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ORIGINAL PAPER

Jacobi–Sobolev Orthogonal Polynomials and Spectral Methods for Elliptic Boundary Value Problems

Xuhong Yu[1] · Zhongqing Wang[1] · Huiyuan Li[2]

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Abstract
Generalized Jacobi polynomials with indexes $\alpha, \beta \in \mathbb{R}$ are introduced and some basic properties are established. As examples of applications, the second- and fourth-order elliptic boundary value problems with Dirichlet or Robin boundary conditions are considered, and the generalized Jacobi spectral schemes are proposed. For the diagonalization of discrete systems, the Jacobi–Sobolev orthogonal basis functions are constructed, which allow the exact solutions and the approximate solutions to be represented in the forms of infinite and truncated Jacobi series. Error estimates are obtained and numerical results are provided to illustrate the effectiveness and the spectral accuracy.

Keywords Generalized Jacobi polynomials · Spectral method · Jacobi–Sobolev orthogonal basis functions · Elliptic boundary value problems · Error analysis

Mathematics Subject Classification 33C45 · 65N35 · 35J25 · 65N15

1 Introduction

During the past decades, significant progress has been made in research on Jacobi spectral methods for solving various differential equations; see, e.g., [3–5, 8–10, 17, 18]. In the early days, one usually considered approximations to differential equations by the classical Jacobi orthogonal polynomials $\{P_k^{\alpha,\beta}(x)\}_{k\geq 0}$ with indexes $\alpha, \beta > -1$; see [2, 20]. To overcome the restrictions of indexes $\alpha, \beta > -1$, some authors extended the classical Jacobi orthogonal polynomials to generalized Jacobi polynomials/functions, which allow α, β to be any real numbers. This essentially widens the applications of Jacobi spectral methods

✉ Zhongqing Wang
zqwang@usst.edu.cn

Xuhong Yu
xhyu@usst.edu.cn

Huiyuan Li
huiyuan@iscas.ac.cn

[1] University of Shanghai for Science and Technology, Shanghai 200093, China

[2] State Key Laboratory of Computer Science/Laboratory of Parallel Computing, Institute of Software, Chinese Academy of Sciences, Beijing 100190, China

[6, 7]. In fact, Shen [16] introduced an efficient Legendre orthogonal basis function $\phi_k(x) = \frac{1}{\sqrt{4k+6}}(L_k(x) - L_{k+2}(x))$, which can also be regarded as generalized Jacobi polynomials with indexes $(\alpha, \beta) = (-1, -1)$.

Since for non-integers $\alpha, \beta < -1$, $P_k^{\alpha,\beta}(x)$ is a function, but not a polynomial in the definition of [6, 7]. This will bring some difficulties to the error analysis sometimes. For this purpose, Li and Xu [11] recently introduced a family of new generalized Jacobi polynomials $\{J_k^{\alpha,\beta}(x)\}_{k\geq 0}$ with indexes $\alpha, \beta \in \mathbb{R}$, which inherit all properties of the classical Jacobi polynomials. It is worth pointing out that $J_k^{\alpha,\beta}(x)$ is always a polynomial for any $\alpha, \beta \in \mathbb{R}$ and $k \geq 0$. In this paper, we will present some properties of this kind of new generalized Jacobi polynomials, including the orthogonality and recurrence relations, etc.

As applications of generalized Jacobi polynomials, we consider the second- and fourth-order elliptic boundary value problems with Dirichlet or Robin boundary conditions, and propose the corresponding generalized Jacobi spectral methods. As is well known, the utilization of generalized Jacobi polynomials usually leads to a sparse (e.g., penta-diagonal) and well-conditioned algebraic system. In order to diagonalize the discrete system (see, e.g., [1, 12–14, 19]), we will construct the Jacobi–Sobolev orthogonal polynomials. Sobolev orthogonal polynomials have been studied extensively in the past quarter-century, and the interested readers may refer to Marcellan and Xu [15] for a survey of the main developments up to now. By using these Jacobi–Sobolev orthogonal polynomials as basis functions, the discrete system becomes a diagonal matrix. Accordingly, the exact solutions and the approximate solutions can be represented in the forms of infinite and truncated Jacobi series. We also present some rigorous error estimates and provide some numerical results to illustrate the effectiveness and the spectral accuracy.

This paper is organized as follows. In the next section, we first make conventions on the frequently used notations, and then introduce the properties of generalized Jacobi polynomials with arbitrary indexes $\alpha, \beta \in \mathbb{R}$. The Jacobi–Sobolev orthogonal basis functions are constructed and the related Jacobi spectral methods are proposed in Sect. 3. Section 4 is then devoted to the convergence analysis. Numerical results are presented in Sect. 5 to demonstrate the effectiveness and accuracy of the suggested approaches.

2 Generalized Jacobi Polynomials

Let $I = (-1, 1)$ and $\omega(x)$ be a certain weight function. We define

$$L_\omega^2(I) = \{v \mid v \text{ is measurable on } I \text{ and } \|v\|_\omega < \infty\},$$

with the following inner product and norm:

$$(u, v)_\omega = \int_I u(x)v(x)\omega(x)\mathrm{d}x, \quad \|v\|_\omega = (u, v)_\omega^{\frac{1}{2}}, \quad \forall u, v \in L_\omega^2(I).$$

For simplicity, we denote $\partial_x^k v := \frac{\mathrm{d}^k v}{\mathrm{d}x^k}$. For any integer $r \geq 0$, we define the weight Sobolev space $H_\omega^r(I)$ as

$$H_\omega^r(I) = \{v \mid \partial_x^k v \in L_\omega^2(I), 0 \leq k \leq r\},$$

with the following semi-norm and norm:

$$|v|_{r,\omega} = \|\partial_x^r v\|_\omega, \qquad \|v\|_{r,\omega} = \left(\sum_{k=0}^{r} |v|_{k,\omega}^2\right)^{\frac{1}{2}}.$$

We omit the subscript ω whenever $\omega(x) \equiv 1$. We denote by $\mathbb{R}$, $\mathbb{N}$ and $\mathbb{Z}^-$ the sets of real numbers, positive integers and negative integers, respectively. For $a \in \mathbb{R}$ and $n \geq 1$, let $(a)_n := a(a+1)\cdots(a+n-1)$ be the rising factorial in the Pochhammer symbol. Furthermore, $\mathcal{P}_N(I)$ stands for the set of all algebraic polynomials of degree at most N.

For $\alpha, \beta > -1$, the classical Jacobi polynomials are defined by (see [20])

$$P_n^{\alpha,\beta}(x) = \sum_{k=0}^{n} \frac{(k+\alpha+1)_{n-k}(n+\alpha+\beta+1)_k}{(n-k)!k!}\left(\frac{x-1}{2}\right)^k, \qquad n \geq 0. \tag{1}$$

With a slight modification of Theorems 3.22 and 3.23 of [7], we have the following recurrence relationships between the classical Jacobi polynomials $P_n^{\alpha,\beta}(x)$ and $P_n^{\alpha+1,\beta+1}(x)$:

$$(1-x^2)P_{n-1}^{\alpha+1,\beta+1}(x) = \widetilde{A}_n^{\alpha,\beta}P_{n-1}^{\alpha,\beta}(x) + \widetilde{B}_n^{\alpha,\beta}P_n^{\alpha,\beta}(x) - \widetilde{C}_n^{\alpha,\beta}P_{n+1}^{\alpha,\beta}(x), \tag{2}$$

and

$$P_n^{\alpha,\beta}(x) = P_n^{\alpha+1,\beta+1}(x) + \widehat{A}_n^{\alpha,\beta}P_{n-1}^{\alpha+1,\beta+1}(x) - \widehat{B}_n^{\alpha,\beta}P_{n-2}^{\alpha+1,\alpha+1}(x), \tag{3}$$

where

$$\widetilde{A}_n^{\alpha,\beta} = \frac{4(n+\alpha)(n+\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)}, \qquad \widetilde{B}_n^{\alpha,\beta} = \frac{4n(\alpha-\beta)}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$\widetilde{C}_n^{\alpha,\beta} = \frac{4n(n+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \qquad \widehat{A}_n^{\alpha,\beta} = \frac{\alpha-\beta}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$\widehat{B}_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}.$$

The classical Jacobi polynomials $P_n^{\alpha,\beta}(x)$ can be extended to cases with any $\alpha, \beta \in \mathbb{R}$ and the same representation as (1). Moreover, $P_n^{\alpha,\beta}(x)$ is always a polynomial in x for all $\alpha, \beta \in \mathbb{R}$. However, if $-n-\alpha-\beta \in \{1, 2, \ldots, n\}$, i.e., $(n+\alpha+\beta+1)_n = 0$, then a reduction of the degree of $P_n^{\alpha,\beta}(x)$ occurs. To avoid the degree reduction, Li and Xu [11] introduced a kind of new generalized Jacobi polynomials (GJPs) by

$$J_n^{\alpha,\beta}(x) := \sum_{k=n_0}^{n} \frac{(k+\alpha+1)_{n-k}}{(n-k)!k!(n+\alpha+\beta+k+1)_{n-k}}\left(\frac{x-1}{2}\right)^k, \qquad n \geq 0, \tag{4}$$

where

$$n_0 := n_0(n,\alpha,\beta) = \begin{cases} -n-\alpha-\beta, & -n-\alpha-\beta \in \{1, 2, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that the leading coefficient $k_n^{\alpha,\beta}$ of $J_n^{\alpha,\beta}(x)$ is equal to $\frac{1}{2^n n!}$. For notational convenience, let $\mathcal{N} := \mathbb{Z}^- \cup (-1, +\infty)$ and

$$\widehat{\alpha} = \begin{cases} -\alpha, & \alpha \leq -1, \\ 0, & \alpha > -1, \end{cases} \qquad \overline{\alpha} = \begin{cases} -\alpha, & \alpha \leq -1, \\ \alpha, & \alpha > -1. \end{cases}$$

By comparison of the corresponding powers of $x-1$ in $J_n^{\alpha,\beta}(x)$ and $P_n^{\alpha,\beta}(x)$, we have

Lemma 2.1 (see [11]) *For $\alpha, \beta \in \mathcal{N}$,*

$$J_n^{\alpha,\beta}(x) = \frac{1}{(n+\alpha+\beta+1)_n} P_n^{\alpha,\beta}(x), \qquad \text{if } n_0 = 0, \tag{5}$$

$$J_n^{\alpha,\beta}(1) = \frac{(\alpha+1)_n}{n!(n+\alpha+\beta+1)_n} \delta_{n_0,0}, \tag{6}$$

$$\partial_x J_n^{\alpha,\beta}(x) = \frac{1}{2} J_{n-1}^{\alpha+1,\beta+1}(x), \tag{7}$$

and

$$J_n^{\alpha,\beta}(x) = \frac{1}{(n-\widehat{\alpha}-\widehat{\beta}+1)_{\widehat{\alpha}+\widehat{\beta}}} \Big(\frac{x-1}{2}\Big)^{\widehat{\alpha}} \Big(\frac{x+1}{2}\Big)^{\widehat{\beta}} J_{n-\widehat{\alpha}-\widehat{\beta}}^{\overline{\alpha},\overline{\beta}}(x), \quad n \geq \widehat{\alpha}+\widehat{\beta}. \tag{8}$$

According to (8), we know that

$$\begin{aligned} &\partial_x^\nu J_n^{\alpha,\beta}(1) = 0, \qquad n \geq -\alpha \in \mathbb{N},\ \nu = 0, 1, \ldots, -\alpha - 1, \\ &\partial_x^\nu J_n^{\alpha,\beta}(-1) = 0, \qquad n \geq -\beta \in \mathbb{N},\ \nu = 0, 1, \ldots, -\beta - 1. \end{aligned} \tag{9}$$

Moreover, by (4) we have $J_0^{\alpha,\beta}(x) = 1$ and

$$J_1^{\alpha,\beta}(x) = \begin{cases} \dfrac{1}{2}(x-1), & \alpha+\beta+2 = 0, \\ \dfrac{1}{2}(x-1) + \dfrac{\alpha+1}{\alpha+\beta+2}, & \text{otherwise.} \end{cases}$$

With a slight modification of (3.110) in [18], we obtain the following recurrence relation.

