$$
\mathcal{C}(A)=\lambda\mathrm{tr}(A)\,I+2\mu\,A\,, \qquad (10)
$$

the expression $\tilde{\chi}$

$$
\tilde{\chi}=C_\Gamma e_\Gamma-\varepsilon_\Gamma(D^2b\,e_\Gamma)\,, \qquad (11)
$$

the parameters $\beta=(\lambda+2\mu)^{-1}$, $\zeta=\beta(\frac{2}{3}\mu+\lambda)$, $\kappa=\frac{\lambda_0}{\rho c\bar{\alpha}\beta}$, and $\eta=\frac{\zeta\bar{\alpha}\tau_0}{\lambda_0}$. Here ρ is shell density, c is specific heat, and λ_0 is thermal conductivity.

Then, the displacement $\mathbf{e}\in C([0,\infty);V)$ and thermal variables θ, $\varphi\in C([0,\infty);L_2(\Gamma))$ satisfy the following system of shell equations which holds on $\Gamma\times(0,\infty)$:

$$
\begin{aligned}
&\partial_{tt}w-\gamma\Delta_\Gamma\partial_{tt}w+\Delta^2_\Gamma w+\frac{\gamma}{2}\mathrm{div}_\Gamma(D^2b\,\partial_{tt}e_\Gamma) \qquad (12)\\
&\quad-\zeta\Delta_\Gamma\theta-\zeta(4H^2-2K)\theta-2\zeta\gamma^{-1}H\varphi+P_1(e_\Gamma)+Q_1(w)=0\\
&(I+\gamma(D^2b)^2)\partial_{tt}e_\Gamma-\beta[\gamma^{-1}\mathrm{div}_\Gamma\mathcal{C}(\varepsilon_\Gamma(e_\Gamma))+D^2b\,\mathrm{div}_\Gamma\mathcal{C}(\tilde{\chi}) \qquad (13)\\
&\quad-\mathrm{div}_\Gamma(D^2b\,\mathcal{C}(\tilde{\chi}))]+\zeta(\mathrm{div}_\Gamma(D^2b\,\theta)-D^2b\,\nabla_\Gamma\theta)+\zeta\gamma^{-1}\nabla_\Gamma\varphi\\
&\quad-\frac{\gamma}{2}D^2b\nabla_\Gamma\partial_{tt}w+P_2(w)+Q_2(e_\Gamma)=0
\end{aligned}
$$

where P_1 denotes coupling terms and Q_1 denotes lower order terms in the plate equation; and P_2, Q_2 in the wave equation:

$$
\begin{aligned}
P_1(e_\Gamma) &= \beta[2\lambda H\gamma^{-1}\mathrm{div}_\Gamma e_\Gamma+2\mu\gamma^{-1}\mathrm{tr}(D^2b\,\varepsilon_\Gamma(e_\Gamma))\\
&\quad-2\mu\langle(D^2b)^2..D^3b,e_\Gamma\rangle+2\mu\mathrm{div}_\Gamma\mathrm{div}_\Gamma(\tilde{\chi})+4\mu H\mathrm{tr}(D^3be_\Gamma\,D^2b)\\
&\quad-\lambda\Delta_\Gamma\langle2\nabla_\Gamma H,e_\Gamma\rangle-\lambda(4H^2-2K)\langle2\nabla_\Gamma H,e_\Gamma\rangle
\end{aligned}
$$

$$
\begin{aligned}
Q_1(w) &= \beta[k_\gamma w + 4\mu \mathrm{div}_\Gamma((D^2b)^2\nabla_\Gamma w) + \lambda\Delta_\Gamma((4H^2-2K)w) \\
&\quad +2\mu \mathrm{div}_\Gamma \mathrm{div}_\Gamma((D^2b)^2 w) + 2\mu \mathrm{div}_\Gamma(K\nabla_\Gamma w) + \lambda(4H^2-2K)\Delta_\Gamma w] \\
&\qquad +2\mu \mathrm{tr}(S_\Gamma w (D^2b)^2) + 4\mu H \mathrm{tr}((D^2b)^3 w) \\
P_2(w) &= \beta[-2\lambda\gamma^{-1}\nabla_\Gamma(Hw) + 2\mu\gamma^{-1}\mathrm{div}_\Gamma(w\, D^2b) \\
&\qquad +\lambda 2\nabla_\Gamma H(\Delta_\Gamma w - (4H^2-2K)w) - 2\mu(D^2b)^2 .. D^3bw] \\
&\qquad -2\mu \mathrm{div}_\Gamma(D^2b\, S_\Gamma w) + 2\mu D^2b\, \mathrm{div}_\Gamma(S_\Gamma w) \\
Q_2(e_\Gamma) &= -2\beta\mu\gamma^{-1}(K\, e_\Gamma + 2(D^2b)^2 e_\Gamma)
\end{aligned}
$$

