AN IMPROVEMENT ON SAATY'S AHP

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Abstract: AHP is a commonly used method in analyzing multi-factor evaluation or multi-attribute decision-making problems. However, AHP has some serious logistic mistakes. Since it cannot maintain the independence of alternatives, AHP cannot lead to an ordering of alternatives that is consistent with their ordering before the values of the assessments or the quantity of alternatives change. Therefore, using a numerical illustration, the mistake of traditional AHP is found out. An improvement on AHP which can keep the consistency of the alternatives' ordering results is put forward in this paper.

Key words: AHP; Mistake; Improvement

1. INTRCDUCTION

Analytic Hierarchy Process (AHP) is a system analysis method put forward by professor Saaty in the 1980s. It is a connection of quantitative and qualitative analysis method and is commonly used to study multi-objective or multi-attribute problems in many fields.

By quantifying the process of thinking and subjective judges of human beings, computation could be greatly reduced; meanwhile, AHP strives to maintain the consistency of the decision-maker's process of thinking and the principles of decision-making and therefore tries to solve satisfactorily the complex social and economic problems which could not be fully quantified. This method has now been a key component of the science of decision-making. Although this method is widely applied, it has its mistakes. Besides the usually cited problems of inconsistency in comparison matrixes resulted from the definitions of Saaty's 1-9 scales [1]~[6], we find one serious mistake in AHP: it is Logic wrong in calculating the final priority weights of alternatives. As a result, decision results derived by applying this method could seriously deviate from the one the decision maker actually desires. Furthermore, the new priority ranking of the alternatives will be completely different from the old when the group of evaluation alters.

In this paper, Section two explains the principles and calculation of AHP. Section three proves the logical mistake in AHP theoretically and numerically. Section four gives corrections to AHP and gives a new correct comprehensive ranking method. And section five concludes.

2. THE TRADITIONAL AHP

For a typical hierarchy, the overall goal is situated at the highest level; element (attributes) with similar nature are grouped at the same interim levels and decision variables (alternatives) are situated at the lowest level. See Figure. 1. By means of pairwise comparisons of the elements using the scales as suggested by Saaty, reciprocal matrixes for all clusters can be formulated. In order to measure the level of consistency of a reciprocal matrix, a consistency test has been proposed. After finding the maximum eigenvalue and the corresponding eigenvector of each reciprocal matrix in each cluster, together with some manipulations in matrix algebra, a ranking of the alternatives can be obtained.

The steps of AHP are:

Step 1: Set up a hierarchy model:

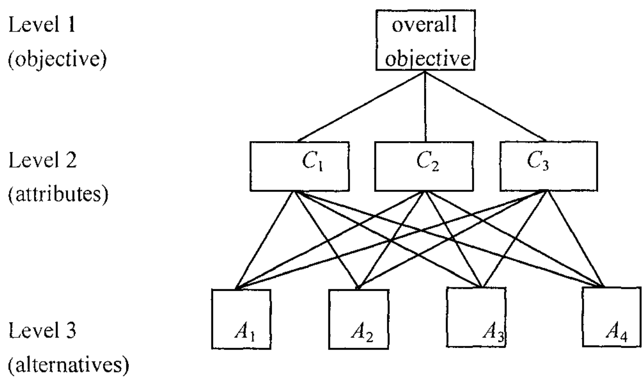

Figure. 1 The hierarchy model

Step 2:Set up the comparison matrix of each level.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix}$$

where, a_{ij} is an exact number representing the scale for the relative importance of the i-th sub-element over the j-th sub-element.

