MODELING OF TOPOLOGY VARIATIONS IN ELASTICITY

Serguei A. Nazarov*
Institute of Mechanical Engineering Problems, Laboratory of Mathematical Methods, Russian Academy of Sciences, V.O. Bol'shoi 61, 199178 St. Petersburg, Russia
serna@snark.ipme.ru

Jan Sokolowski†
Institut Elie Cartan, Laboratoire de Mathematiques, Université Henri Poincaré Nancy I, BP 239, 54506 Vandoeuvre-Les-Nancy Cedex, France
Jan.Sokolowski@iecn.u-nancy.fr

Abstract Two approaches are proposed for the modeling of deformation of elastic solids with small geometrical defects. The first approach is based on the theory of self adjoint extensions of differential operators. In the second approach function spaces with separated asymptotics and point asymptotic conditions are introduced, and the variational formulation is established. For both approaches the accuracy estimates are derived. Finally, the spectral problems are considered and the error estimates for eigenvalues are given.

Keywords: Shape optimization, topology optimization, asymptotic analysis, elliptic operators, singular perturbations

Introduction

It seems that in the literature on shape optimisation there is a lack of general numerical method or technique, beside the level set method, that can be applied in the process of optimisation of an arbitrary shape functional (SF) for simultaneous boundary and topology variations. In the paper [22] (see also [19]) the so–called topological derivative (TD) of an arbitrary SF is introduced. TD usually determines whether a

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change of topology by nucleation of a small hole, or in similar setting of a small inclusion at a given point $x \in \Omega$, would result in improving the value $J(\Omega)$ of a given SF or not. In the paper the *boundary topology variations* are considered for boundary value problems for elastic solids. The singular perturbations of the geometrical domain Ω are defined by small arcs $\gamma_h^1, ..., \gamma_h^I$ of the length $O(h)$ on the boundary $\partial\Omega$.

We propose two efficient approaches to the modeling of topological variations. First approach is developped in the framework of the self-adjoint extensions of differential operators, the second uses the function spaces with the detached asymptotics. In both cases, the main idea consists in modeling of small defects or inhomogeneities by concentrated *actions*, the so-called potentials of zero-radii. In this way the solution $u(\varepsilon, h)$ with *singular* behaviour for $\varepsilon \to 0+$ is replaced by a function with the singularities at the centres $P^1, ..., P^I$ of the defects. The modern framework of analysis of elliptic boundary value problems in non smooth domains allows for the the relatively complete theory of singular solutions and provides the techniques of derivation of error estimates for asymptotic approximations. We can use the known results in this field for the solution of shape and topology optimization problems in an *inverse order.* First, the localization and integral atributes of openings are determined, followed by the appropriate changes of the topology of geometrical domains. The proposed two different approaches to topology optimization have some positive features. The first approach deals with selfadjoint operators, so can be readily extended to the evolution boundary value problems. The second approach, based on the *generalized Green's formulae*, results in the variational problem formulation with the solution given by a stationary point of an auxiliary functional close in its form to the energy functional.

1. Problem Formulation

Let us consider the deformations of plane heterogeneous anisotropic elastic body $\Omega \subset \mathbb{R}^2$ clamped on small parts of the boundary $\Gamma = \partial\Omega$ in the form of closed connected curves $\gamma_h^1, ..., \gamma_h^I$. Instead of tensor notation, we make use of the matrix notation which we describe briefly. The constitutive relations in the elasticity theory are written with the elastic fields in the form of columns. First, two matrices are introduced,

$$D(x)^\top = \begin{bmatrix} x_1 & 0 & \alpha x_2 \\ 0 & x_2 & \alpha x_1 \end{bmatrix} , \qquad d(x)^\top = \begin{bmatrix} 1 & 0 & -\alpha x_2 \\ 0 & 1 & \alpha x_1 \end{bmatrix} , \tag{1}$$

where $\alpha = 2^{-1/2}$ is the normalizing coefficients, and $\top$ stands for transposition. The first matrix is used to define the column of strains from the displacement column $u = (u_1, u_2)^\top$,

$$\varepsilon(u) := (\varepsilon_{11}(u), \varepsilon_{22}(u), \alpha^{-1}\varepsilon_{12}(u))^\top = D(\nabla)u \; . \tag{2}$$