The thermal variables φ and θ satisfy the following coupled heat-like equations:

$$
\frac{1}{\kappa}\partial_t\varphi - \Delta_\Gamma\varphi - 2H\theta + \eta(2H\partial_t w + \mathrm{div}_\Gamma \partial_t e_\Gamma) = f_1 \tag{14}
$$

$$
\begin{aligned}
\frac{1}{\kappa}\partial_t\theta - \Delta_\Gamma\theta - \eta(\Delta_\Gamma\partial_t w + \mathrm{tr}(C_\Gamma\partial_t e_\Gamma) - \mathrm{div}_\Gamma(D^2b\partial_t e_\Gamma) \\
+(4H^2-2K)\partial_t w) = f_2
\end{aligned} \tag{15}
$$

where f_1 and f_2 represent heat sources or sinks. We have the following free boundary conditions on $\Upsilon_1 \times (0,\infty)$:

$$
\begin{aligned}
&\Big(\mathcal{C}(w\, D^2b + \varepsilon_\Gamma(e_\Gamma)) - \zeta\varphi I\Big)\cdot\nu = 0 \\
&\langle(\lambda\beta\nabla_\Gamma, \mathrm{tr}(D^3b\, e_\Gamma)\rangle + 4\mu\beta(D^2b)^2\nabla_\Gamma w + 2\mu\beta K\nabla_\Gamma w - \nabla_\Gamma(\Delta_\Gamma w)\nu \\
&+\langle\gamma(D^2b\partial_{tt}e_\Gamma - 2\nabla_\Gamma\partial_{tt}w) + 2\mu\beta B^3_\Gamma(C_\Gamma e_\Gamma - \varepsilon_\Gamma(D^2b\, e_\Gamma) + (D^2b)^2\, w), \nu\rangle \\
&\qquad -\langle\beta\lambda\nabla_\Gamma((4H^2-2K)w) - \zeta\nabla_\Gamma\theta, \nu\rangle + 2\mu B^2_\Gamma w = 0 \\
&\lambda\beta \mathrm{tr}(D^3b\, e_\Gamma) + \lambda\beta(4H^2-2K)w - \Delta_\Gamma w - \zeta\theta \\
&\quad +2\mu\beta\langle(C_\Gamma e_\Gamma - \varepsilon_\Gamma(D^2b\, e_\Gamma) + (D^2b)^2)\nu, \nu\rangle + 2\mu\beta B^1_\Gamma w = 0 \\
&\langle\theta, \nu\rangle = -\lambda_2(\theta - \tilde\theta), \qquad \langle\varphi, \nu\rangle = \lambda_2(\varphi - \tilde\varphi)
\end{aligned}
$$

clamped boundary conditions on $\Upsilon_0 \times (0,\infty)$:

$$
w = \frac{\partial}{\partial\nu}w = 0, \quad e_\Gamma = 0, \quad \theta = \varphi = 0 \tag{16}
$$

Here λ_2 is the coefficient from Newton's law of cooling, and $\tilde\theta = \bar\alpha\tilde\tau_1$, $\tilde\varphi = \bar\alpha\tilde\tau_2$, where $\tilde\tau$ is the temperature of the external medium of the shell.