Usually we can use the Saaty's scale for pairwise comparison as follow:

Table. 1 The Saaty's scale for pairwise comparison

Saaty's scale	The relative importance of the two sub-elements
1	Equal importance
3	Moderate importance of one over another
5	Strong importance
7	Very strong importance
9	Extreme importance

Step 3: Consistency test

If the comparison matrix is "perfectly consistent", the scale of comparison matrix should be satisfied with:

$$a_{ii}=1;\ a_{ij} = \frac{1}{a_{ji}};\ a_{ij} = \frac{a_{ik}}{a_{jk}},\ i,j,k = 1,2,\ldots,m \tag{1}$$

In AHP, the decision maker should be consistent in the preference ratings give in the pairwise comparison matrix. Before using the scale, the comparison matrix should be checked for consistency. The focus of this paper is not the consistency of the comparison matrix. So all comparison matrixes in this paper are consistent matrix.

Step 4: Calculation of priority weights of each level

According to Saaty (1980), the priority weight of each level can be derived from the normalized eigenvector of corresponding matrix as follow:

$$AW = \lambda_{\max} W \tag{2}$$

where, $\lambda_{\max}$ and W represent the maximum eigenvalue and the corresponding normalized eigenvector of comparison matrix A. We have:

$$\sum_{i=1}^{n} W_i = 1 \tag{3}$$

Step 5: Calculation of final priority of alternatives

According to above the priority weights of each level, we can get the final priority of alternatives by using matrix algebra:

$$W_i = \sum_{j=1}^{m} W_j^c W_i^j \ , \quad i = 1,2,\ldots,n \tag{4}$$

where, W_i represents the final priority weight of the i-th alternative; W_j^c represents the priority weight of the j-th attribute; W_i^j represents the priority weight of the i-th alternative for the j-th attribute.

Obviously there should be:

$$\sum_{i=1}^{n} W_i = \sum_{i=1}^{n}\sum_{j=1}^{m} W_j^c W_i^j = 1 \tag{5}$$

3. THE MISTAKE OF TRADITIONAL AHP

We will find the Logic mistake of AHP through following analysis and illustration.

3.1 Numerical Illustration of AHP

A firm will make a decision. In this decision problem, the firm has five alternatives,A_1, A_2, A_3, A_4, A_5. The firm would evaluate the five alternatives from three attributes, C_1, C_2, C_3. Next, we will apply AHP to evaluate the above five alternatives.

First, set up the AHP hierarchy model as Fig. 2 shows:

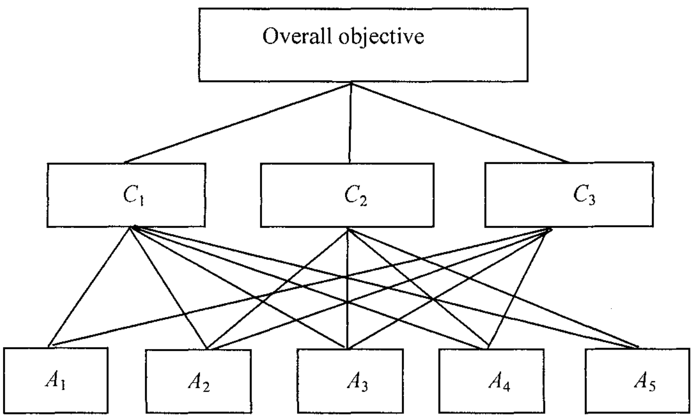

Figure. 2 The hierarchy model of the firm

Now, set up the comparison matrix to show the importance of the alternatives, and calculate the priority weights of the attributes, W_j^c. The calculation results are shown in Table. 2.

Table. 2 For the overall objective, the relative importance (weights) of attributes

Overall Objective	C_1	C_2	C_3	W_j^c
C_1	1	1/5	1/3	0.111
C_2	5	1	5/3	0.556
C_3	3	3/5	1	0.333

For each attribute, construct the comparison matrixes at the alternative level (All are strictly consistent matrixes), and calculate the priority weights of each alternative relative to attribute k, W_j^k (k=1, 2, 3). The calculation results are shown in Table. 3~5.

