Research Article

Finite-amplitude ferro-convection and electro-convection in a rotating fluid

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Abstract
The onset of Rayleigh–Bénard convection in a horizontal ferrofluid layer subjected to rotation with respect to the vertical axis is theoretically investigated in the paper. Using a truncated Fourier series representation, an analytically intractable fifth-order Lorenz model that has two quadratic nonlinearities is derived and then reduced to the analytically tractable Landau equation with cubic nonlinearity using the method of multi-scales. The critical Rayleigh number of the linear stability theory and that of the energy stability method is drawn from other works and compared with that obtained by the weakly nonlinear stability method reported in the paper. It is found that the critical Rayleigh number obtained by the two nonlinear theories predicts subcritical motions. Further, the results on electro-convection in a rotating fluid layer are extracted from the corresponding problem of ferro-convection by establishing an one-to-one correspondence between the governing equations of the two problems.

Keywords Ferro-convection · Electro-convection · Energy stability · Rayleigh–Bénard · Lorenz model

1 Introduction

The theory of rotating Newtonian fluids and Rayleigh–Bénard convection in such fluids is a well-studied topic [1–6]. The corresponding problem in ferrofluids (called rotating ferro-convection) and in dielectric fluids (called rotating electro-convection) can be interesting for reasons more than one.

Ferrofluids are synthesized suspensions of micron-sized magnetic particles in a carrier liquid of suitable choice. The carrier liquids generally used are synthetic oil, water, kerosene, or such other liquids. The suspended magnetic particles that possess a fixed magnetic moment are coated with a surfactant, such as oleic acid, to avoid coagulation. In the presence of an external magnetic field, the resulting orientation of the magnetic particles leads to a net magnetization of the fluid which depends on both the applied magnetic field and the temperature of the fluid. The applied magnetic field exerts a force on the fluid which is known as the Kelvin force. When the magnetic field strength is very high, the magnetization achieves its saturation value and the particles get aligned with the applied field. The magnetization generally depends on the magnetic field, temperature, and density of the ferrofluid. Hence, a local variation in magnetic field/temperature may give rise to thermo-magnetic convection which is equivalent to the natural convection occurring due to variation in temperature only. This phenomenon has attracted the interest of many researchers in past five decades and has widespread scientific applications [7–15].

The linear and nonlinear ferro-convection in nonrotating fluids occupying a very shallow enclosure has now been extensively studied [16–39]. The linear and nonlinear problems of rotating ferro-convection have also attracted

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attention of many [40–45]. The nonlinear stability analysis of rotating ferro-convection reported above has been studied using weakly nonlinear stability analysis (based on Lorenz model or Ginzburg–Landau equation) or by energy stability analysis.

The work on nonlinear, rotating electro-convection is nonexistent, though there are few works on nonlinear electro-convection in the absence of rotation (Siddheshwar and Radhakrishna [46] and reference therein). The one-to-one correspondence between the linear problems of ferro-convection and electro-convection lead Siddheshwar [47, 48] to take a unified approach at studying both.

The exhaustive literature survey on ferro-convection and electro-convection reveals that the following aspects are as yet unconsidered:

1. Recognizing that linear and nonlinear electro-convection problems in rotating and nonrotating systems have an analogy with the corresponding problems of ferro-convection.
2. Comparing the critical Rayleigh numbers of linear stability, weakly nonlinear stability, and energy stability of rotating ferro-convection as well as rotating electro-convection - existence of subcritical motions.
3. Connecting the analytically intractable Lorenz model of nonrotating and rotating ferro-convection with the corresponding Landau model that is analytically tractable.
4. Analyzing whether the analytical solution of the Landau equation provides insights into the nature of the amplitude of convection in transient and steady-state regimes.

The objective of the present paper is to consider the aforementioned aspects that concern nonlinear rotating ferro-convection (and corresponding electro-convection problem by analogy).

2 Problem formulation for the rotating ferro-convection

Consider a horizontal layer of ferrofluid confined between two infinitely extended parallel plates, $z = 0$ and $z = d$. We choose the right-hand system of coordinates (x, y, z), where z-axis is assumed in such a way that the gravitational force, $\mathbf{g}$, is taken as $\mathbf{g} = -g_0\hat{\mathbf{k}}$. The layer is subjected to rotation with respect to the z-axis with a constant angular speed $\mathbf{\Omega}$. The motion described here occurs in a way as it appears to an observer at rest in a frame rotating about the same axis and with the same angular velocity (uniformly rotating frame of reference). An external magnetic field $\mathbf{H}$ is applied in the z-direction. An adverse temperature gradient across the fluid layer is maintained by imposing a uniform temperature difference ΔT between the plates. Schematic of the same is described in Fig. 1.