Lemma 2.2 *For any* $\alpha, \beta \in \mathbb{R}, n \geq 1$ *and* $-n-\alpha-\beta \notin \{-1, 0, 1, 2, \ldots, n+1\}$,

$$a_n^{\alpha,\beta} J_{n+1}^{\alpha,\beta}(x) = (b_n^{\alpha,\beta} x + c_n^{\alpha,\beta}) J_n^{\alpha,\beta}(x) - d_n^{\alpha,\beta} J_{n-1}^{\alpha,\beta}(x), \tag{10}$$

where

$$\begin{aligned} a_n^{\alpha,\beta} &= 2(n+1)(n+\alpha+\beta+1)_{n+2}(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1), \\ b_n^{\alpha,\beta} &= (n+\alpha+\beta+1)_{n+2}(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1), \\ c_n^{\alpha,\beta} &= (\alpha^2-\beta^2)(n+\alpha+\beta+1)_{n+1}(2n+\alpha+\beta-1)(2n+\alpha+\beta)(2n+\alpha+\beta+1), \\ d_n^{\alpha,\beta} &= 2(n+\alpha)(n+\beta)(n+\alpha+\beta)_{n+3}. \end{aligned}$$

Remark 2.1 For $-n-\alpha-\beta \in \{-1, 0, 1, 2, \ldots, n+1\}$, we may derive the generalized Jacobi polynomials $J_n^{\alpha,\beta}(x)$ directly from the definition of (4).

Let $\omega^{\alpha,\beta}(x) = (1-x)^\alpha (1+x)^\beta$. It can be checked readily that the generalized Jacobi polynomials satisfy the following orthogonality relation:

$$\int_I J_m^{\alpha,\beta}(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) \mathrm{d}x = \gamma_n^{\alpha,\beta} \delta_{m,n}, \qquad \alpha, \beta \in \mathcal{N}, \quad m, n \geq \widehat{\alpha}+\widehat{\beta}, \tag{11}$$

where $\delta_{m,n}$ is the Kronecker symbol, and the normalization constant is given by

$$\gamma_n^{\alpha,\beta} = \frac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{n!(n+\alpha+\beta+1)_{n+1}\Gamma(2n+\alpha+\beta+1)}. \tag{12}$$

Moreover, by (7) and (11) we get

$$\int_I \partial_x^k J_m^{\alpha,\beta}(x)\partial_x^k J_n^{\alpha,\beta}(x)\omega^{\alpha+k,\beta+k}(x)\mathrm{d}x = \frac{1}{4^k}\gamma_{n-k}^{\alpha+k,\beta+k}\delta_{m,n}, \qquad m,n \geq \widehat{\alpha+k} + \widehat{\beta+k} + k. \tag{13}$$

Lemma 2.3 *For any $\alpha, \beta \in \mathcal{N}$, the generalized Jacobi polynomials satisfy*

$$J_n^{\alpha,\beta}(x) = J_n^{\alpha+1,\beta+1}(x) + A_n^{\alpha,\beta}J_{n-1}^{\alpha+1,\beta+1}(x) - B_n^{\alpha,\beta}J_{n-2}^{\alpha+1,\beta+1}(x), \tag{14}$$

where

$$A_n^{\alpha,\beta} = \frac{\alpha-\beta}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)},$$
$$B_n^{\alpha,\beta} = \frac{(n+\alpha)(n+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}.$$

Proof This formula follows from (2), (3), (5) and (8) directly. We omit the details.

As special cases, one can verify that

$$\begin{aligned} J_n^{-1,-1}(x) &= J_n^{0,0}(x) - \frac{1}{4(2n-1)(2n-3)}J_{n-2}^{0,0}(x), \quad n \geq 2, \\ J_n^{-2,-2}(x) &= J_n^{0,0}(x) - \frac{1}{2(2n-1)(2n-5)}J_{n-2}^{0,0}(x) + \frac{1}{16(2n-3)(2n-5)^2(2n-7)}J_{n-4}^{0,0}(x), \quad n \geq 4. \end{aligned} \tag{15}$$

Lemma 2.4 *For $\alpha, \beta \in \mathbb{R}$, the generalized Jacobi polynomials $J_n^{\alpha,\beta}(x)$ are the eigenfunctions of the following singular Sturm–Liouville differential operators*:

$$\partial_x(\omega^{\alpha+1,\beta+1}(x)\partial_x J_n^{\alpha,\beta}(x)) + \lambda_n^{\alpha,\beta}\omega^{\alpha,\beta}(x)J_n^{\alpha,\beta}(x) = 0 \quad \textit{with} \quad \lambda_n^{\alpha,\beta} = n(n+\alpha+\beta+1), \tag{16}$$

provided that $n_0 = 0$ or $n_0 \leq -\alpha$ with $\alpha \in \mathbb{Z}^-$.

Proof Clearly,

$$\begin{aligned} &\partial_x(\omega^{\alpha+1,\beta+1}(x)\partial_x J_n^{\alpha,\beta}(x)) + \lambda_n^{\alpha,\beta}\omega^{\alpha,\beta}(x)J_n^{\alpha,\beta}(x) \\ &= \omega^{\alpha,\beta}(x)[(1-x^2)\partial_x^2 J_n^{\alpha,\beta}(x) + ((\beta-\alpha) - (\alpha+\beta+2)x)\partial_x J_n^{\alpha,\beta}(x) + \lambda_n^{\alpha,\beta}J_n^{\alpha,\beta}(x)]. \end{aligned}$$

Next, using the expression (4), we derive that

$$
\begin{aligned}
&(1-x^2)\partial_x^2 J_n^{\alpha,\beta}(x)+((\beta-\alpha)-(\alpha+\beta+2)x)\partial_x J_n^{\alpha,\beta}(x)+\lambda_n^{\alpha,\beta}J_n^{\alpha,\beta}(x)\\
&=-\sum_{k=n_0}^{n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+1)}{(n-k)!(k-2)!\Gamma(k+\alpha+1)\Gamma(2n+\alpha+\beta+1)}\Big(\frac{x-1}{2}\Big)^k\\
&\quad-\sum_{k=n_0}^{n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+1)}{(n-k)!(k-2)!\Gamma(k+\alpha+1)\Gamma(2n+\alpha+\beta+1)}\Big(\frac{x-1}{2}\Big)^{k-1}\\
&\quad-(\alpha+\beta+2)\sum_{k=n_0}^{n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+1)}{(n-k)!(k-1)!\Gamma(k+\alpha+1)\Gamma(2n+\alpha+\beta+1)}\Big(\frac{x-1}{2}\Big)^k\\
&\quad-(\alpha+1)\sum_{k=n_0}^{n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+1)}{(n-k)!(k-1)!\Gamma(k+\alpha+1)\Gamma(2n+\alpha+\beta+1)}\Big(\frac{x-1}{2}\Big)^{k-1}\\
&\quad+n(n+\alpha+\beta+1)\sum_{k=n_0}^{n}\frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+k+1)}{(n-k)!k!\Gamma(k+\alpha+1)\Gamma(2n+\alpha+\beta+1)}\Big(\frac{x-1}{2}\Big)^k\\
&=\frac{n_0(n_0+\alpha)\Gamma(n+\alpha+1)\Gamma(n+\alpha+\beta+n_0+1)}{(n-n_0)!n_0!\Gamma(n_0+\alpha+1)\Gamma(2n+\alpha+\beta+1)}\Big(\frac{x-1}{2}\Big)^{n_0-1}.
\end{aligned}
\tag{17}
$$

Hence, if $n_0=0$, then we obtain the result of (16). Moreover, if $\alpha\in\mathbb{Z}^-$ and $0\le n_0\le -\alpha$, then $\Gamma(n+\alpha+1)=0$. This also leads to the desired result.

3 Applications

This section is devoted to the applications of generalized Jacobi polynomials. We consider the second- and fourth-order elliptic boundary value problems with Dirichlet or Robin boundary conditions, and propose the generalized Jacobi spectral schemes. For the diagonalization of discrete systems, we construct the Jacobi–Sobolev orthogonal basis functions, which allow the exact solutions and the approximate solutions to be represented in the forms of infinite and truncated Jacobi series.

3.1 Second-Order Problem with Dirichlet Boundary Condition

Consider the following second-order elliptic boundary value problem:

$$
\begin{cases}
-u''(x)+\mu u(x)=f(x), & \mu\ge 0,\ x\in I,\\
u(-1)=u(1)=0.
\end{cases}
\tag{18}
$$

Let

$$
H_0^1(I)=\{v\in H^1(I)\mid v(\pm1)=0\},\qquad \mathcal{P}_N^0(I)=\mathcal{P}_N(I)\cap H_0^1(I).
$$

Clearly, $\mathcal{P}_N^0(I)=\mathrm{span}\{J_n^{-1,-1}(x):2\le n\le N\}$. A weak formulation of (18) is to find $u\in H_0^1(I)$ such that

$$
A_\mu(u,v):=(u',v')+\mu(u,v)=(f,v),\qquad \forall v\in H_0^1(I).
\tag{19}
$$

Clearly, if $f\in(H_0^1(I))'$, then by the Lax–Milgram lemma, (19) admits a unique solution.

The generalized Jacobi spectral scheme for (19) is to find $u_N\in\mathcal{P}_N^0(I)$ such that

$$A_\mu(u_N,\phi)=(f,\phi),\qquad \forall\phi\in\mathcal{P}_N^0(I). \tag{20}$$

To propose a fully diagonalized approximation scheme for (20), we need to construct new basis functions $\{\varphi_n\}_{n\geq 2}$, which are mutually orthogonal with respect to the Sobolev inner product $A_\mu(\cdot,\cdot)$.