Proof: *Elastic Equations.* We begin with the stress-strain relationships for a general shell body. Let T be a transformation of the body S_h. We have that the stress

$$
\sigma = \mathcal{C}\varepsilon(T) - \mathcal{C}\varepsilon^\tau(T)\,, \tag{17}
$$

where ε^τ denotes the thermal strain. By Hypothesis 1 (v), we can write that $\varepsilon^\tau = \bar\alpha\tau I$, where τ is the temperature and $\bar\alpha$ is the coefficient of thermal expansion. Next, we note that the body is assumed to be isotropic, so that $\mathcal{C}(A) = \lambda \mathrm{tr}(A) + 2\mu A$ where λ and μ are the Lamé coefficients. Finally, we use all this information to compute the potential energy

$$\mathcal{E}_p = \frac{1}{2}\int_{S_h} \varepsilon..\sigma = \frac{1}{2}\left[\lambda \int_{S_h} (\mathrm{tr}\varepsilon(T))^2 + 2\mu \int_{S_h} \mathrm{tr}(\varepsilon(T)^2) \right.$$
$$\left. -\bar{\nu}\int_{S_h} \mathrm{tr}(\varepsilon(T))\bar{\alpha}\tau\right], \qquad \text{where } \lambda + \tfrac{2}{3}\mu = \bar{\nu}.$$

REMARK 4 *At this point in the computation of the elastic energy, it is customary to impose the hypothesis of plane stresses:* $\sigma..(\nabla b \otimes \nabla b) = 0$ *(which in local coordinates is denoted* $\sigma_{33} = 0$*). As is well understood (see, e.g. [1]), this assumption implies a change of Lamé coefficient* λ *to* $\frac{E\nu}{1-\nu^2}$*, while* μ *remains unchanged. The same situation arises in the case of plates, we refer to [7] for further details. This modified expression for* λ *is more in line with both experimental evidence and asymptotic models. This does not affect any of the mathematical arguments to follow, so the imposition of this hypothesis is left to the discretion of the reader.*

Let us denote $\mathcal{E}_p = \mathcal{E}_{p,e} + \mathcal{E}_{p,t}$, with the elastic contribution to the potential energy $\mathcal{E}_{p,e}$ given as calculated in [2] using Lemma 2 as

$$\begin{aligned}
\mathcal{E}_{p,e} \;&=\; h\frac{\lambda}{2}\,|2H\,w + \mathrm{div}_\Gamma e_\Gamma|^2_{L_2(\Gamma)} + h\mu\int_\Gamma \mathrm{tr}\left[(\varepsilon_\Gamma(e_\Gamma) + D^2b\,w + V_\Gamma e_\Gamma)^2\right] \\
&+h\frac{\lambda\gamma}{2}\left|\Delta_\Gamma w + \mathrm{tr}(C_\Gamma e_\Gamma) - \mathrm{div}_\Gamma(D^2b\,e_\Gamma) + (4H^2 - 2K)w\right|^2_{L_2(\Gamma)} \\
&+\mu\gamma h\int_\Gamma \mathrm{tr}\left[\left(S_\Gamma w + C_\Gamma e_\Gamma - \varepsilon_\Gamma(D^2b\,e_\Gamma) + G_\Gamma w + w(D^2b)^2\right)^2\right]d\Gamma
\end{aligned}$$

where $\gamma = \frac{h^2}{12}$. We compute the thermal contribution $\mathcal{E}_{p,t}$ explicitly from Lemma 2, the expansion of τ (5), and the definition of φ and θ (6) :

$$\mathcal{E}_{p,t} = \frac{\bar{\nu}}{2}\int_{S_h} \mathrm{tr}(\varepsilon(T))\bar{\alpha}\tau = \frac{\bar{\nu}}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_{\Gamma^z} \mathrm{tr}(\varepsilon(T))\bar{\alpha}(\tau_1\circ p + z\tau_2\circ p) \tag{18}$$
$$= \frac{\bar{\nu}}{2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\int_\Gamma ((\mathrm{tr}(\varepsilon(T))\theta_1\circ p)\circ T_z)j(z) + ((z\mathrm{tr}(\varepsilon(T))\theta_2\circ p)\circ T_z)j(z).$$

after using the Federer decomposition and a change of variable. Now, Hypothesis 1 (ii) allows us to say $j(z) \approx 1$; and noting that the functions p and T_z are inverses by definition and evaluating the z-integral, we have

$$\mathcal{E}_{p,t} = \frac{h\bar{\nu}}{2}\int_\Gamma (\mathrm{div}_\Gamma e_\Gamma + 2Hw)\varphi \tag{19}$$
$$+\frac{h^3\bar{\nu}}{24}\int_\Gamma (\Delta_\Gamma w + \mathrm{tr}(C_\Gamma e_\Gamma) - \mathrm{div}_\Gamma(D^2b\,e_\Gamma + (4H^2 - 2K)w)\theta.$$