Table. 3 For C_1, the relative priority weight of alternatives

C_1	A_1	A_2	A_3	A_4	A_5	W_i^1
A_1	1	3	5	4	7	0.519
A_2	1/3	1	5/3	4/3	7/3	0.173
A_3	1/5	3/5	1	4/5	7/5	0.104
A_4	1/4	3/4	5/4	1	7/4	0.13
A_5	1/7	3/7	5/7	4/7	1	0.074

Table. 4 For C_2, the relative priority weight of alternatives

C_2	A_2	A_3	A_4	A_5	W_i^2
A_2	1	1/7	1/3	1/5	0.063
A_3	7	1	7/3	7/5	0.438
A_4	3	3/7	1	3/5	0.188
A_5	5	5/7	5/3	1	0.313

Next, calculate the final priority weights of the alternatives, W_i:

$$W_i = \sum_{j=1}^{3} W_j^c W_i^j$$

Table. 5 For C_3, the relative priority weight of alternatives

C_3	A_1	A_2	A_3	A_4	W_i^3
A_1	1	1	3	3	0.375
A_2	1	1	3	3	0.375
A_3	1/3	1/3	1	1	0.125
A_4	1/3	1/3	1	1	0.125

The results are shown in Table. 6.

Table. 6 The final priority weights of alternatives

Attribute / Alternative	C_1	C_2	C_3	W_i	Ranking results
	0.111	0.556	0.333		
A_1	0.519	0	0.375	0.183	2
A_2	0.173	0.063	0.375	0.179	4
A_3	0.104	0.438	0.125	0.296	1
A_4	0.13	0.188	0.125	0.16	5
A_5	0.074	0.313	0	0.182	3

The final ranking of the alternatives is:

$$A_3 > A_1 > A_5 > A_2 > A_4 \text{。}$$

3.2 The Logic Mistake in AHP

Now if, for certain reasons, alternative A1 could not be carried out any more, the ranking of the remaining four alternatives should be:

$$A_3 > A_5 > A_2 > A_4 \text{。}$$

However, if now we apply AHP to evaluate the following four alternatives once again, we will derive a completely different ranking order. The calculation results are shown in Table.7, 8, 9 and 10, with the weights of the alternatives being the same as in Table. 2.

Table. 7 For C_1, the relative priority weights of alternatives

C_1	A_2	A_3	A_4	A_5	W_i^1
A_2	1	5/3	4/3	7/3	0.36
A_3	3/5	1	4/5	7/5	0.216
A_4	3/4	5/4	1	7/4	0.27
A_5	3/7	5/7	4/7	1	0.154

Table. 8 For C_2, the relative priority weights of alternatives

C_2	A_2	A_3	A_4	A_5	W_i^2
A_2	1	1/7	1/3	1/5	0.063
A_3	7	1	7/3	7/5	0.438
A_4	3	3/7	1	3/5	0.188
A_5	5	5/7	5/3	1	0.313

Table. 9 For C_3, the relative priority weights of alternatives

C_3	A_2	A_3	A_4	W_i^3
A_2	1	3	3	0.6
A_3	1/3	1	1	0.2
A_4	1/3	1	1	0.2

Table. 10 The final priority weights of alternatives

Attribute / Alternative	C_1	C_2	C_3	W_i	Ranking results
	0.111	0.556	0.333		
A_2	0.36	0.063	0.6	0.275	2
A_3	0.216	0.438	0.2	0.334	1
A_4	0.27	0.188	0.2	0.201	3
A_5	0.154	0.313	0	0.191	4

The new ranking of the remaining alternatives become: $A_3 > A_2 > A_4 > A_5$, which is greatly different from the previous ranking of $A_3 > A_5 > A_2 > A_4$.

Now we assume it is A_3, instead of A_1 that could not be carried out any more. According to AHP, we can get the weights of the remaining four alternatives as shown in Table.11.

Table. 11 The final priority weights of alternatives

Attribute / Alternative	C_1	C_2	C_3	W_i	Ranking results
	0.111	0.556	0.333		
A_1	0.579	0	0.429	0.207	4
A_2	0.193	0.111	0.429	0.226	3
A_4	0.145	0.333	0.143	0.249	2
A_5	0.083	0.556	0	0.318	1

According to AHP, the ranking of the remaining four alternatives is: $A_5 > A_4 > A_2 > A_1$. This ranking is greatly different from the previous ranking of $A_1 > A_5 > A_2 > A_4$.