Lemma 3.1 *Let $\varphi_2(x)=(3)_2J_2^{-1,-1}(x)$ and $\varphi_n(x)\in\mathcal{P}_n^0(I)$ be the Sobolev orthogonal generalized Jacobi polynomials such that $\varphi_n(x)-(n+1)_nJ_n^{-1,-1}(x)\in\mathcal{P}_{n-1}^0(I)$, and*

$$A_\mu(\varphi_m,\varphi_n)=\eta_n\delta_{m,n},\qquad m,\ n\geq 2. \tag{21}$$

Then we have

$$\varphi_n(x)=(n+1)_nJ_n^{-1,-1}(x)-d_n\varphi_{n-2}(x),\qquad n\geq 2, \tag{22}$$

where $\varphi_n(x)\equiv 0$ $(n<2)$, $\eta_n=0$ $(n<2)$, $d_n=0$ $(n<4)$, and

$$\eta_n=2(2n-1)+\frac{4\mu(2n-1)}{(2n-3)(2n+1)}-d_n^2\eta_{n-2},\quad n\geq 2,\qquad d_n=-\frac{2\mu}{(2n-3)\eta_{n-2}},\quad n\geq 4. \tag{23}$$

Proof According to the orthogonality assumption (21) and the Gram–Schmidt orthogonalization procedure, we have

$$\varphi_n(x)=(n+1)_nJ_n^{-1,-1}(x)-\sum_{m=2}^{n-1}\frac{A_\mu((n+1)_nJ_n^{-1,-1},\varphi_m)}{\eta_m}\varphi_m(x),\qquad n\geq 3. \tag{24}$$

We first use mathematical induction to verify (22). Clearly, by (19) and (13), we deduce that for any $n>m\geq 2$,

$$A_\mu((n+1)_nJ_n^{-1,-1},\varphi_m)=\mu(n+1)_n(J_n^{-1,-1},\varphi_m). \tag{25}$$

Hence, by (25), (15) and (11) we derive

$$A_\mu((4)_3J_3^{-1,-1},\varphi_2)=\mu(120J_3^{-1,-1},12J_2^{-1,-1})=0.$$

This implies $\varphi_3(x)=(4)_3J_3^{-1,-1}(x)$. Similarly, by (25), (15) and (11) we get that for $n=4$ and $m=2,3$,

$$\begin{aligned}&A_\mu((n+1)_nJ_n^{-1,-1},\varphi_m)\\&=\mu((n+1)_nJ_n^{-1,-1},(m+1)_mJ_m^{-1,-1})\\&=\mu(n+1)_n(m+1)_m\Big(J_n^{0,0}-\tfrac{1}{4(2n-1)(2n-3)}J_{n-2}^{0,0},J_m^{0,0}-\tfrac{1}{4(2m-1)(2m-3)}J_{m-2}^{0,0}\Big)\\&=-\tfrac{2\mu}{2n-3}\delta_{m,n-2}.\end{aligned}$$

Therefore, $\varphi_4(x)=(5)_4J_4^{-1,-1}(x)-d_4\varphi_2(x)$ with $d_4=-\frac{2\mu}{5\eta_2}$.

Next, assume that for any $4\leq m\leq n-1$ and $n\geq 5$,

$$\varphi_m(x) = (m+1)_m J_m^{-1,-1}(x) - d_m \varphi_{m-2}(x).$$

We shall prove $\varphi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_n \varphi_{n-2}(x)$ for $n \geq 5$. Clearly, by (25), (15), (11) and the induction assumption, we derive that for $n > m$ and $n \geq 5$,

$$\begin{aligned}
&A_\mu((n+1)_n J_n^{-1,-1}, \varphi_m) \\
&= \mu((n+1)_n J_n^{-1,-1}, (m+1)_m J_m^{-1,-1}) - \mu d_m((n+1)_n J_n^{-1,-1}, \varphi_{m-2}) \\
&= \mu(n+1)_n(m+1)_m \Big(J_n^{0,0} - \tfrac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0}, J_m^{0,0} - \tfrac{1}{4(2m-1)(2m-3)} J_{m-2}^{0,0} \Big) \\
&\quad - \mu d_m (n+1)_n \Big(J_n^{0,0} - \tfrac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0}, \varphi_{m-2} \Big) \\
&= -\frac{2\mu}{2n-3} \delta_{m,n-2},
\end{aligned}$$

which along with (24), implies

$$\varphi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_n \varphi_{n-2}(x), \qquad d_n = -\frac{2\mu}{(2n-3)\eta_{n-2}}, \qquad n \geq 5.$$

It remains to confirm the constant η_n, $n \geq 2$. Clearly, by (21), (19), (13), (15) and (11), we know that for $n = 2, 3$,

$$\begin{aligned}
\eta_n &= A_\mu(\varphi_n, \varphi_n) = A_\mu((n+1)_n J_n^{-1,-1}, (n+1)_n J_n^{-1,-1}) \\
&= ((n+1)_n \partial_x J_n^{-1,-1}, (n+1)_n \partial_x J_n^{-1,-1}) \\
&\quad + \mu((n+1)_n)^2 \left(J_n^{0,0} - \frac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0}, J_n^{0,0} - \frac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0} \right) \\
&= 2(2n-1) + \frac{4\mu(2n-1)}{(2n-3)(2n+1)}.
\end{aligned} \tag{26}$$

Thus $\eta_2 = 6 + \frac{12}{5}\mu$ and $\eta_3 = 10 + \frac{20}{21}\mu$ as in (23). Moreover, by (26) we obtain readily that for any $n \geq 4$,

$$A_\mu((n+1)_n J_n^{-1,-1}, (n+1)_n J_n^{-1,-1}) = 2(2n-1) + \frac{4\mu(2n-1)}{(2n-3)(2n+1)}.$$

On the other hand, by (22) and (21) we have

$$A_\mu((n+1)_n J_n^{-1,-1}, (n+1)_n J_n^{-1,-1}) = A_\mu(\varphi_n + d_n \varphi_{n-2}, \varphi_n + d_n \varphi_{n-2}) = \eta_n + d_n^2 \eta_{n-2}.$$

Hence

$$\eta_n = 2(2n-1) + \frac{4\mu(2n-1)}{(2n-3)(2n+1)} - d_n^2 \eta_{n-2}, \qquad n \geq 4.$$

This ends the proof.

Obviously, $\mathcal{P}_N^0(I) = \{\varphi_n \mid 2 \le n \le N\}$. Thus the variational forms (19) and (20) together with the orthogonality of $\{\varphi_n\}_{n\ge 2}$ lead to the following main theorem in this subsection.

Theorem 3.1 *Let $u(x)$ and $u_N(x)$ be the solutions of (19) and (20), respectively. Then both $u(x)$ and $u_N(x)$ have the explicit representations in $\{\varphi_n\}_{n\ge 2}$,*

$$u(x) = \sum_{n=2}^{\infty} \hat{u}_n \varphi_n(x), \qquad u_N(x) = \sum_{n=2}^{N} \hat{u}_n \varphi_n(x),$$
$$\hat{u}_n = \frac{1}{\eta_n} A_\mu(u, \varphi_n) = \frac{1}{\eta_n}(f, \varphi_n), \qquad n \ge 2.$$

3.2 Second-Order Problem with Robin Boundary Condition

Consider the following second-order elliptic boundary value problem:

$$\begin{cases} -u''(x) + \mu u(x) = f(x), & \mu > 0,\ x \in I, \\ u'(1) + \nu_+ u(1) = a_+, & \nu_+ \ge 0, \\ -u'(-1) + \nu_- u(-1) = a_-, & \nu_- \ge 0. \end{cases} \tag{27}$$

A weak formulation of (27) is to find $u \in H^1(I)$ such that

$$\begin{aligned} A_{\mu,\nu}(u, v) &:= (u', v') + \mu(u, v) + \nu_+ u(1)v(1) + \nu_- u(-1)v(-1) \\ &= (f, v) + a_+ v(1) + a_- v(-1), \qquad \forall v \in H^1(I). \end{aligned} \tag{28}$$

The Lax–Milgram lemma guarantees a unique solution to (28) if $f \in (H^1(I))'$.

Clearly, $\mathcal{P}_N(I) = \{J_n^{-1,-1} \mid 0 \le n \le N\}$. The generalized Jacobi spectral scheme for (28) is to find $u_N \in \mathcal{P}_N(I)$, such that

$$A_{\mu,\nu}(u_N, \phi) = (f, \phi) + a_+ \phi(1) + a_- \phi(-1), \qquad \forall \phi \in \mathcal{P}_N(I). \tag{29}$$

To propose a fully diagonalized approximation scheme for (29), we need to construct new basis functions $\{\psi_n\}_{n\in\mathbb{N}}$, which are mutually orthogonal with respect to the Sobolev inner product $A_{\mu,\nu}(\cdot,\cdot)$.

Lemma 3.2 *Let $\psi_0(x) = J_0^{-1,-1}(x)$ and $\psi_n(x) \in \mathcal{P}_n(I)$ be the Sobolev orthogonal Jacobi polynomials such that $\psi_n(x) - (n+1)_n J_n^{-1,-1}(x) \in \mathcal{P}_{n-1}(I)$, and*

$$A_{\mu,\nu}(\psi_m, \psi_n) = \rho_n \delta_{m,n}, \qquad m, n \ge 0. \tag{30}$$

Then we have $\psi_1(x) = 2J_1^{-1,-1}(x) - d_{1,1}\psi_0(x)$, and

$$\psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_{n,1}\psi_{n-1}(x) - d_{n,2}\psi_{n-2}(x), \qquad n \ge 2, \tag{31}$$

where

$$\begin{aligned} &\rho_0 = 2\mu + \nu_+ + \nu_-, \qquad \rho_1 = 2 + \tfrac{2}{3}\mu + 2\mu(d_{1,1}+1)^2 + \nu_+ d_{1,1}^2 + \nu_-(2 + d_{1,1})^2, \\ &\rho_n = 2(2n-1) - d_{n,1}^2 \rho_{n-1} - d_{n,2}^2 \rho_{n-2} + \tfrac{4\mu(2n-1)}{(2n-3)(2n+1)}, \qquad n \ge 2, \\ &d_{1,1} = -\tfrac{2(\mu+\nu_-)}{\rho_0}, \qquad d_{2,1} = \tfrac{2\mu(d_{1,1}+1)}{\rho_1}, \qquad d_{2,2} = -\tfrac{2\mu}{\rho_0}, \\ &d_{n,1} = \tfrac{2\mu d_{n-1,1}}{(2n-3)\rho_{n-1}}, \qquad d_{n,2} = -\tfrac{2\mu}{(2n-3)\rho_{n-2}}, \qquad n \ge 3. \end{aligned}$$

Proof According to the orthogonality assumption (30) and the Gram–Schmidt orthogonalization procedure, we have

$$\psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - \sum_{m=0}^{n-1} \frac{A_{\mu,\nu}((n+1)_n J_n^{-1,-1}, \psi_m)}{\rho_m} \psi_m(x), \qquad n \geq 1. \tag{32}$$

We first use mathematical induction to verify (31). Clearly, by (28) and (13), we deduce that for any $n > m \geq 0$,

$$A_{\mu,\nu}(J_n^{-1,-1}, \psi_m) = \mu(J_n^{-1,-1}, \psi_m) + \nu_+ J_n^{-1,-1}(1)\psi_m(1) + \nu_- J_n^{-1,-1}(-1)\psi_m(-1). \tag{33}$$

Moreover, by the definition of (4), we have $J_1^{-1,-1}(x) = J_1^{0,0}(x) - \frac{1}{2}J_0^{0,0}(x)$ and

$$J_0^{-1,-1}(\pm 1) = 1, \qquad J_1^{-1,-1}(-1) = -1, \qquad J_1^{-1,-1}(1) = 0, \qquad J_n^{-1,-1}(\pm 1) = 0, \quad n \geq 2. \tag{34}$$

Hence, by (33), (34) and (11), we get

$$\begin{aligned} A_{\mu,\nu}(2J_1^{-1,-1}, \psi_0) &= \mu(2J_1^{0,0} - J_0^{0,0}, J_0^{0,0}) + 2\nu_+ J_1^{-1,-1}(1)J_0^{-1,-1}(1) + 2\nu_- J_1^{-1,-1}(-1)J_0^{-1,-1}(-1) \\ &= -2(\mu + \nu_-). \end{aligned}$$