Here $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2})$ is the gradient, and $\varepsilon_{jk}(u)$ are Cartesian components of the strain tensor (the multipliers α in (1) and (2) make the norms of vector and tensor of strains equal). The second matrix generates the rigid motions $d(x)b$ of the body Ω for any column $b \in \mathbb{R}^3$. The Hooke's law

$$\sigma(u; x) = A(x)\varepsilon(u; x) \; , \tag{3}$$

represents the column of stresses in function of the strains (2), and includes the symmetric and positively definite (3×3)-matrix function A of elastic moduli, which is supposed to be smooth function of the variable x. In view of (1)-(3), the equilibrium equations and the boundary conditions of traction free type are given as follows

$$L(x, \nabla)u(h, x) := D(-\nabla)^\top A(x) D(\nabla)u(h, x) = f(x) \; , \quad x \in \Omega \; , \tag{4}$$

$$B(x, \nabla)u(h, x) := D(n(x))^\top A(x) D(\nabla)u(h, x) = 0 \; , \quad x \in \Sigma_h \; . \tag{5}$$

Here, $n = (n_1, n_2)^\top$ is the unit column of external normal vector to the contour Γ, the contour is supposed to be sufficiently smooth for the sake of simplicity of the presentation. In (5)

$$\Sigma_h = \Gamma \setminus \{\gamma_h^1 \cup ... \cup \gamma_h^I\} \; ,$$

and γ_h^j are arcs of the length hl_j, with the centres $P^j \in \partial\Omega$, where $h \in (0, h_0]$ is a small parameter and $l_1, ..., l_I$ are fixed constants. The elastic body is clamped on the sets γ_h^j,

$$u(h, x) = 0 \; , \quad x \in \gamma_h^1 \cup ... \cup \gamma_h^I \; . \tag{6}$$

The Dirichlet condition (6) provides the Korn inequality

$$\|u \; ; \; H^1(\Omega)\| \leq K(h) \|D(\nabla)u \; ; \; L_2(\Omega)\| \; , \tag{7}$$

however the dependence of the multiplier $K(h)$ on the parameter h is to be clarified.

Proposition 1 *Let* $I \geq 2$. *For any field* $u \in H^1(\Omega)^2$ *verifying the Dirichlet conditions* (6), *the Korn inequality holds with the multiplier* $K(h)$ *such that*

$$K(h) \leq c|\ln h| \; . \tag{8}$$

The estimate is asymptotically exact. In (8) *the constant* c *is independent of* u *and* $h \in (0, h_0]$ *with* $h_0 < 1$.

Note that in the case $I = 1$ the Korn inequality (7) is still valid, but the multiplier $K(h)$ becomes of order h^{-1} (cf. [21]) and thus, it does not satisfy estimate (8).

2. Modeling of Singularly Perturbed Boundary Value Problem

We consider the functional

$$\mathcal{F}(u;h) = \int_{\Omega} J(x;u(h,x))dx \ . \tag{9}$$

Asymptotic structures for specific problems with the logarithmic growth of fundamental solutions, turn out to be quite complex and therefore, of limited practical interest for analysis of functional (9). The main particularity of the asymptotic analysis, beside the presence of boundary layers near the arcs $\gamma_h^1, ..., \gamma_h^I$, is the form of asymptotic terms which are rational functions of the large parameter $|\ln h|$. Such phenomenon was discovered by Il'in for the scalar problem in [6] (see also [7] and [13]). A simplification caused regarding the asymptotics with respect to the parameter $|\ln h|^{-1}$ is not sufficient to provide the analysis, since the leading terms of asymptotics do not reflect the distribution of contact regions and do not exhibit the interactions between the regions.
We propose an appproach, based on the modeling of problem (4)-(6), by means of auxiliary boundary value problems with the boundary conditions of traction free type on the punctured contour $\partial\Omega \setminus \{P^1, ..., P^I\}$ and with the prescribed class of singularities at the points $P^1, ..., P^I$. Such singularities are obtained by an application of forces concentrated at the points, and therefore, imitate the reaction of the elastic body at the obstacles $\gamma_h^1, ..., \gamma_h^I$. Thus, in the setting, the models take into account the interaction between the elastic body with the rigid foundation. On the other hand, the proposed singularities are not included in the energy class $H^1(\Omega)^2$, however the resulting singular solutions are still in the space $L_q(\Omega)$, for $q \geq 1$. The main profit from our point of view for such modeling is the possibility, with the singular solutions, for asymptotically exact approximation of functional (9), under the condition that for some $q \in [1, \infty)$ and for any $u, v \in L_q(\Omega)^2$ the following inequality is valid