3.3 Analysis The Mistake in AHP

The key to judge whether AHP is mistaken is to see whether equation (3)~(5) holds. That is, whether there is need to normalize the sub-elements of the eigenvector of the matrix.

It is right to normalize the weights of the attributes, although actually, the ranking of the alternatives won't be affected whether we do the normalization or not. However, it is not correct to normalize the priority weights of the alternatives. For example, assume the decision matrix of a multi-attribute decision making problem is:

$$A = \begin{Bmatrix} 0.6 & 0.02 \\ 0.2 & 0.06 \end{Bmatrix}$$

where x_{ij} represents the utility that the decision maker derives from alternative i with respect to attribute j. The weights of the attributes both are 0.5.

If the decision maker did not know the values in the decision matrix, when he makes decisions by applying AHP, for the two attributes, he will give the following two comparison matrixes:

$$D_1 = \begin{Bmatrix} 1 & 3 \\ 1/3 & 1 \end{Bmatrix}, \quad D_2 = \begin{Bmatrix} 1 & 1/3 \\ 3 & 1 \end{Bmatrix}$$

Calculating the weights of the alternatives with respect to the two attributes respectively, we have: $W^1 = \{3/4, 1/4\}^T$, $W^2 = \{1/4, 3/4\}^T$

Then the final priority weights of the alternatives are: 1/2. That is, the two alternatives are the same to the decision maker. However, from the decision matrix, it is obvious that alternative 1 is more desirable than alternative 2. The cause of this discrepancy lies in the fact that the utilities of the alternatives in respect to attribute 2 have been enlarged relative to that in respect to attribute 1. For example, the utility for alternative 1 in respect to attribute 2, relative to attribute 1, has been enlarged from 1/30 to 1/3, and the utility for alternative 2 in respect to attribute 2, relative to attribute 1, has been enlarged from 3/10 to 3. In all, the fundamental cause of this discrepancy lies in the logic mistake in AHP in the respect of normalization and final priority weight calculation.

Further suppose the alternative the firm has is more than five. Suppose the firm has another X-5 ($X > 5$) alternatives: $A_6, A_7, ..., A_X$, and the weights of these alternatives all are zero except for one attribute, e.g., C_1 (or C_2, C_3). Suppose the weights of the X-5 alternatives are the same for attribute C1, and are equal to the weight of alternative A_1, or any other alternative, with respect to attribute C_1.

Now give the comparison matrix. The elements in the matrix, a_{ij} ($i, j = 1,2,3,4,5$), do not vary with the number of alternatives. That is,

$$W_1^1 = W_6^1 = \ldots = W_X^1 \circ$$

According to equation (3), there is:

$$\sum_{i=1}^{X} W_i^1 = 1 \tag{6}$$

Therefore, when the number of alternatives increases, although the relative values of the priority weights (a_{ij}) of the alternatives with respect to attribute C_1 remain the same, the absolute values decrease with the number of alternatives. That is, when the number of alternatives increases, there are more alternatives whose weights are not zero in respect to attribute C_1, with the sum of the weights still being 1. Hence, the final priority weight varies with the number of alternatives. When the number of alternatives decreases, the opposite will occur. Thus, we can see that AHP could not keep the independence of alternatives since the priority weights are affected by other alternatives.

However, from intuition, we know that alternatives should be independent. For example, in the overall evaluations of students, student i is better than student j, no matter whether they are evaluated in the whole class or in the whole grade or in the whole school. It is unbelievable if student i is better than j when evaluated in the whole class but obvious worse than j when evaluated in the whole grade. While the latter conclusion is right what we would reach when applying AHP.

4. IMPROVEMENT ON AHP

The cause of the mistake in AHP lies in the fact that it could not maintain the independence of alternatives. Therefore, it is of crucial importance to keep the relative utility of the attributes constant in order to correct AHP.