This means that

$$\psi_1(x) = 2J_1^{-1,-1}(x) - d_{1,1}\psi_0(x), \qquad d_{1,1} = -\frac{2(\mu+\nu_-)}{\rho_0}.$$

Similarly, using (33), (34), (30) and (11), we get that for $n \geq 2$ and $m = 0, 1$,

$$\begin{aligned} &A_{\mu,\nu}((n+1)_n J_n^{-1,-1}, \psi_0) = \mu((n+1)_n J_n^{-1,-1}, J_0^{-1,-1}) \\ &= \mu(n+1)_n \left(J_n^{0,0} - \tfrac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0}, J_0^{0,0} \right) = -2\mu\delta_{0,n-2}, \end{aligned} \tag{35}$$

and

$$\begin{aligned} &A_{\mu,\nu}((n+1)_n J_n^{-1,-1}, \psi_1) = \mu((n+1)_n J_n^{-1,-1}, 2J_1^{-1,-1} - d_{1,1}\psi_0) \\ &= \mu(n+1)_n \left(J_n^{0,0} - \tfrac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0}, 2J_1^{0,0} - (d_{1,1}+1)J_0^{0,0} \right) \\ &= -\tfrac{2}{3}\mu\delta_{1,n-2} + 2\mu(d_{1,1}+1)\delta_{0,n-2}. \end{aligned} \tag{36}$$

Therefore, there exist the constants $d_{2,1}$ and $d_{2,2}$ such that

$$\psi_2(x) = (3)_2 J_2^{-1,-1}(x) - d_{2,1}\psi_1(x) - d_{2,2}\psi_0(x), \qquad d_{2,1} = \frac{2\mu(d_{1,1}+1)}{\rho_1}, \qquad d_{2,2} = -\frac{2\mu}{\rho_0}.$$

Next, assume that for any $2 \leq m \leq n-1$ and $n \geq 3$,

$$\psi_m(x) = (m+1)_m J_m^{-1,-1}(x) - d_{m,1}\psi_{m-1}(x) - d_{m,2}\psi_{m-2}(x).$$

We shall prove $\psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_{n,1}\psi_{n-1}(x) - d_{n,2}\psi_{n-2}(x)$ for $n \geq 3$. Clearly, by (33), (34), (15), (11) and the induction assumption, we derive that for $n > m \geq 2$,

$$\begin{aligned}
&A_{\mu,\nu}((n+1)_n J_n^{-1,-1}, \psi_m)\\
&= \mu((n+1)_n J_n^{-1,-1}, (m+1)_m J_m^{-1,-1} - d_{m,1}\psi_{m-1} - d_{m,2}\psi_{m-2})\\
&= \mu(n+1)_n (J_n^{0,0} - \frac{1}{4(2n-1)(2n-3)} J_{n-2}^{0,0},\\
&\quad (m+1)_m (J_m^{0,0} - \frac{1}{4(2m-1)(2m-3)} J_{m-2}^{0,0}) - d_{m,1}(m)_{m-1} J_{m-1}^{0,0})\\
&= -\frac{2\mu}{2n-3}\delta_{m,n-2} + \frac{2\mu d_{n-1,1}}{2n-3}\delta_{m-1,n-2}.
\end{aligned} \tag{37}$$

This together with (32) leads to

$$\begin{aligned}
&\psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_{n,1}\psi_{n-1}(x) - d_{n,2}\psi_{n-2}(x), \qquad n \geq 3,\\
&d_{n,1} = \tfrac{2\mu d_{n-1,1}}{(2n-3)\rho_{n-1}}, \qquad d_{n,2} = -\tfrac{2\mu}{(2n-3)\rho_{n-2}}.
\end{aligned}$$

It remains to confirm the constant ρ_n with $n \geq 0$. Clearly, by (28), (7), (34) and (11), we know that

$$\begin{aligned}
\rho_0 &= A_{\mu,\nu}(\psi_0, \psi_0) = (\partial_x J_0^{-1,-1}, \partial_x J_0^{-1,-1}) + \mu(J_0^{-1,-1}, J_0^{-1,-1})\\
&\quad + \nu_+ J_0^{-1,-1}(1) J_0^{-1,-1}(1) + \nu_- J_0^{-1,-1}(-1) J_0^{-1,-1}(-1)\\
&= 2\mu + \nu_+ + \nu_-,
\end{aligned}$$

and

$$\begin{aligned}
\rho_1 &= A_{\mu,\nu}(\psi_1, \psi_1) = (\partial_x\psi_1, \partial_x\psi_1) + \mu(\psi_1, \psi_1) + \nu_+\psi_1(1)\psi_1(1) + \nu_-\psi_1(-1)\psi_1(-1)\\
&= (\partial_x(2J_1^{-1,-1} - d_{1,1}J_0^{-1,-1}), \partial_x(2J_1^{-1,-1} - d_{1,1}J_0^{-1,-1}))\\
&\quad + \mu(2J_1^{-1,-1} - d_{1,1}J_0^{-1,-1}, 2J_1^{-1,-1} - d_{1,1}J_0^{-1,-1}) + \nu_+ d_{1,1}^2 + \nu_-(2 + d_{1,1})^2\\
&= 2 + \frac{2}{3}\mu + 2\mu(d_{1,1}+1)^2 + \nu_+ d_{1,1}^2 + \nu_-(2+d_{1,1})^2.
\end{aligned}$$

Moreover, by (28), (7) and (11), we obtain readily that for any $n \geq 2$,

$$A_{\mu,\nu}((n+1)_n J_n^{-1,-1}, (n+1)_n J_n^{-1,-1}) = 2(2n-1) + \frac{4\mu(2n-1)}{(2n-3)(2n+1)}.$$

On the other hand, by (31) and (30) we have

$$
\begin{aligned}
&A_{\mu,\nu}((n+1)_n J_n^{-1,-1}, (n+1)_n J_n^{-1,-1}) \\
&= A_{\mu,\nu}(\psi_n + d_{n,1}\psi_{n-1} + d_{n,2}\psi_{n-2}, \psi_n + d_{n,1}\psi_{n-1} + d_{n,2}\psi_{n-2}) \\
&= \rho_n + d_{n,1}^2\rho_{n-1} + d_{n,2}^2\rho_{n-2}.
\end{aligned}
$$

Hence

$$
\rho_n = 2(2n-1) + \frac{4\mu(2n-1)}{(2n-3)(2n+1)} - d_{n,1}^2\rho_{n-1} - d_{n,2}^2\rho_{n-2}, \qquad n \geq 2.
$$

The proof is completed.

Obviously, $\mathcal{P}_N(I) = \{\psi_n \mid 0 \leq n \leq N\}$. Thus the variational forms (28) and (29) together with the orthogonality of $\{\psi_n\}_{n\geq 0}$ lead to the following main theorem in this subsection.

Theorem 3.2 *Let $u(x)$ and $u_N(x)$ be the solutions of (28) and (29), respectively. Then both $u(x)$ and $u_N(x)$ have the explicit representations in $\{\psi_n\}_{n\geq 0}$,*

$$
\begin{aligned}
&u(x) = \sum_{n=0}^{\infty} \hat{u}_n\psi_n(x), \qquad\qquad u_N(x) = \sum_{n=0}^{N} \hat{u}_n\psi_n(x), \\
&\hat{u}_n = \frac{1}{\rho_n}A_{\mu,\nu}(u,\psi_n) = \frac{1}{\rho_n}[(f,\psi_n) + a_+\psi_n(1) + a_-\psi_n(-1)], \qquad n \geq 0.
\end{aligned}
$$

3.3 Second-Order Problem with a Harmonic Potential

Consider the following second-order problem with a harmonic potential:

$$
\begin{cases}
-u''(x) + \mu x^2 u(x) = f(x), & \mu \geq 0,\ x \in I, \\
u(-1) = u(1) = 0.
\end{cases} \tag{38}
$$

A weak formulation of (38) is to find $u \in H_0^1(I)$ such that

$$
B_\mu(u,v) := (u',v') + \mu(x^2u,v) = (f,v), \qquad \forall v \in H_0^1(I). \tag{39}
$$

The generalized Jacobi spectral scheme for (39) is to find $u_N \in \mathcal{P}_N^0(I)$ such that

$$
B_\mu(u_N,\phi) = (f,\phi), \qquad \forall \phi \in \mathcal{P}_N^0(I). \tag{40}
$$

To propose a fully diagonalized approximation scheme for (39), we need to construct new basis functions $\{\Psi_n\}_{n\geq 2}$, which are mutually orthogonal with respect to the Sobolev inner product $B_\mu(\cdot,\cdot)$.

Lemma 3.3 *Let $\Psi_2(x) = (3)_2 J_2^{-1,-1}(x)$ and $\Psi_n(x) \in \mathcal{P}_n^0(I)$ be the Sobolev orthogonal generalized Jacobi polynomials such that $\Psi_n(x) - (n+1)_n J_n^{-1,-1}(x) \in \mathcal{P}_{n-1}^0(I)$, and*

$$
B_\mu(\Psi_m,\Psi_n) = \varrho_n\delta_{m,n}, \qquad m,n \geq 2. \tag{41}
$$

Then we have

$$
\Psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_{n,1}\Psi_{n-2}(x) - d_{n,2}\Psi_{n-4}(x), \qquad n \geq 2, \tag{42}
$$

where $\Psi_n(x) \equiv 0\ (n<2),\ \varrho_n = 0\ (n<2),\ d_{n,1} = 0\ (n<4),\ d_{n,2} = 0\ (n<6)$, *and*

$$\begin{aligned}
\varrho_n &= 2(2n-1) + \frac{4\mu(2n-1)(n^2-n-3)}{(2n-5)(2n-3)(2n+1)(2n+3)} - d_{n,1}^2\varrho_{n-2} - d_{n,2}^2\varrho_{n-4}, \qquad n \geq 2,\\
d_{n,1} &= \frac{1}{\varrho_{n-2}}\left(\frac{2\mu(1+d_{n-2,1})(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)} - \frac{2\mu(n^2-n-3)}{(2n-5)(2n-3)(2n+1)}\right), \qquad n \geq 4,\\
d_{n,2} &= -\frac{2\mu(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)\varrho_{n-4}}, \qquad n \geq 6.
\end{aligned} \tag{43}$$

Proof According to the orthogonality assumption (41) and the Gram–Schmidt orthogonalization procedure, we have

$$\Psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - \sum_{m=2}^{n-1} \frac{B_\mu((n+1)_n J_n^{-1,-1}, \Psi_m)}{\varrho_m}\Psi_m(x), \qquad n \geq 3. \tag{44}$$

We first use mathematical induction to verify (42). Clearly, by (39) and (13), we deduce that for any $n > m \geq 2$,

$$\begin{aligned}
B_\mu((n+1)_n J_n^{-1,-1}, \Psi_m) &= (n+1)_n(\partial_x J_n^{-1,-1}, \partial_x \Psi_m) + \mu(n+1)_n(x^2 J_n^{-1,-1}, \Psi_m)\\
&= \mu(n+1)_n(x^2 J_n^{-1,-1}, \Psi_m).
\end{aligned} \tag{45}$$