$$|\mathcal{J}(u;h) - \mathcal{J}(v;h)| \leq \tag{10}$$
$$\leq c_{\mathcal{F}} \|u - v \ ; \ L_q(\Omega)\| \Big(\|u \ ; \ L_q(\Omega)\|^{q-1} + \|v \ ; \ L_q(\Omega)\|^{q-1} \Big)$$

with the constant $c_{\mathcal{F}}$ independent of $h \in (0, h_0]$ and u, v.
Modeling defects in media by an application of extensions of differential operators which give rise to the singular solutions comes back to the work [3] and is developped in [20], [16], [18], [17], [9] and in other publications, for problems of mathematical physics and general elliptic systems. There are two possibilities for realization of such ideas. First

of all, the operator $L(x, \nabla)$ in (4), considered as an unbounded operator in the space $L_2(\Omega)^2$, subsists the restriction of the domain of definition, which becomes smaller compared to intrinsic domain $H^1(\Omega)^2$. In this way the domain of the adjoint becomes wider, and finally the selfadjoint operator is selected in the form of an intermediate operator. Under the proper choice of extension parameters, the selfadjoint operator asymptotically acquires the attributes of singularly perturbed problem such as the energy functional and the spectre (see [17], [9], [19]). Since the selected operator is selfadjoint, the classical semigroup theory can be used to construct solutions for the associated evolution problems. On the other hand, for our problem, the domain of selfadjoint extension depends on the large parameter $|\ln h|$, which could leads to ill posed problems for numerical methods when applied for solution of shape optimization or shape inverse problems (see [8], [22], [23], [5]). This difficulty can be avoided by application of slightly different technique, including the space with separated asymptotics (see [18] and others). Roughly speaking, the boundary value problem is defined in larger class, compared with the energy space $H^1(\Omega)^2$. In the class, the behaviour of functions at points $P^1, ..., P^I$ is prescribed a priori. The coefficients of asymptotic expansions satisfy some additional relations, in order to ensure the unique solvability of boundary value problems. Matching conditions for parameters of selfadjoint extensions, and the relations called asymptotic point conditions, result in the exactly same solutions obtained by the first and the second approach.

3. Modeling with Self Adjoint Extensions

We denote by $\chi_1, ..., \chi_J$ the cutoff functions with mutually disjoint supports, equal to one in neighbourhoods of the points $P^1, ..., P^I$, respectively, and by $T^j = (T^{j1}, T^{j2})$ the Poisson kernel, i.e., the (2×2)-matrix function, each column $T^{jk}, k = 1, 2$ is a solution of the elasticity boundary value problem in the half-plane $\{x \ : \ n(P^j)^\top x > 0\}$ under unit force concentrated at the point P^j and directed in the positive direction of $\mathcal{O}x_k$ (linear combinations of the Boussinesq-Cerruti solutions problems).

The unbounded operator $\mathcal{L}$ in $L_2(\Omega)^2$ defined by the differential expression $L(x, \nabla)$ with the domain

$$\mathcal{D}(\mathcal{L}) = \{v \in H^2(\Omega)^2 \ : \ B(x, \nabla)v = 0 \quad \text{on } \partial\Omega \ , \quad v(P^1) = ... = v(P^I) = 0\} \tag{11}$$

is closed and symmetric, however the adjoint $\mathcal{L}^*$ has the larger domain compared to (11),

$$\mathcal{D}(\mathcal{L}^*) = \{v \in \mathfrak{D} \ : \ B(x,\nabla)v = 0 \quad \text{on } \partial\Omega \setminus \{P^1,...,P^I\}\} \ , \tag{12}$$

$$\mathfrak{D} = \{v(x) = \widetilde{v}(x) + \sum_{i=1}^{I} \chi_i(x)\left[b^i + T^i(x-P^i)a^i\right]$$

$$\widetilde{v} \in H^2(\Omega)^3 \ , \ \widetilde{v}^1(P^1) = ... = \widetilde{v}^I(P^I) = 0 \ , a^i, b^i \in \mathbb{R}^3\} \ .$$