To show the correct calculation steps for AHP, we take the previous example once again.

(1) Set up a hierarchy model as the previous example shows.

(2) Set up the comparison matrix, here we do not set up the comparison matrix on the attribute level any more, we only set up the comparison matrix on the alternative level for a given attribute.

(3) Select an alternative for which the weight of no attribute is zero as the benchmark, and then give the comparison matrix for the attributes in the benchmark alternative.

Select alternative A_3 as the benchmark. Calculate the weights of the alternatives, α_3^j. See Table.12.

Table. 12 Relative weights of the attributes of the benchmark alternative A3

A_3	C_1	C_2	C_3	α_j^3
C_1	1	1/7	1	0.111
C_2	7	1	7	0.778
C_3	1	1/7	1	0.111

(4) Let the relative total utility of the benchmark alternative be 1, i.e., $U_* = 1$. Calculate the relative total utility of the other alternatives.

$$U_i = \sum_{j=1}^{n} \alpha_*^j \frac{W_i^j}{W_*^j} \tag{7}$$

Where * represents the benchmark alternative and n represents the number of alternatives.

(5) Rank the alternatives according to the relative total utilities of the alternatives.

As for our previous example, the relative total utilities of the alternatives are as follows.

Table. 13 Ranking of the alternatives (the benchmark being A3)

Attribute / Alternatives	C_1	C_2	C_3	U_i	Ranking result
A_1	0.556	0	0.333	0.889	2
A_2	0.185	0.111	0.333	0.63	4
A_3	0.111	0.778	0.111	1	1
A_4	0.139	0.333	0.111	0.583	5
A_5	0.079	0.556	0	0.635	3

The ranking of the alternatives is: $A_3 > A_1 > A_5 > A_2 > A_4$.

If alternative A1 could not be implemented any more for certain reasons, calculate the relative total utilities of the remaining alternatives once again, with the benchmark still being A_3. The calculation results are shown in Table.14.

The ranking of the remaining alternatives is: $A_3 > A_5 > A_2 > A_4$.

There is no change in the ranking of the remaining alternatives when alternative A_1 is taken off.

Table.14 Ranking of the alternatives (the benchmark being A_3)

Attribute / Alternative	C_1	C_2	C_3	U_i	Ranking results
A_2	0.185	0.111	0.333	0.63	3
A_3	0.111	0.778	0.111	1	1
A_4	0.139	0.333	0.111	0.583	4
A_5	0.079	0.556	0	0.635	2

Now, if it is not alternative A_1, but A_3, that could not be implemented any more, we could recalculate the relative total utilities of the remaining alternatives, with another alternative as the benchmark, for example, A_2. Note that now the comparison matrix that shows the relative weights of the attributes of the benchmark alternative for the total utility has changed. The ranking of the alternatives is shown in Table.15.

Table.15 Ranking of the alternatives (the benchmark being A_2)

Attribute / Alternative	C_1	C_2	C_3	U_i	Ranking results
A_1	0.882	0	0.529	1.412	1
A_2	0.294	0.176	0.529	1	3
A_4	0.221	0.529	0.176	0.927	4
A_5	0.126	0.880	0	1.008	2

The ranking of the remaining alternatives is: $A_1 > A_5 > A_2 > A_4$.

There is no change in the ranking of the remaining alternatives when alternative A_3 is taken off.

5. CONCLUSIONS

The paper shows that the prevalent AHP has a serious mistake that makes the alternatives dependent on others, so that when there is one alternative taken off or more alternatives considered, there will be discrepancy of the other alternatives as compared with before.

Our improvement on AHP, however, could maintain the independency of alternatives, so that when the number of alternatives changes, the ranking of the other alternatives remain the same as before. Although our method does not calculate the weights of the attributes, this idea or information is already reflected in the calculation of the final priority ranking indexes, or the calculation of the relative weights of the attributes of the benchmark alternative for the total utility.

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