Thus, collecting (17) and (19) gives the desired expression for the potential energy of the shell. For the kinetic energy of the shell we have, from [2] and the Kirchhoff hypothesis:

$$\begin{aligned}\mathcal{E}_k &= \frac{\rho h}{2}\int_\Gamma |\partial_t e_\Gamma|^2 + |\partial_t w|^2 \\ &\quad + \frac{\rho h\gamma}{2}\int_\Gamma |D^2 b\,\partial_t e_\Gamma|^2 + |\nabla_\Gamma \partial_t w|^2 + |D^2 b\,\partial_t e_\Gamma - \nabla_\Gamma \partial_t w|^2.\end{aligned} \tag{20}$$

The kinetic energy of the thermal variables will be discussed later.

Among all kinematically admissible displacements, the actual motion of the shell will make stationary the Lagrangian

$$\mathcal{L}(\mathbf{e}) = \int_0^t \mathcal{E}_k(\mathbf{e}) - \mathcal{E}_{p,e}(\mathbf{e}) + \mathcal{E}_{p,t}(\mathbf{e})\,.$$

Note that we take the variation with respect to $\mathbf{e}$ only: $\frac{\partial}{\partial\psi}\mathcal{L}(\mathbf{e}+\psi\,\hat{\mathbf{e}})\Big|_{\psi=0} = 0$. This results in the following weak form of the equations:

$$\begin{aligned}
&\int_0^t \Big[-\rho[2(\partial_t e_\Gamma, \partial_t \hat{e_\Gamma})_\Gamma + 2\gamma((D^2 b)\partial_t e_\Gamma, (D^2 b)\partial_t \hat{e_\Gamma})_\Gamma \\
&-\gamma(\nabla_\Gamma \partial_t w, (D^2 b)\partial_t \hat{e_\Gamma})_\Gamma \\
&-\gamma((D^2 b)\partial_t e_\Gamma, \nabla_\Gamma \partial_t \hat{w})_\Gamma + 2(\partial_t w, \partial_t \hat{w})_\Gamma + 2\gamma(\nabla_\Gamma \partial_t w, \nabla_\Gamma \partial_t \hat{w})_\Gamma] \\
&2\lambda\gamma(\Delta_\Gamma w, \Delta_\Gamma \hat{w})_\Gamma + 4\mu\gamma\int_\Gamma \mathrm{tr}((S_\Gamma w + G_\Gamma w)(S_\Gamma \hat{w} + G_\Gamma \hat{w})) \\
&+2\lambda(\mathrm{div}_\Gamma e_\Gamma, \mathrm{div}_\Gamma \hat{e_\Gamma})_\Gamma + 4\mu\int_\Gamma \mathrm{tr}(\varepsilon_\Gamma(e_\Gamma)\varepsilon_\Gamma(\hat{e_\Gamma})) \\
&-2\mu(D^2 b\, e_\Gamma, D^2 b\, \hat{e_\Gamma})_\Gamma + 4\lambda(Hw, \mathrm{div}_\Gamma \hat{e_\Gamma})_\Gamma \\
&+4\lambda(\mathrm{div}_\Gamma e_\Gamma, H\hat{w})_\Gamma + 4\mu(w, \mathrm{tr}(\varepsilon_\Gamma(\hat{e_\Gamma})\, D^2 b))_\Gamma + 4\mu(\mathrm{tr}(\varepsilon_\Gamma(e_\Gamma)\, D^2 b), \hat{w})_\Gamma \\
&+2(\sqrt{k_\gamma} w, \sqrt{k_\gamma}\hat{w})_\Gamma + 2\lambda\gamma(\mathrm{div}_\Gamma(D^2 b\, e_\Gamma), \mathrm{div}_\Gamma(D^2 b\, \hat{e_\Gamma}))_\Gamma \\
&+4\mu\gamma\int_\Gamma \mathrm{tr}(\varepsilon_\Gamma(D^2 b\, e_\Gamma)\varepsilon_\Gamma(D^2 b\, \hat{e_\Gamma})) - 2\lambda\gamma(\mathrm{div}_\Gamma(D^2 b\, e_\Gamma), \mathrm{tr}(C_\Gamma \hat{e_\Gamma}))_\Gamma \\
&-2\lambda\gamma(\mathrm{tr}(C_\Gamma e_\Gamma), \mathrm{div}_\Gamma(D^2 b\, \hat{e_\Gamma}))_\Gamma + 2\lambda\gamma(\mathrm{tr}(C_\Gamma e_\Gamma), \mathrm{tr}(C_\Gamma \hat{e_\Gamma}))_\Gamma \\
&+4\mu\gamma\int_\Gamma \mathrm{tr}(C_\Gamma e_\Gamma\, C_\Gamma \hat{e_\Gamma}) - 4\mu\gamma\int_\Gamma \mathrm{tr}(C_\Gamma e_\Gamma\, \varepsilon_\Gamma(D^2 b\, \hat{e_\Gamma})) \\
&-4\mu\gamma\int_\Gamma \mathrm{tr}(\varepsilon_\Gamma(D^2 b\, e_\Gamma)\, C_\Gamma \hat{e_\Gamma}) - 2\lambda\gamma(\mathrm{tr}(D^3 b\, e_\Gamma), \Delta_\Gamma \hat{w})_\Gamma \\
&-2\lambda\gamma(\Delta_\Gamma w, \mathrm{tr}(D^3 b \hat{e_\Gamma}))_\Gamma \qquad (21) \\
&+4\mu\gamma\int_\Gamma \mathrm{tr}(C_\Gamma e_\Gamma\, S_\Gamma \hat{w}) + 4\mu\gamma\int_\Gamma \mathrm{tr}(S_\Gamma w\, C_\Gamma \hat{e_\Gamma}) \\
&-4\mu\gamma\int_\Gamma \mathrm{tr}(S_\Gamma w\, \varepsilon_\Gamma(D^2 b\, \hat{e_\Gamma})) - 4\mu\gamma\int_\Gamma \mathrm{tr}(\varepsilon_\Gamma(D^2 b\, e_\Gamma)\, S_\Gamma \hat{w})
\end{aligned}$$