The following representation is well known

$$T^j(x) = -\mathcal{T}^{j0} \ln|x| + \mathcal{T}^{j1}(|x|^{-1}x) \ , \tag{13}$$

where $\mathcal{T}^{j1}$ is a smooth matrix function on the semisphere and $\mathcal{T}^{j0}$ is a constant (2×2)-matrix, symmetric and positive definite.
By comparison of formulae (11) and (12) we can see that the defect of the operator $\mathcal{L}$ is $(3I : 3I)$. The coefficients b^i_1, b^i_2, b^i_3 and b_j from (12) are collected in the column $\mathbf{f} \in \mathbb{R}^{3I}$, the remaining coefficients are collected in the column $\mathbf{a}$.
Lemma 1 *Let* $\mathbf{S}$ *be a symmetric* $(3I \times 3I)$*-matrix. The restriction* $\mathbf{L}$ *of the operator* $\mathcal{L}^*$ *to the linear subset of* (12)

$$\mathcal{D}(\mathbf{L}) = \{v \in \mathcal{D}(\mathbf{L}^*) \ : \ \mathbf{f} = \mathbf{S}\mathbf{a}\} \tag{14}$$

is a selfadjoint operator in $L_2(\Omega)^2$. *If the matrix* $\mathbf{S}$ *is not singular, then under condition* $I > 1$ *the equation*

$$\mathbf{L}\mathbf{v} = f \tag{15}$$

admits the unique solution for each $f \in L_2(\Omega)^2$.
The proper choice of parameters of selfadjoint extension, i.e., the selection of the matrix $\mathbf{S}$ is performed in Section 9.5 in such a way that the solution $\mathbf{v}$ of equation (15) becomes an approximation of the solution to problem (4)-(6).

4. Modeling in Spaces with Separated Asymptotics

The linear set (12) with the norm

$$\|v \ ; \ \mathfrak{D}\| = (\|\widetilde{v} \ ; \ H^2(\Omega)\|^2 + \|\mathbf{a} \ ; \ \mathbb{R}^{3I}\|^2 + \|\mathbf{f} \ ; \ \mathbb{R}^{3I}\|^2)^{\frac{1}{2}} \ .$$

becomes the Hilbert space. Two projection operators are introduced $\pi^{\pm} \ : \ \mathfrak{D} \to \mathbb{R}^{3I}$, which take from the function v the columns of coefficients

$$\pi^- v = \mathbf{a} \ , \quad \pi^+ v = \mathbf{f} \ .$$

Let us consider the boundary value problem of linear elasticity with the asymptotic conditions at the points $P^1, ..., P^I$,

$$\begin{aligned} L(x,\nabla)\mathfrak{v}(x) = f(x)\ , \quad x \in \Omega\ , \\ B(x,\nabla)\mathfrak{v}(x) = 0\ , \quad x \in \partial\Omega \setminus \{P^1, ..., P^I\}\ , \\ S\pi^-\mathfrak{v} - \pi^+\mathfrak{v} = 0 \in \mathbb{R}^{3I}\ . \end{aligned} \tag{16}$$

It is easy to see that for the same matrix S in (14) and in (16) the solutions $\mathbf{v} \in \mathcal{D}(\mathbf{L})$ of (15) and $\mathfrak{v} \in \mathfrak{D}$ coincide.