Moreover, by Lemma 2.3 with $\alpha = \beta = -1$, we get for $n \geq 2$

$$\begin{aligned}
x^2 J_n^{-1,-1}(x) = {} & 4(n+1)(n+2)J_{n+2}^{0,0}(x) + \frac{n^2-n-3}{(2n-3)(2n+3)}J_n^{0,0}(x)\\
& - \frac{n^2-n-3}{4(2n+1)(2n-1)(2n-3)(2n-5)}J_{n-2}^{0,0}(x)\\
& - \frac{(n-2)(n-3)}{16(2n-1)(2n-3)^2(2n-5)^2(2n-7)}J_{n-4}^{0,0}(x).
\end{aligned} \tag{46}$$

Hence, by (45), (46), (15) and (11) we derive

$$B_\mu((4)_3 J_3^{-1,-1}, \Psi_2) = \mu(120x^2 J_3^{-1,-1}, 12J_2^{-1,-1}) = 0.$$

This implies $\Psi_3(x) = (4)_3 J_3^{-1,-1}(x)$. Similarly, we get that for $n = 4$ and $m = 2, 3$,

$$\begin{aligned}
&B_\mu((n+1)_n J_n^{-1,-1}, \Psi_m) = \mu((n+1)_n x^2 J_n^{-1,-1}, (m+1)_m J_m^{-1,-1})\\
&= \left(\frac{2\mu(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)} - \frac{2\mu(n^2-n-3)}{(2n-5)(2n-3)(2n+1)}\right)\delta_{m,n-2} - \frac{2\mu(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)}\delta_{m,n-4}\\
&= \frac{6\mu}{(2n+1)(2n-3)(2n-7)}\delta_{m,n-2} - \frac{2\mu(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)}\delta_{m,n-4}.
\end{aligned}$$

Therefore,

$$\Psi_4(x) = (5)_4 J_4^{-1,-1}(x) - d_{4,1}\Psi_2(x), \qquad d_{4,1} = \tfrac{2\mu}{15\varrho_2},$$
$$\Psi_5(x) = (6)_5 J_5^{-1,-1}(x) - d_{5,1}\Psi_3(x), \qquad d_{5,1} = \tfrac{2\mu}{77\varrho_3}.$$

In the same manner, we can verify the results of (42) for $n = 6$ with the constants $d_{6,1}$ and $d_{6,2}$ as in (43).

Next, assume that for any $6 \le m \le n-1$ and $n \ge 7$,

$$\Psi_m(x) = (m+1)_m J_m^{-1,-1}(x) - d_{m,1}\Psi_{m-2}(x) - d_{m,2}\Psi_{m-4}(x).$$

We shall prove $\Psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_{n,1}\Psi_{n-2}(x) - d_{n,2}\Psi_{n-4}(x)$ for $n \ge 7$. Clearly, by (45), (46), (15), (11) and the induction assumption, we derive that for $n > m$ and $n \ge 7$,

$$\begin{aligned} &B_\mu((n+1)_n J_n^{-1,-1}, \Psi_m) = \mu((n+1)_n x^2 J_n^{-1,-1}, (m+1)_m J_m^{-1,-1} - d_{m,1}\Psi_{m-2} - d_{m,2}\Psi_{m-4}) \\ &= \left(\frac{2\mu(1+d_{n-2,1})(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)} - \frac{2\mu(n^2-n-3)}{(2n-5)(2n-3)(2n+1)}\right)\delta_{m,n-2} \\ &\quad - \frac{2\mu(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)}\delta_{m,n-4}, \end{aligned}$$

which along with (44), implies

$$\begin{aligned} &\Psi_n(x) = (n+1)_n J_n^{-1,-1}(x) - d_{n,1}\Psi_{n-2}(x) - d_{n,2}\Psi_{n-4}(x), \\ &d_{n,1} = \frac{1}{\varrho_{n-2}}\left(\frac{2\mu(1+d_{n-2,1})(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)} - \frac{2\mu(n^2-n-3)}{(2n-5)(2n-3)(2n+1)}\right), \\ &d_{n,2} = -\frac{2\mu(n-2)(n-3)}{(2n-3)(2n-5)(2n-7)\varrho_{n-4}}. \end{aligned}$$

It remains to confirm the constant ϱ_n, $n \ge 2$. Clearly, by (41), (39), (13), (46), (15) and (11), we know that for $n = 2, 3$,

$$\begin{aligned} \varrho_n &= B_\mu(\Psi_n, \Psi_n) = ((n+1)_n)^2 B_\mu(J_n^{-1,-1}, J_n^{-1,-1}) \\ &= ((n+1)_n)^2\big((\partial_x J_n^{-1,-1}, \partial_x J_n^{-1,-1}) + \mu(x^2 J_n^{-1,-1}, J_n^{-1,-1})\big) \\ &= (2n-1)^2(L_{n-1}, L_{n-1}) + \mu\left(\frac{n^2-n-3}{(2n-3)(2n+3)}L_n - \frac{n^2-n-3}{(2n-5)(2n+1)}L_{n-2}, L_n - L_{n-2}\right) \\ &= 2(2n-1) + \frac{4\mu(2n-1)(n^2-n-3)}{(2n-5)(2n-3)(2n+1)(2n+3)}. \end{aligned} \tag{47}$$

Thus $\varrho_2 = 6 + \frac{12}{35}\mu$ and $\varrho_3 = 10 + \frac{20}{63}\mu$ as in (43). Moreover, by (47) we obtain readily that for any $n \ge 4$,

$$B_\mu((n+1)_n J_n^{-1,-1}, (n+1)_n J_n^{-1,-1}) = 2(2n-1) + \frac{4\mu(2n-1)(n^2-n-3)}{(2n-5)(2n-3)(2n+1)(2n+3)}.$$

On the other hand, by (42) and (41) we have

$$\begin{aligned}&B_\mu((n+1)_nJ_n^{-1,-1},(n+1)_nJ_n^{-1,-1})\\&=B_\mu(\Psi_n+d_{n,1}\Psi_{n-2}+d_{n,2}\Psi_{n-4},\Psi_n+d_{n,1}\Psi_{n-2}+d_{n,2}\Psi_{n-4})\\&=\varrho_n+d_{n,1}^2\varrho_{n-2}+d_{n,2}^2\varrho_{n-4}.\end{aligned}$$

Hence

$$\varrho_n=2(2n-1)+\frac{4\mu(2n-1)(n^2-n-3)}{(2n-5)(2n-3)(2n+1)(2n+3)}-d_{n,1}^2\varrho_{n-2}-d_{n,2}^2\varrho_{n-4},\qquad n\ge 4.$$

This ends the proof.

Obviously, $\mathcal{P}_N^0(I)=\{\Psi_n\mid 2\le n\le N\}$. Thus the variational forms (39) and (40) together with the orthogonality of $\{\Psi_n\}_{n\ge2}$ lead to the following main theorem in this subsection.

Theorem 3.3 *Let $u(x)$ and $u_N(x)$ be the solutions of (39) and (40), respectively. Then both $u(x)$ and $u_N(x)$ have the explicit representations in $\{\Psi_n\}_{n\ge2}$,*

$$\begin{aligned}&u(x)=\sum_{n=2}^{\infty}\hat{u}_n\Psi_n(x),\qquad u_N(x)=\sum_{n=2}^{N}\hat{u}_n\Psi_n(x),\\&\hat{u}_n=\frac{1}{\varrho_n}B_\mu(u,\Psi_n)=\frac{1}{\varrho_n}(f,\Psi_n),\qquad n\ge2.\end{aligned}$$

3.4 Fourth-Order Problem with Dirichlet Boundary Condition

Consider the following fourth-order elliptic boundary value problem:

$$\begin{cases}\partial_x^4u(x)-\lambda\partial_x^2u(x)+\mu u(x)=f(x), & \lambda,\mu\ge0,\ x\in I,\\ \partial_xu(\pm1)=u(\pm1)=0.\end{cases}\tag{48}$$

Let

$$H_0^2(I)=\{v\in H^2(I)\mid\partial_xv(\pm1)=v(\pm1)=0\},\qquad\mathcal{V}_N(I)=\mathcal{P}_N(I)\cap H_0^2(I).$$

Clearly, $\mathcal{V}_N(I)=\mathrm{span}\{J_n^{-2,-2}(x):4\le n\le N\}$. A weak formulation of (48) is to find $u\in H_0^2(I)$ such that

$$B_{\lambda,\mu}(u,v):=(\partial_x^2u,\partial_x^2v)+\lambda(\partial_xu,\partial_xv)+\mu(u,v)=(f,v),\qquad\forall v\in H_0^2(I).\tag{49}$$

Clearly, if $f\in(H_0^2(I))'$, then by the Lax–Milgram lemma, (49) admits a unique solution.

The generalized Jacobi spectral scheme for (49) is to find $u_N\in\mathcal{V}_N(I)$ such that

$$B_{\lambda,\mu}(u_N,\phi)=(f,\phi),\qquad\forall\phi\in\mathcal{V}_N(I).\tag{50}$$

To propose a fully diagonalized approximation scheme for (49), we need to construct new basis functions $\{\Phi_n\}_{n\ge4}$, which are mutually orthogonal with respect to the Sobolev inner product $B_{\lambda,\mu}(\cdot,\cdot)$.

Lemma 3.4 *Let $\Phi_4(x)=(4)_3J_4^{-2,-2}(x)$ and $\Phi_n(x)\in\mathcal{V}_n(I)$ be the Sobolev orthogonal generalized Jacobi polynomials such that $\Phi_n(x)-(n)_{n-1}J_n^{-2,-2}(x)\in\mathcal{V}_{n-1}(I)$, and*

$$B_{\lambda,\mu}(\Phi_m, \Phi_n) = \sigma_n \delta_{m,n}, \qquad m, n \geq 4. \tag{51}$$

Then we have

$$\Phi_n(x) = (n)_{n-1} J_n^{-2,-2}(x) - d_{n,1}\Phi_{n-2}(x) - d_{n,2}\Phi_{n-4}(x), \qquad n \geq 5, \tag{52}$$

where $\Phi_n(x) \equiv 0$ $(n \leq 3)$, $\sigma_n = 0$ $(n \leq 3)$, $d_{n,1} = 0$ $(n \leq 5)$, $d_{n,2} = 0$ $(n \leq 7)$, *and*

$$\begin{aligned}
\sigma_n &= \frac{2n-3}{2} + \frac{\lambda(2n-3)}{(2n-1)(2n-5)} + \frac{3\mu(2n-3)}{(2n-7)(2n-5)(4n^2-1)} - d_{n,1}^2\sigma_{n-2} - d_{n,2}^2\sigma_{n-4}, \qquad n \geq 4,\\
d_{n,1} &= -\frac{1}{\sigma_{n-2}}\left(\frac{\lambda}{2(2n-5)} + \frac{2\mu}{(2n-1)(2n-5)(2n-9)} + \frac{\mu d_{n-2,1}}{2(2n-5)(2n-7)(2n-9)}\right), \qquad n \geq 6,\\
d_{n,2} &= \frac{\mu}{2(2n-5)(2n-7)(2n-9)\sigma_{n-4}}, \qquad n \geq 8.
\end{aligned} \tag{53}$$