$$
\begin{aligned}
&-2\lambda\gamma((4H^2-2K)w, \operatorname{tr}(D^3b\,\hat{e_\Gamma}))_\Gamma \\
&-2\lambda\gamma(\operatorname{tr}(D^3b\,e_\Gamma), (4H^2-2K)\hat{w})_\Gamma \\
&-4\mu\gamma(\operatorname{tr}(D^3b\,e_\Gamma(D^2b)^2), \hat{w})_\Gamma - 4\mu\gamma(w, \operatorname{tr}(D^3b\,\hat{e_\Gamma}(D^2b)^2))_\Gamma \\
&+2\lambda\gamma((4H^2-2K)w, \Delta_\Gamma\hat{w})_\Gamma + 2\lambda\gamma(\Delta_\Gamma w, (4H^2-2K)\hat{w})_\Gamma \\
&+4\mu\gamma(w, \operatorname{tr}(S_\Gamma\hat{w}(D^2b)^2))_\Gamma + 4\mu\gamma(\operatorname{tr}(S_\Gamma w(D^2b)^2), \hat{w})_\Gamma \\
&+2\bar{\nu}(\varphi, \operatorname{div}_\Gamma\hat{e_\Gamma})_\Gamma + 4\bar{\nu}(\varphi, H\,\hat{w})_\Gamma + 2\bar{\nu}\gamma(\theta, \Delta_\Gamma\hat{w})_\Gamma \\
&+2\bar{\nu}\gamma(\theta, \operatorname{tr}(C_\Gamma\hat{e_\Gamma})_\Gamma - 2\bar{\nu}\gamma(\theta, \operatorname{div}_\Gamma(D^2b\,\hat{e_\Gamma}))_\Gamma \\
&\left.+2\bar{\nu}\gamma((4H^2-2K)\theta, \hat{w})_\Gamma\right]\, dt = 0
\end{aligned}
$$

We integrate the expression (20) in order to derive the equations (12) and (13) of Theorem 3. These calculations are presented explicitly in [3], so we omit the details. We note that the regularity of the weak solution e is high enough to permit the necessary integration by parts to derive the strong form – this is proved in Proposition 4.2 of [3]. After this, inspection of (20) reveals that there are two fourth-order terms in w. The first, $(\Delta_\Gamma w, \Delta_\Gamma\hat{w})$, will yield the required tangential biharmonic operator Δ_Γ^2 in the strong form. However, the next term is also fourth-order, and we would like to combine the two in analogy to the case of plates, where the second term becomes the biharmonic plus a boundary integral. In fact, in [3] we show that