Proposition 2

1) *For functions* $\mathfrak{v}, \mathfrak{u} \in \mathfrak{D}$ *the generalized Green's formula is valid*

$$\begin{aligned} (L\mathfrak{v}, \mathfrak{u})_\Omega + (B\mathfrak{v}, \mathfrak{u})_{\partial\Omega} + \langle S\pi^-\mathfrak{v} - \pi^+\mathfrak{v}, \pi^-\mathfrak{u}\rangle = \\ = (\mathfrak{v}, L\mathfrak{u})_\Omega + (\mathfrak{v}, B\mathfrak{u})_{\partial\Omega} + \langle \pi^-\mathfrak{v}, S\pi^-\mathfrak{u} - \pi^+\mathfrak{u}\rangle\ , \end{aligned} \tag{17}$$

where $(\cdot,\cdot)_\Xi$ *and* $\langle\cdot,\cdot\rangle$ *are scalar products in the spaces* $L_2(\Xi)^2$ *and* $\mathbb{R}^{3I}$, *respectively.*

2) *The function* $\mathfrak{v} \in \mathfrak{D}$ *is a solution to problem* (16) *if and only if it is a stationary point of the functional*

$$\mathfrak{E}(\mathfrak{v}) = \frac{1}{2}(L\mathfrak{v}, \mathfrak{v})_\Omega + \frac{1}{2}(B\mathfrak{v}, \mathfrak{v})_{\partial\Omega} + \frac{1}{2}\langle S\pi^-\mathfrak{v} - \pi^+\mathfrak{v}, \pi^-\mathfrak{v}\rangle - (f, \mathfrak{v})_\Omega\ . \tag{18}$$

If $\det S \neq 0$ *and the condition* $I \geq 1$ *is satisfied, then the stationary point of functional* (18) *is uniquely determined.*

The symmetric generalized Green's formula shows that the boundary value problem is formally selfadjoint.

The second assertion in Proposition 1 furnishes the variational formulation of problem (16) over the Hilbert space $\mathfrak{D}$, and shows the uniqueness of solutions under the same conditions as in the case of equation (15).

5. How to Determine the Model Parameters

The solution $\mathbf{v} = \mathfrak{v}$ of equation (15) or of problem (16) satisfies system (4) and boundary conditions (5), however, in general, leaves a discrepancy in the boundary conditions (6). In order to construct an approximation for the solution $u(h,x)$ in the vicinity of the points $P^1, ..., P^I$, the method of matched asymptotic expansions is applied (see [7], [11], and cf. [13], [18]). Thus, selecting for the outer asymptotic expansion $\mathbf{v} = \mathfrak{v}$, we construct the inner expansions $w^j(\xi^j)$, employing the fast variables $\xi^j = h^{-1}(x - P^j)$. The dilatation of coordinates in the limit $h \to +0$ implies the rectifying of the boundary, freezing of coefficients at the point P^j, and the volume forces vanish from the equilibrium equations. In the other words, the boundary value problem for w^j consist of the homogeneous elasticity system

$$D(-\nabla_\xi)^\top A(P^j)D(\nabla_\xi)w^j(\xi) = 0 \ , \quad \xi \in \mathbb{R}^2_j \ , \tag{19}$$

the boundary conditions of traction free type

$$D(n^j)^\top A(P^j)D(\nabla_\xi)w^j(\xi) = 0 \ , \quad \xi \in \partial\mathbb{R}^2_j \ , \quad |\xi| > l_j/2 \ , \tag{20}$$

the Dirichlet conditions for $j = 1, .., I$

$$w^j(\xi) = 0 \ , \quad \xi \in \partial\mathbb{R}^2_j \ , \quad |\xi| < l_j/2 \ . \tag{21}$$

Here $n^j = n^j(P^j)$ is normal vector on $\partial\Omega$ evaluated at points P^j; $\mathbb{R}^2_j$ is the half-plane $\{\xi \in \mathbb{R}^2 \ : \ \xi^\top n^j < 0\}$.
Since we are going to glue w^j with the singular solution $\mathbf{v} = \mathfrak{v}$ with the logarithmic singularity, it is necessary to allow for the logarithmic growth of $w^j(\xi)$ for $|\xi| \to +\infty$. Such solutions of homegeneous problem (19)-(21) are well known (see [2], [1] and others). The solutions ressemble capacitary potentials in the theory of harmonic functions (see e.g., [10]), belong to the space $H^2_{\mathrm{loc}}(\overline{\mathbb{R}}^2_j)^2$ and admit the following asymptotic representation at the infinity

$$w^j(\xi) = T^j(\xi)a^j + c^j + O(|\xi^j|^{-1}) \ , \quad |\xi^j| \to +\infty \ . \tag{22}$$