Proof According to the orthogonality assumption (51) and the Gram–Schmidt orthogonalization procedure, we have

$$\Phi_n(x) = (n)_{n-1} J_n^{-2,-2}(x) - \sum_{m=4}^{n-1} \frac{B_{\lambda,\mu}((n)_{n-1} J_n^{-2,-2}, \Phi_m)}{\sigma_m}\Phi_m(x), \qquad n \geq 5. \tag{54}$$

We first use mathematical induction to verify (52). Clearly, by (49) and (13), we deduce that for any $n > m \geq 4$,

$$B_{\lambda,\mu}((n)_{n-1} J_n^{-2,-2}, \Phi_m) = \lambda(n)_{n-1}(\partial_x J_n^{-2,-2}, \partial_x \Phi_m) + \mu(n)_{n-1}(J_n^{-2,-2}, \Phi_m). \tag{55}$$

Moreover, by (15), we get for $n \geq 4$,

$$\partial_x J_n^{-2,-2}(x) = \frac{1}{2}\left(J_{n-1}^{0,0}(x) - \frac{1}{4(2n-3)(2n-5)} J_{n-3}^{0,0}(x)\right). \tag{56}$$

Hence, by (55), (56), (15) and (11) we derive

$$B_{\lambda,\mu}((5)_4 J_5^{-2,-2}, \Phi_4) = \lambda(5)_4(\partial_x J_5^{-2,-2}, \partial_x J_4^{-2,-2}) + \mu(5)_4(J_5^{-2,-2}, J_4^{-2,-2}) = 0.$$

This implies $\Phi_5(x) = (5)_4 J_5^{-2,-2}(x)$. Similarly, we get that for $n > m$ and $m = 4, 5$,

$$\begin{aligned}
&B_{\lambda,\mu}((n)_{n-1} J_n^{-2,-2}, \Phi_m)\\
&= (n)_{n-1}(m)_{m-1}\left(\lambda(\partial_x J_n^{-2,-2}, \partial_x J_m^{-2,-2}) + \mu(J_n^{-2,-2}, J_m^{-2,-2})\right)\\
&= -\left(\frac{\lambda}{2(2n-5)} + \frac{2\mu}{(2n-1)(2n-5)(2n-9)}\right)\delta_{m,n-2} + \frac{\mu}{2(2n-5)(2n-7)(2n-9)}\delta_{m,n-4}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\Phi_6(x) &= (6)_5 J_6^{-2,-2}(x) - d_{6,1}\Phi_4(x), \qquad d_{6,1} = -\tfrac{33\lambda+4\mu}{462\sigma_4},\\
\Phi_7(x) &= (7)_6 J_7^{-2,-2}(x) - d_{7,1}\Phi_5(x), \qquad d_{7,1} = -\tfrac{65\lambda+4\mu}{1\,170\sigma_5}.
\end{aligned}$$

In the same manner, we can verify the results of (52) for $n = 8$ with the constants $d_{8,1}$ and $d_{8,2}$ as in (53).

Next, assume that for any $4 \le m \le n-1$ and $n \ge 9$,

$$\Phi_m(x) = (m)_{m-1} J_m^{-2,-2}(x) - d_{m,1}\Phi_{m-2}(x) - d_{m,2}\Phi_{m-4}(x).$$

We shall prove $\Phi_n(x) = (n)_{n-1} J_n^{-2,-2}(x) - d_{n,1}\Phi_{n-2}(x) - d_{n,2}\Phi_{n-2}(x)$ for $n \ge 9$. Clearly, by (55), (56), (15), (11) and the induction assumption, we derive that for $n > m$ and $n \ge 9$,

$$\begin{aligned}
&B_{\lambda,\mu}((n)_{n-1}J_n^{-2,-2}, \Phi_m)\\
&= \lambda(n)_{n-1}(m)_{m-1}(\partial_x J_n^{-2,-2}, \partial_x J_m^{-2,-2}) + \mu(n)_{n-1}(m)_{m-1}(J_n^{-2,-2}, J_m^{-2,-2}) - \mu d_{m,1}(n)_{n-1}(m-2)_{m-3}(J_n^{-2,-2}, J_{m-2}^{-2,-2})\\
&= -\left(\frac{\lambda}{2(2n-5)} + \frac{2\mu}{(2n-1)(2n-5)(2n-9)} + \frac{\mu d_{n-2,1}}{2(2n-5)(2n-7)(2n-9)}\right)\delta_{m,n-2}\\
&\quad + \frac{\mu}{2(2n-5)(2n-7)(2n-9)}\delta_{m,n-4},
\end{aligned}$$

which along with (54) implies

$$\begin{aligned}
&\Phi_n(x) = (n)_{n-1}J_n^{-2,-2}(x) - d_{n,1}\Phi_{n-2}(x) - d_{n,2}\Phi_{n-4}(x), \qquad n \ge 8,\\
&d_{n,1} = -\frac{1}{\sigma_{n-2}}\left(\frac{\lambda}{2(2n-5)} + \frac{2\mu}{(2n-1)(2n-5)(2n-9)} + \frac{\mu d_{n-2,1}}{2(2n-5)(2n-7)(2n-9)}\right),\\
&d_{n,2} = \frac{\mu}{2(2n-5)(2n-7)(2n-9)\sigma_{n-4}}.
\end{aligned}$$

It remains to confirm the constant σ_n, $n \ge 4$. Clearly, by (52), (51), (13), (56), (15) and (11), we know that for $n = 4, 5$,

$$\begin{aligned}
\sigma_n &= B_{\lambda,\mu}(\Phi_n, \Phi_n) = ((n)_{n-1})^2 B_{\lambda,\mu}(J_n^{-2,-2}, J_n^{-2,-2})\\
&= ((n)_{n-1})^2\left((\partial_x^2 J_n^{-2,-2}, \partial_x^2 J_n^{-2,-2}) + \lambda(\partial_x J_n^{-2,-2}, \partial_x J_n^{-2,-2}) + \mu(J_n^{-2,-2}, J_n^{-2,-2})\right)\\
&= \frac{2n-3}{2} + \frac{\lambda(2n-3)}{(2n-1)(2n-5)} + \frac{3\mu(2n-3)}{(2n-7)(2n-5)(4n^2-1)}.
\end{aligned} \tag{57}$$

Thus $\sigma_4 = \frac{5}{2} + \frac{5}{21}\lambda + \frac{5}{63}\mu$ and $\sigma_5 = \frac{7}{2} + \frac{7}{45}\lambda + \frac{7}{495}\mu$. Moreover, by (57) we obtain readily that for any $n \ge 6$,

$$B_{\lambda,\mu}((n)_{n-1}J_n^{-2,-2}, (n)_{n-1}J_n^{-2,-2}) = \frac{2n-3}{2} + \frac{\lambda(2n-3)}{(2n-1)(2n-5)} + \frac{3\mu(2n-3)}{(2n-7)(2n-5)(4n^2-1)}.$$

On the other hand, by (52) and (51) we have

$$\begin{aligned}
&B_{\lambda,\mu}((n)_{n-1}J_n^{-2,-2}, (n)_{n-1}J_n^{-2,-2})\\
&= B_{\lambda,\mu}(\Phi_n + d_{n,1}\Phi_{n-2} + d_{n,2}\Phi_{n-4}, \Phi_n + d_{n,1}\Phi_{n-2} + d_{n,2}\Phi_{n-4})\\
&= \sigma_n + d_{n,1}^2\sigma_{n-2} + d_{n,2}^2\sigma_{n-4}.
\end{aligned}$$

Hence

$$\sigma_n = \frac{2n-3}{2} + \frac{\lambda(2n-3)}{(2n-1)(2n-5)} + \frac{3\mu(2n-3)}{(2n-7)(2n-5)(4n^2-1)} - d_{n,1}^2\sigma_{n-2} - d_{n,2}^2\sigma_{n-4}, \qquad n \geq 6.$$

This ends the proof.

Obviously, $\mathcal{V}_N(I) = \{\Phi_n \mid 4 \leq n \leq N\}$. Thus the variational forms (49) and (50) together with the orthogonality of $\{\Phi_n\}_{n\geq 2}$ lead to the following main theorem in this subsection.

Theorem 3.4 *Let $u(x)$ and $u_N(x)$ be the solutions of (49) and (50), respectively. Then both $u(x)$ and $u_N(x)$ have the explicit representations in $\{\Phi_n\}_{n\geq 4}$,*

$$u(x) = \sum_{n=4}^{\infty} \hat{u}_n \Phi_n(x), \qquad u_N(x) = \sum_{n=4}^{N} \hat{u}_n \Phi_n(x),$$
$$\hat{u}_n = \frac{1}{\sigma_n} B_{\lambda,\mu}(u, \Phi_n) = \frac{1}{\sigma_n}(f, \Phi_n), \qquad n \geq 4.$$

4 Convergence Analysis

In this section, we first derive the approximation properties of generalized Jacobi orthogonal projections, and then present the numerical errors of schemes (20) and (29).

4.1 Orthogonal Projections

Let $\alpha, \beta \in \mathcal{N}$ and

$$\mathcal{Q}_N^{\alpha,\beta}(I) := \text{span}\left\{ J_{\hat{\alpha}+\hat{\beta}}^{\alpha,\beta}(x), J_{\hat{\alpha}+\hat{\beta}+1}^{\alpha,\beta}(x), \ldots, J_N^{\alpha,\beta}(x) \right\}.$$

According to (9), we have

$$\mathcal{Q}_N^{\alpha,\beta}(I) = \{\phi \in \mathcal{P}_N(I) : \partial_x^i \phi(-1) = \partial_x^j \phi(1) = 0,\ 0 \leq i \leq -\alpha - 1,\ 0 \leq j \leq -\beta - 1\}.$$

The orthogonal projection $\pi_{N,\alpha,\beta} : L^2_{\omega^{\alpha,\beta}}(I) \to \mathcal{Q}_N^{\alpha,\beta}(I)$ is defined by

$$(\pi_{N,\alpha,\beta} u - u,\ \phi)_{\omega^{\alpha,\beta}} = 0, \qquad \forall \phi \in \mathcal{Q}_N^{\alpha,\beta}(I), \tag{58}$$

or equivalently,

$$\pi_{N,\alpha,\beta} u = \sum_{n=\hat{\alpha}+\hat{\beta}}^{N} \hat{u}_n^{\alpha,\beta} J_n^{\alpha,\beta}(x), \qquad \hat{u}_n^{\alpha,\beta} = \frac{1}{\gamma_n^{\alpha,\beta}} \int_I u(x) J_n^{\alpha,\beta}(x) \omega^{\alpha,\beta}(x) \mathrm{d}x.$$

According to Theorem 1.8.2 of [17], one verifies readily that if $\alpha, \beta \in \mathcal{N}$, $\partial_x^r u \in L^2_{\omega^{\alpha+r,\beta+r}}(I)$ and integers $1 \leq r \leq N+1$, then

$$\|\partial_x^k(\pi_{N,\alpha,\beta} u - u)\|_{\omega^{\alpha+k,\beta+k}} \leq c N^{k-r} \|\partial_x^r u\|_{\omega^{\alpha+r,\beta+r}}, \qquad 0 \leq k \leq r, \tag{59}$$

where c stands for a generic positive constant independent of any function and N.