$$
\begin{aligned}
\int_\Gamma \operatorname{tr}((S_\Gamma w + G_\Gamma w)(S_\Gamma\hat{w} + G_\Gamma\hat{w}))\, d\Gamma &= \int_\Gamma \Delta_\Gamma w \Delta_\Gamma\hat{w}\, d\Gamma \\
&+ \int_\Gamma \langle K\nabla_\Gamma w, \nabla_\Gamma\hat{w}\rangle\, d\Gamma + 2\int_\Gamma \langle D^2b\nabla_\Gamma w, D^2b\nabla_\Gamma\hat{w}\rangle\, d\Gamma \\
&+ \int_\Upsilon (B_\Gamma^1 w \frac{\partial}{\partial\nu}\hat{w} - B_\Gamma^2 w\hat{w})\, d\Upsilon
\end{aligned}
$$

with B_Γ^1 and B_Γ^2 being defined as

$$
\begin{aligned}
B_\Gamma^1 w &= -(\tau\otimes\tau)..(S_\Gamma w + G_\Gamma w) \\
B_\Gamma^2 w &= \langle\nabla_\Gamma\left((\tau\otimes\nu)..(S_\Gamma w + G_\Gamma w)\right), \tau\rangle\,.
\end{aligned} \tag{22}
$$

The operators B_Γ^1 and B_Γ^2 are simply the tangential versions of the same operators which appear in the modeling of Kirchhoff plates [7]. One can show this explicitly by choosing a local basis $\nu = (\nu_1, \nu_2)$; $\tau = (-\nu_2, \nu_1)$ and substituting appropriately. The additional boundary operator B_Γ^3 which appears in the free boundary conditions of Theorem 3 is given by

$$
B_\Gamma^3 A = \partial_t\langle\tau, A\nu\rangle + \langle\operatorname{div}_\Gamma A, \nu\rangle = \langle\nabla_\Gamma(\tau\otimes\nu..A), \tau\rangle + \langle\operatorname{div}_\Gamma A, \nu\rangle\,. \tag{23}
$$

This operator comes from integration of cross-terms involving $S_\Gamma\hat{w}$.

Thermal Equations. Next, we must obtain the equations of motion for the thermal variables φ and θ. Recall that τ is the temperature of the body, measured from a reference temperature τ_0. By combining Fourier's law of heat conduction, the entropy balance law, the second law of thermodynamics for irreversible processes, and using the fact that the change of temperature is small to linearize, we have the following equation for heat transfer in a three-dimensional isotropic, elastic body (see [7], p. 29, and [8], Chapter 1):

$$\Delta\tau - \frac{1}{\kappa}\partial_t\tau - \frac{\eta}{\bar{\alpha}}\partial_t\left(\varepsilon(T)..I\right) = -\frac{\mathcal{H}}{\lambda_0} \tag{24}$$

with

$$\kappa = \frac{\lambda_0}{c\rho}, \qquad \eta = \left(\lambda + \frac{2}{3}\mu\right)\frac{\bar{\alpha}^2\tau_0}{\lambda_0}, \tag{25}$$

where $\lambda_0 > 0$ is the coefficient of thermal conductivity (assumed to be constant), c is the specific heat, ρ is the density of the material, and $\mathcal{H}$ are heat sources and sinks inside the body.

Recalling the definition (5), the equality (2), re-writing the thermal loads as $\mathcal{H} = \mathcal{H}_1 \circ p + b\mathcal{H}_2 \circ p$ (justified again by the assumption that the change in temperature is small), and substituting gives

$$\begin{aligned}
&\Delta(\tau_1 \circ p) + \Delta(b\,\tau_2 \circ p) - \frac{1}{\kappa}\partial_t(\tau_1 \circ p + b\tau_2 \circ p)\\
&\quad -\frac{\eta}{\alpha}\partial_t(\mathrm{div}_\Gamma e_\Gamma + 2Hw)\circ p + \frac{\eta}{\bar{\alpha}}\partial_t(b\,(\Delta_\Gamma w + \mathrm{tr}(C_\Gamma e_\Gamma)\\
&\qquad -\mathrm{div}_\Gamma(D^2 b\, e_\Gamma) + (4H^2 - 2K)w)\circ p) = -\frac{\mathcal{H}_1 \circ p}{\lambda_0} - b\frac{\mathcal{H}_2 \circ p}{\lambda_0}.
\end{aligned}$$