The column a^j in (22) can be arbitrary, however,

$$c_j = M^j a^j \ , \tag{23}$$

where the symetric (2×2)-matrix M^j is called *Wiener elastic capacity matrix* for the half-plane clamped along the interval $[-l_j/2, l_j/2]$. When we return to the coordinates x, by comparison of representations obtained from (22)-(23) and (13)

$$\begin{aligned} & w^i(h^{-1}(x - P^i)) = \\ & = T^i(x - P^i)a^i + (\mathcal{T}^{i0} \ln h + M^i)a^i + O(h|x - P^i|^{-1}) \ , \end{aligned} \tag{24}$$

with the expansion of the field $v = \mathbf{v} = \mathfrak{v}$ given in (12), the following equalities arise

$$b^i = \{\mathcal{T}^{j0} \ln h + M^i\}a^i \ , \quad i = 1, ..., I \ . \tag{25}$$

(26)

which in vector notation takes the form $\mathbf{b} = \mathbf{S}\mathbf{a}$, used already in (14) and indirectly in (16). Thus, the matrix $\mathbf{S}$ is diagonal by blocks and contains (3×3)-matrices separated in (25) by curly braces. In view of

the properties of $\mathcal{T}^{j0}$ listed after the formula (13), the matrix $\mathbf{S}$ is symmetric and negative definite for sufficiently small $h \in (0, h_0]$.
The relations (25) are derived by matching the outer expansion $\mathbf{v} = \mathfrak{v}$ with the inner expansions $w^j(\xi^j), j = 1, ..., I$. Therefore, by the Korn inequality (7), (8), proximity to the true solution $u(h, x)$ of the global asymptotic approximation in the energy norm can be established. The global asymptotic approximation is obtained by glueing of the expansions in the standard way (cf. [7] and [13], [18]). However, in view of the assumption (10) for the modeling of functional (9) the estimate for the difference $u - \mathbf{v} = u - \mathfrak{v}$ in the norm $L_q(\Omega)^2$ is required. Such an estimate can be established, taking into account the embedding $H^1(\Omega) \subset L_q(\Omega)$, by direct evaluation of the $L_q(\Omega)$-norms of the remainders in the representations (24).

THEOREM 1 *If u and $\mathbf{v} = \mathfrak{v}$ are solutions to problems* (4) -(6) *and* (15)=(16), *respectively, with the same right-hand side $f \in L_2(\Omega)^2$, then*

$$\|u - \mathbf{v}; L_q(\Omega)\| \leq c_\varkappa h |\ln h|^{\varkappa + 5/2} \|f; L_2(\Omega)\| \ . \tag{27}$$

Functional (9) *admits the estimate*

$$|\mathcal{F}(u; h) - \int_\Omega J(x; \mathbf{v}(\ln h)) dx| \leq C_\varkappa \mu_q(h) \|f; L_2\|^q \ , \tag{28}$$

where $\varkappa$ is arbitrary positive, the constants $c_\varkappa$ and $C_\varkappa$ are independent of f and $h \in (0, h_0]$, and

$$\mu_q(h) = h |\ln h|^{q(\varkappa + 5/2)} \text{ for } q \in [1, 2]; \ \mu_q(h) = h^{2/q} \text{ for } q > 2 \ . \tag{29}$$

According to the Clapeyron's Theorem *the potential energy = the elastic energy − the work of external forces* takes the form

$$\mathcal{E}(u; f) = \frac{1}{2}(AD(\nabla)u, D(\nabla)u)_\Omega - (f, u)_\Omega = -\frac{1}{2}\int_\Omega u^\top f dx \ , \tag{30}$$

and Theorem 1 can be used to show that functional (30) evaluated on the solutions to problem (4)-(6), with the precision $O(\mu_q(h)\|f; L_2(\Omega)\|)$ is approximated by the energy functionals, for the problems (15), (16),

$$\begin{aligned} \mathbf{E}(\mathbf{v}; f) &= \frac{1}{2}(\mathbf{L}\mathbf{v}, \mathbf{v})_\Omega - (f, \mathbf{v})_\Omega \\ \mathfrak{E}(\mathfrak{v}; f) &= \frac{1}{2}(D(-\nabla)^\top AD(\nabla)\mathfrak{v}, \mathfrak{v})_\Omega + \frac{1}{2}\langle \mathbf{S}\pi^- \mathfrak{v} - \pi^+ \mathfrak{v}, \pi^- \mathfrak{v}\rangle - (f, \mathfrak{v})_\Omega \end{aligned}$$