Next, for integers $m \geq 1$, let

$$H_0^m(I) = \{v \in H^m(I) \mid \partial_x^k v(\pm 1) = 0,\ 0 \leq k \leq m-1\}.$$

It is obvious that $(\partial_x^m \cdot,\ \partial_x^m \cdot)$ is an inner product on $H_0^m(I)$ and $\mathcal{Q}_N^{-m,-m}(I) \subseteq H_0^m(I)$. We define the orthogonal projection $\pi_N^{m,0} : H_0^m(I) \to \mathcal{Q}_N^{-m,-m}(I)$ by

$$(\partial_x^m(\pi_N^{m,0}u - u),\ \partial_x^m \phi) = 0, \qquad \forall \phi \in \mathcal{Q}_N^{-m,-m}(I). \tag{60}$$

By (58) and integration by parts, we know that for any $u \in H_0^m(I)$,

$$\begin{aligned}&(\partial_x^m(\pi_{N,-m,-m}u - u), \partial_x^m \phi) = (-1)^m(\pi_{N,-m,-m}u - u, \partial_x^{2m}\phi)\\ &= (-1)^m(\pi_{N,-m,-m}u - u, \omega^{m,m}\partial_x^{2m}\phi)_{\omega^{-m,-m}} = 0, \qquad \forall \phi \in \mathcal{Q}_N^{-m,-m}(I).\end{aligned} \tag{61}$$

This means $\pi_N^{m,0}u(x) = \pi_{N,-m,-m}u(x)$ for any $u \in H_0^m(I)$. Therefore, by (59) we get that for integers $1 \le r \le N+1$,

$$\|\partial_x^k(\pi_N^{m,0}u - u)\|_{\omega^{-m+k,-m+k}} \le cN^{k-r}\|\partial_x^r u\|_{\omega^{-m+r,-m+r}}, \qquad 0 \le k \le r. \tag{62}$$

In order to analyze the numerical error of scheme (25), we have to consider an unusual mapping. To do this, for any $u \in H^1(I)$ we set

$$\widetilde{u}(x) = u(x) - \frac{1}{2}u(1)(x+1) + \frac{1}{2}u(-1)(x-1) \in H_0^1(I),$$

and define the mapping ${}_*\pi_N^1$ by

$${}_*\pi_N^1 u(x) = \pi_N^{1,0}\widetilde{u}(x) + \frac{1}{2}u(1)(x+1) - \frac{1}{2}u(-1)(x-1) \in \mathcal{P}_N(I).$$

Clearly, ${}_*\pi_N^1 u(\pm 1) = u(\pm 1)$. Moreover, by virtue of (62) with $m = 1$, we have that if $\partial_x^r u \in L^2_{\omega^{r-1,r-1}}(I)$ and integers $1 \le r \le N+1$, then for $k = 0, 1$,

$$\begin{aligned}\|\partial_x^k({}_*\pi_N^1 u - u)\|_{\omega^{k-1,k-1}} &= \|\partial_x^k(\pi_N^{1,0}\widetilde{u} - \widetilde{u})\|_{\omega^{k-1,k-1}}\\ &\le cN^{k-r}\|\partial_x^r \widetilde{u}\|_{\omega^{r-1,r-1}} \le cN^{k-r}\|\partial_x^r u\|_{\omega^{r-1,r-1}}.\end{aligned} \tag{63}$$

4.2 Convergence Analysis

Theorem 4.1 *Let u and u_N be the solutions of (19) and (20), respectively. If $u \in H^r_{\omega^{r-1,r-1}}(I)$ and integers $1 \le r \le N+1$, then for $\mu > 0$,*

$$\|\partial_x^k(u - u_N)\| \le cN^{k-r}\|\partial_x^r u\|_{\omega^{r-1,r-1}}, \qquad k = 0, 1, \tag{64}$$

while for $\mu = 0$,

$$\|\partial_x^k(u - u_N)\|_{\omega^{k-1,k-1}} \le cN^{k-r}\|\partial_x^r u\|_{\omega^{r-1,r-1}}, \qquad k = 0, 1. \tag{65}$$

Proof Clearly, $\mathcal{P}_N^0(I) = \mathcal{Q}_N^{-1,-1}(I)$. Hence, by (19) and (60) with $m = 1$, we get

$$(\partial_x \pi_N^{1,0}u, \partial_x \phi) + \mu(\pi_N^{1,0}u, \phi) = (f, \phi) + \mu(\pi_N^{1,0}u - u, \phi), \quad \phi \in \mathcal{P}_N^0(I). \tag{66}$$

Subtracting (20) from (66) yields

$$(\partial_x(\pi_N^{1,0}u - u_N), \partial_x \phi) + \mu(\pi_N^{1,0}u - u_N, \phi) = \mu(\pi_N^{1,0}u - u, \phi), \quad \phi \in \mathcal{P}_N^0(I). \tag{67}$$

Taking $\phi = \pi_N^{1,0}u - u_N$ in (67), and using the Cauchy–Schwartz inequality, we find

$$\|\partial_x(\pi_N^{1,0}u - u_N)\|^2 + \mu\|\pi_N^{1,0}u - u_N\|^2 = \mu(\pi_N^{1,0}u - u, \pi_N^{1,0}u - u_N) \le \mu\|\pi_N^{1,0}u - u\|\|\pi_N^{1,0}u - u_N\|.$$

This along with (62) leads to

$$\|\partial_x(\pi_N^{1,0}u - u_N)\|^2 + \mu\|\pi_N^{1,0}u - u_N\|^2 \le c\|\pi_N^{1,0}u - u\|^2 \le cN^{-2r}\|\partial_x^r u\|^2_{\omega^{r-1,r-1}}.$$

If $\mu > 0$, then by using (62) again, we obtain the desired result of (64) for $k = 0, 1$. If $\mu = 0$, then we have $u_N(x) = \pi_N^{1,0}u(x)$. Hence, by (62) we derive the desired result of (65).

Theorem 4.2 *Let u and u_N be the solutions of (28) and (29), respectively. If $u \in H^r_{\omega^{r-1,r-1}}(I)$ and integers $1 \le r \le N+1$, then*

$$\|\partial_x(u - u_N)\| + \mu\|u - u_N\| + \nu_+|u(1) - u_N(1)| + \nu_-|u(-1) - u_N(-1)| \le cN^{1-r}\|\partial_x^r u\|_{\omega^{r-1,r-1}}. \tag{68}$$

Proof By (28) we get

$$\begin{aligned}&(\partial_{x*}\pi_N^1 u, \partial_x\phi) + \mu({}_*\pi_N^1 u, \phi) + \nu_+ u(1)\phi(1) + \nu_- u(-1)\phi(-1)\\ &= (\partial_x({}_*\pi_N^1 u - u), \partial_x\phi) + \mu({}_*\pi_N^1 u - u, \phi) + (f, \phi) + a_+\phi(1) + a_-\phi(-1), \quad \phi \in \mathcal{P}_N(I).\end{aligned} \tag{69}$$

Subtracting (29) from (69) yields

$$\begin{aligned}&(\partial_x({}_*\pi_N^1 u - u_N), \partial_x\phi) + \mu({}_*\pi_N^1 u - u_N, \phi)\\ &+\nu_+(u(1) - u_N(1))\phi(1) + \nu_-(u(-1) - u_N(-1))\phi(-1)\\ &= (\partial_x({}_*\pi_N^1 u - u), \partial_x\phi) + \mu({}_*\pi_N^1 u - u, \phi), \qquad \phi \in \mathcal{P}_N(I).\end{aligned} \tag{70}$$

Take $\phi = {}_*\pi_N^1 u - u_N$ in (70). Then by (63), the Cauchy–Schwartz inequality and the fact ${}_*\pi_N^1 u(\pm 1) = u(\pm 1)$, we deduce that

$$\begin{aligned}&\|\partial_x({}_*\pi_N^1 u - u_N)\|^2 + \mu\|{}_*\pi_N^1 u - u_N\|^2 + \nu_+(u(1) - u_N(1))^2 + \nu_-(u(-1) - u_N(-1))^2\\ &= (\partial_x({}_*\pi_N^1 u - u), \partial_x({}_*\pi_N^1 u - u_N)) + \mu({}_*\pi_N^1 u - u, {}_*\pi_N^1 u - u_N)\\ &\le \|\partial_x({}_*\pi_N^1 u - u)\|\|\partial_x({}_*\pi_N^1 u - u_N)\| + \mu\|{}_*\pi_N^1 u - u\|\|{}_*\pi_N^1 u - u_N\|.\end{aligned}$$

This along with the Cauchy–Schwartz inequality and (63) leads to

$$\begin{aligned}&\|\partial_x({}_*\pi_N^1 u - u_N)\|^2 + \mu\|{}_*\pi_N^1 u - u_N\|^2 + 2\nu_+(u(1) - u_N(1))^2 + 2\nu_-(u(-1) - u_N(-1))^2\\ &\le \|\partial_x({}_*\pi_N^1 u - u)\|^2 + \mu\|{}_*\pi_N^1 u - u\|^2 \le cN^{2-2r}\|\partial_x^r u\|^2_{\omega^{r-1,r-1}}.\end{aligned}$$

By using (63) again, we get the desired result.

5 Numerical Results

In this section, we examine the effectiveness and the accuracy of the generalized Jacobi spectral method for solving elliptic boundary problems on the interval I.

5.1 Problem (18)

We examine the second-order Dirichlet boundary problem (18) with $\mu = 1$ and consider the following two cases:

- Smooth solution: $u(x) = (1 - x^2)\sin(1 + x)$. In Fig. 1, we list the discrete L^2- and L^∞ -errors. Clearly, the near straight lines indicate an exponential convergence rate.
- Non-smooth solution: $u(x) = (1 - x^2)(1 + x)^h$. In Fig. 2, we plot the discrete L^2-errors vs. $\log_{10} N$ with $h = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$. They show that the bigger the index h becomes, the faster the numerical errors decay. Obviously,

$$\|\partial_x^r((1 - x^2)(1 + x)^h)\|_{\omega^{r-1,r-1}} < \infty, \qquad \forall 1 \le r \le 2h + 1,$$

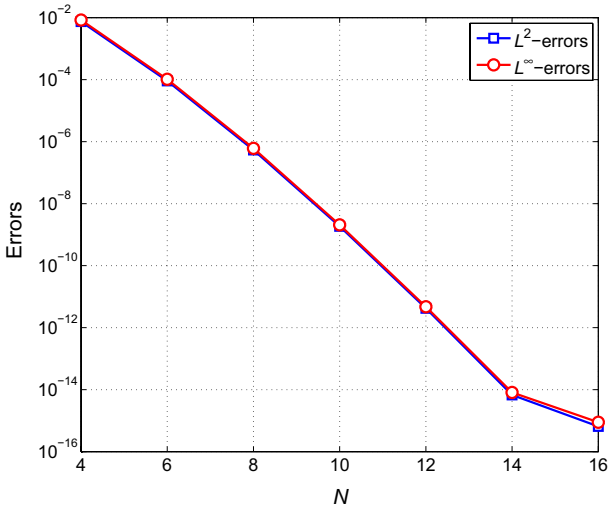

Fig. 1 Errors of $u(x) = (1 - x^2)\sin(1 + x)$

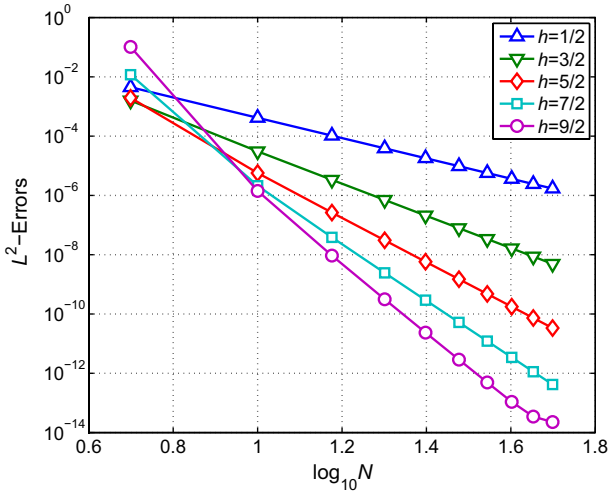

Fig. 2 L^2-errors of $u(x) = (1 - x^2)(1 + x)^h$

the expected L^2-errors given by Theorem 4.1 can be bounded by $cN^{-1-2h+\epsilon}$ for any $\epsilon > 0$. The observed convergence rates shown in Fig. 2 agree with the theoretical result, and the near straight lines indicate an algebraic convergence rate.