Expanding $\Delta(b\,\tau_2 \circ p) = b\,\Delta(\tau_2 \circ p) + \tau_2 \circ p\Delta b + 2\langle\nabla b, \nabla(\tau_2 \circ p)\rangle$ and multiplying by $\bar{\alpha}$ gives

$$\begin{aligned}
&\Delta(\varphi \circ p) + b\,\Delta(\theta \circ p) + 2H\,\theta \circ p + 2\langle\nabla b, \nabla\theta \circ p\rangle\\
&-\frac{1}{\kappa}\partial_t(\varphi \circ p + b\theta \circ p) - \eta\partial_t(\mathrm{div}_\Gamma e_\Gamma + 2Hw)\circ p + \eta\partial_t(b\,(\Delta_\Gamma w\\
&+\mathrm{tr}(C_\Gamma e_\Gamma) - \mathrm{div}_\Gamma(D^2 b\, e_\Gamma) + (4H^2 - 2K)w)\circ p) = f_1 \circ p + b\,f_2 \circ p
\end{aligned} \tag{26}$$

after defining $f_i = -\frac{\bar{\alpha}\mathcal{H}_i}{\lambda_0}$. Notice that equation (26) is of the form $A_F \circ p + bA_B \circ p = f_1 \circ p + b\,f_2 \circ p$ where A_F denotes the thermal change due to the flexure of the shell, and A_B the change due to bending of the shell. This gives us two coupled equations on the three-dimensional body:

$$\begin{aligned}
&[(\Delta - \frac{1}{\kappa}\partial_t)b\varphi + 2H\,\theta] \circ p + 2\langle\nabla b, \nabla\theta \circ p\rangle\\
&-\eta\partial_t(\mathrm{div}_\Gamma e_\Gamma + 2Hw)\circ p = f_1 \circ p
\end{aligned}$$

$$(\Delta - \frac{1}{\kappa}\partial_t)\theta \circ p + \eta\partial_t((\Delta_\Gamma w + \mathrm{tr}(C_\Gamma e_\Gamma) \\ -\mathrm{div}_\Gamma(D^2 b\, e_\Gamma) + (4H^2 - 2K)w) \circ p) = f_2 \circ p\,.$$

Restricting these to the midsurface gives immediately that

$$\Delta_\Gamma\varphi + 2H\theta - \frac{1}{\kappa}\partial_t\varphi - \eta\partial_t(\mathrm{div}_\Gamma e_\Gamma + 2Hw) = f_1$$

$$\Delta_\Gamma\theta - \frac{1}{\kappa}\partial_t\theta - \eta\partial_t(\Delta_\Gamma w + \mathrm{tr}(C_\Gamma e_\Gamma) - \mathrm{div}_\Gamma(D^2 b\, e_\Gamma) + (4H^2 - 2K)w) = f_2$$

as desired, since $\langle \nabla b, \nabla_\Gamma \theta\rangle = 0$. Finally, the boundary conditions on φ and θ are given by Newton's law of cooling.

THEOREM 5 (WELL-POSEDNESS) *The thermoelastic shell model presented in Theorem 3 generates a C_0 semigroup of contractions $\left\{e^{\mathbf{A}t}\right\}_{t\geq 0}$ on the space*

$$\mathcal{H} = H^2(\Gamma) \times H^1_\gamma(\Gamma) \times [H^1(\Gamma)]^2 \times [L_2(\Gamma)]^2 \times L_2(\Gamma) \times L_2(\Gamma)$$

Therefore for initial data $\mathbf{x}^0 = [w^0, w^1, e^0_\Gamma, e^1_\Gamma, \theta^0, \phi^0] \in \mathcal{H}$, the solution $\mathbf{x}(t) = [w, \partial_t w, e_\Gamma, \partial_t e_\Gamma, \theta, \phi]$ is given by $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}^0$.

Proof: Straightforward calculations show that $\mathbf{A}$ is maximal dissipative – that is, $\langle \mathbf{A}X, X\rangle_{\mathcal{H}} \leq 0$ and $\langle \mathbf{A}^* X, X\rangle_{\mathcal{H}} \leq 0$ for all $X \in \mathcal{H}$. Thus, by the Lumer-Phillips theorem, the system of equations (12)-(16) is well-posed.

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