6. Spectral Problems

Let us assume that the elastic body is homogeneous, i.e., the Hooke's matrix A and the material density $\rho > 0$ are independent of the point $x \in \Omega$. The spectral boundary value problem includes the system of partial differential equations

$$L(x, \nabla)u(h, x) = \Lambda(h)u(h, x) \ , \quad x \in \Omega \ , \tag{31}$$

(compare with (4)) with the boundary conditions (5), (6), and admits the sequence of eigenvalues

$$0 < \Lambda_1(h) \leq \Lambda_2(h) \leq \ldots \leq \Lambda_n(h) \leq \ldots \to +\infty, \tag{32}$$

with an orthonormal in $L_2(\Omega)^2$ system of eigenfunctions $u^n(h, \cdot)$. Asymptotic expansion, for $h \to +0$, of eigenvalues in analogical scalar problems are characterized by holomorphic dependence upon the parameter $|\ln h|^{-1}$ (see [12] and [13]where such asymptotics were constructed and justified). For the boundary value problems in the elasticity the asymptotic constructions become more complicated (cf. [4], where, in particular, the mistake in [14]was corrected), and therefore the modeling of spectral problems are the most actual issue.

We are going to compare the spectral sequence (32) with the spectre

$$\sigma(\rho^{-1}\mathbf{L}) = \{\lambda_1, \lambda_2, ...\} \tag{33}$$

of the selfadjoint operator, indicated already in Lemma 1, or equivalently defined by the spectral problem with point conditions at $P^1, ..., P^I$:

$$\begin{aligned} L(\nabla)\mathfrak{v}(x) = \lambda(h)\rho\mathfrak{v}(x) \ , \quad x \in \Omega \ , \\ B(x, \nabla)\mathfrak{v}(x) = 0 \ , \quad x \in \partial\Omega \setminus \{P^1, ..., P^I\} \ , \\ \mathbf{S}\pi^-\mathfrak{v} - \pi^+\mathfrak{v} = 0 \ . \end{aligned} \tag{34}$$

The space $\mathfrak{D}$ in (12) is compactly embedded in $L_2(\Omega)^3$, whence accordingly, the eigenvalues $\lambda_N(h)$ are of finite multiplicity, and the unique accumulation point at infinity. The operator $\mathbf{L}$ is not positive, since the matrix $\mathbf{S}$ determined in Section 9.5 (see (25)) is negative definite for small $h > 0$. Whence, the numbers from the collection (33), in contrast to (32), may be located as well in the negative part of the real axis. Nevertheless, by an application of the approach proposed in [9]the following result is obtained.

THEOREM 2 *For any $T > 0$ there exists $h_T > 0$ such that all eigenvalues $\lambda_1(h), ..., \lambda_{N(T)}(h) \in \sigma(\rho^{-1}\mathbf{L}) \cap (-T, T)$, for $h \in (0, h_T)$, become positive and satisfy the estimate*

$$|\lambda_n(h)\Lambda_n(h)| \leq c_{n,\varkappa}\mu_2(h) \ ,$$

where $\mu_2(h)$ *is defined in* (29) *and the constant* $c_{n,\varkappa}$ *depends on the eigenvalue number* $n = 1, ..., N(T)$ *and* $\varkappa > 0$ *but it is independent of* $h \in (0, h_T]$.

Remark 3

1) *The result remains valid for the spectres of problems* (31), (5), (6) *and* (34) *in the case of nonhomogeneous elastic material (*A *and* ρ *depend on* x*). For determination of the appropriate selfadjoint extension, the differential operator* $\rho(x)^{-1/2}L(x,\nabla)\rho(x)^{-1/2}$ *should be used or the weighted class* $L_2(\Omega)$ *(cf. [9], [15]).*

2) *The information on the asymptotic behaviour of the eigenfunctions is also available, but it is omitted here.*

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