5.2 Problem (27)

We examine the second-order Robin boundary problem (27) with $\mu = \nu_+ = \nu_- = 1$ and consider the following two cases:

- Smooth solution: $u(x) = \sin(1 + x^2)$. In Fig. 3, we list the discrete L^2- and L^∞-errors. Again, the near straight lines indicate an exponential convergence rate.
- Non-smooth solution: $u(x) = (1 + x)^h e^x$. In Fig. 4, we plot the discrete L^2-errors vs. $\log_{10} N$ with $h = \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$. They also show that the bigger the index h becomes, the faster the numerical errors decay. Clearly,

$$\|\partial_x^r((1 + x)^h e^x)\|_{\omega^{r-1,r-1}} < \infty, \qquad 1 \le r \le 2h - 1,$$

the expected L^2-errors given by Theorem 4.2 can be bounded by $cN^{2-2h+\epsilon}$ for any $\epsilon > 0$. The observed convergence rates shown in Fig. 4 agree with the theoretical result, and the near straight lines indicate an algebraic convergence rate again.

5.3 Problem (38)

We examine the second-order problem (38) with a harmonic potential. Take $\mu = 2$ and consider the following two cases:

- Smooth solution: $u(x) = (1 - x^2)\sin(1 + x)$. In Fig. 5, we list the discrete L^2- and L^∞ -errors. The two near straight lines indicate an exponential convergence rate.

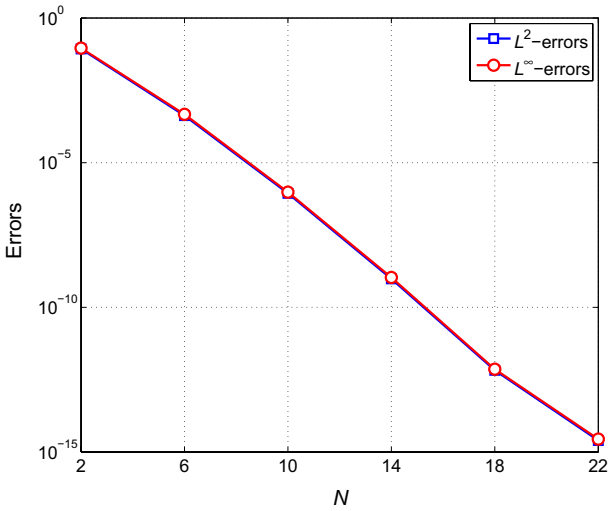

Fig. 3 Errors of $u(x) = \sin(1 + x^2)$

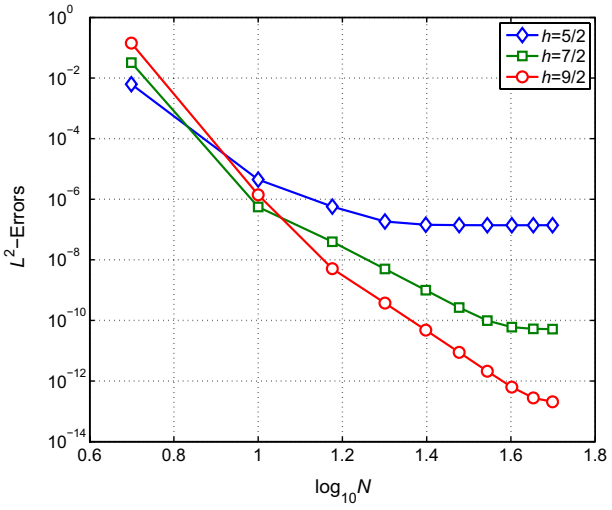

Fig. 4 L^2-errors of $u(x) = (1+x)^h \mathrm{e}^x$

- Non-smooth solution: $u(x) = (1-x^2)(1+x)^h$. In Fig. 6, we plot the discrete L^2-errors vs. $\log_{10} N$ with $h = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$. They also show that the bigger the index h becomes, the faster the numerical errors decay, and the near straight lines indicate an algebraic convergence rate again.

5.4 Problem (48)

We examine the fourth-order Dirichlet boundary problem (48) with $\lambda = 1, \mu = 2$ and consider the following two cases:

- Smooth solution: $u(x) = (1-x^2)^2 \sin(1+x)$. In Fig. 7, we list the discrete L^2- and L^∞ -errors. The two near straight lines indicate an exponential convergence rate.

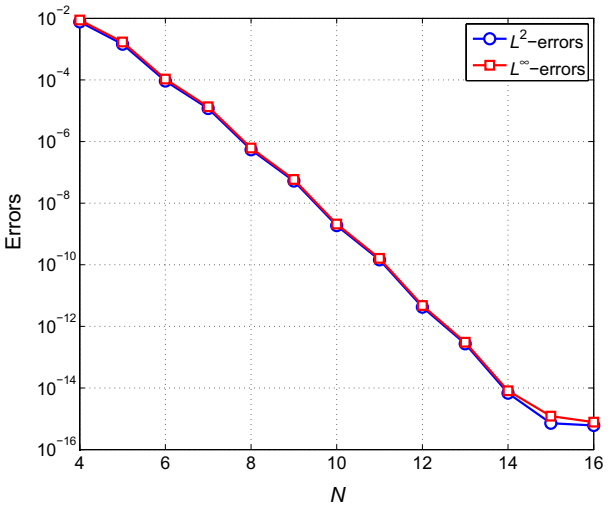

Fig. 5 Errors of $u(x) = (1-x^2)\sin(1+x)$

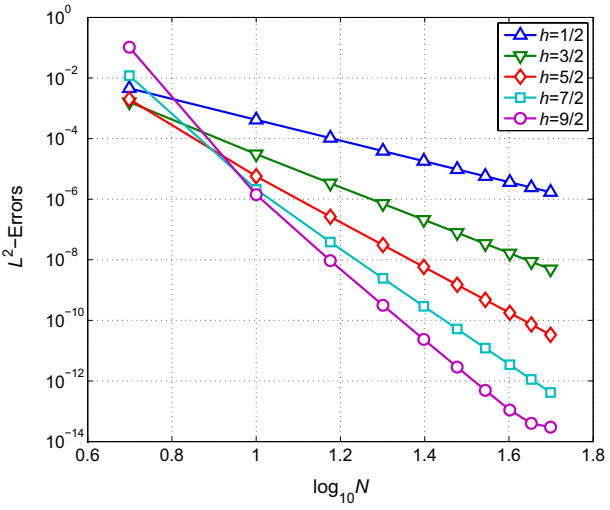

Fig. 6 L^2-errors of $u(x) = (1 - x^2)(1 + x)^h$

- Non-smooth solution: $u(x) = (1 - x^2)^2(1 + x)^h$. In Fig. 8, we plot the discrete L^2-errors vs. $\log_{10} N$ with $h = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$. They also show that the bigger the index h becomes, the faster the numerical errors decay, and the near straight lines indicate an algebraic convergence rate again.

5.5 Condition Numbers

We examine the issue on condition numbers for the resulting algebraic systems. Since we use the Sobolev-orthogonal generalized Jacobi polynomials $\{\varphi_k(x)/\sqrt{\eta_k}\}_{k=2}^N$ and $\{\psi_k(x)/\sqrt{\rho_k}\}_{k=0}^N$ as the basis functions for (20) and (29), respectively, the condition numbers of the total stiff matrices are equal to 1. While in the classical Jacobi spectral method, the basis functions for (20) and (29) are chosen as $\{(1 - x^2)P_n^{0,0}(x)\}_{n=0}^{N-2}$ and $\{P_n^{0,0}(x)\}_{n=0}^N$, respectively. The total stiff matrices have off-diagonal entries. In Table 1 below, we list the condition numbers of the total stiff matrices of the classical Jacobi

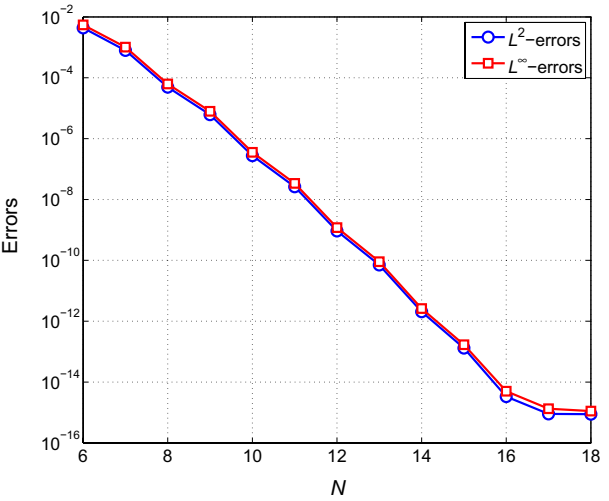

Fig. 7 Errors of $u(x) = (1 - x^2)^2 \sin(1 + x)$

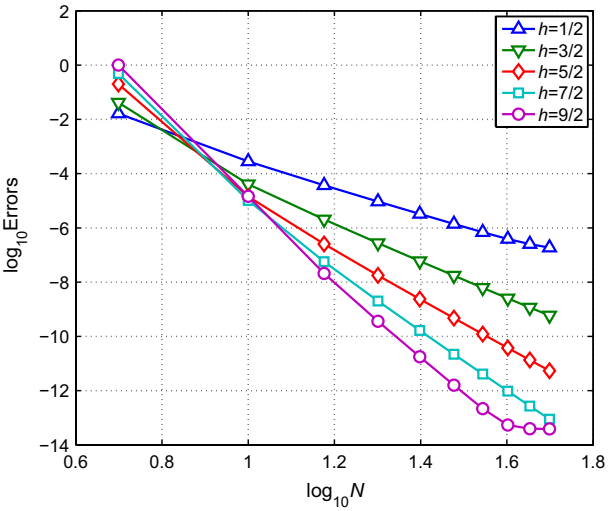

Fig. 8 L^2-errors of $u(x) = (1-x^2)^2(1+x)^h$

Table 1 Condition numbers of the classical Jacobi spectral methods

N	8	16	32	64	128
Eq. (20)	6.073 5E+00	2.793 6E+01	1.255 8E+02	5.423 8E+02	2.279 0E+03
Eq. (29)	5.564 0E+01	3.435 6E+02	2.401 6E+03	1.793 4E+04	1.385 6E+05

spectral method for (20) and (29). It can be observed that the condition numbers of the resulting systems increase asymptotically as $\mathcal{O}(N^2)$ and $\mathcal{O}(N^3)$ for (20) and (29), respectively.

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