For mathematical tractability, all physical quantities are assumed to be independent of the horizontal co-ordinate, y. Thus, the study pertains to two-dimensional ferro-convection confined within a horizontal ferrofluid layer. The height, d, and breadth, b, are such that when $d/b << 1$, and hence, there is no effect of the lateral walls (x) on the dynamics in the bulk of the ferromagnetic fluid.

2.1 Governing equations

To incorporate the effect of uniform rotation into the equation of conservation of linear momentum, the acceleration term can be taken as

$$\left(\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t}\right)_I = \left(\frac{\mathrm{D}\mathbf{q}}{\mathrm{D}t}\right)_R + \mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) - 2(\mathbf{q}\times\mathbf{\Omega})_R, \tag{1}$$

where $\mathbf{q}$ denotes the velocity vector and t is time. The subscripts I and R refer to inertial and rotating frames of reference and $\mathbf{r}$ is the radius vector. The operator $\frac{\mathrm{D}}{\mathrm{D}t}$ in Eq. (1) is given by $\frac{\mathrm{D}}{\mathrm{D}t} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla$. The second and third terms in Eq. (1) are centrifugal and Coriolis forces that occur due to rotation. Since the rotation is uniform, the Euler force is neglected in Eq. (1). The centrifugal force can be expressed as the gradient of a scalar quantity:

$$\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{r}) = -\nabla\left(\frac{1}{2}|\mathbf{\Omega}|^2|\mathbf{r}|^2\right), \tag{2}$$

where $|\mathbf{r}|$ is the distance from the axis of rotation. Thus, the governing equations for studying the rotating ferro-convection problem with applied magnetic field after omitting subscript R are

$$\nabla\cdot\mathbf{q} = 0, \tag{3}$$

$$\rho_0\frac{\partial\mathbf{q}}{\partial t} = -\nabla P + \rho(T)\mathbf{g} + \mu_0(\mathbf{M}\cdot\nabla)\mathbf{H} + \mu\nabla^2\mathbf{q} + 2\rho_0(\mathbf{q}\times\mathbf{\Omega}) \tag{4}$$

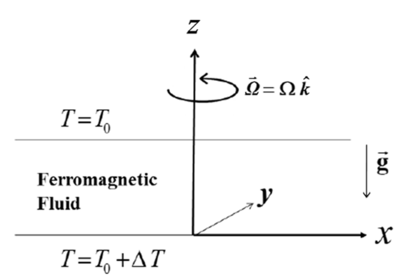

Fig. 1 Schematic of the ferro-convection problem

$$\frac{\partial T}{\partial t}+(\mathbf{q}\cdot\nabla)T=\kappa\nabla^2 T, \tag{5}$$

We are considering only small-scale convective motion and hence $|(\mathbf{q}\cdot\nabla)\mathbf{q}|<<1$. The centrifugal force in Eq. (4) is absorbed into the pressure gradient term. Thus, pressure, P, can be written as $P=p-\frac{\rho_0}{2}|\boldsymbol{\Omega}|^2|\mathbf{r}|^2$, the sum of the hydro-static pressure (p) and that due to the centrifugal force. The physical quantities ρ_0, μ_0, μ, κ, $\mathbf{M}$, $\mathbf{H}$, and T represent the static density, magnetic permeability, effective viscosity, thermal diffusivity, magnetization, magnetic field, and temperature, respectively. In writing Eq. (4), the Boussinesq approximation is assumed to be valid. The state equation may be written as

$$\rho(T)=\rho_0\left[1-\alpha(T-T_0)\right], \tag{6}$$

where α is the thermal expansion coefficient and T_0 is the reference temperature. We presume the ferrofluid to be electrically insulating and write the following Maxwell's equations for the magnetic field, $\mathbf{H}$, and the magnetic induction, $\mathbf{B}$, as

$$\nabla\times\mathbf{H}=\mathbf{0}, \qquad \nabla\cdot\mathbf{B}=0. \tag{7}$$

Furthermore, the relationship between these fields is

$$\mathbf{B}\equiv\mu_0(\mathbf{H}+\mathbf{M}), \tag{8}$$

where $\mathbf{M}=M(T,H)\hat{\mathbf{H}}$. The magnetic equation of state is given by (Finlayson [16])

$$M(T,H)=M_0-K(T-T_0)+\chi(H-H_0), \tag{9}$$

where K and χ are the pyromagnetic and magnetic susceptibility coefficients, $M_0=M(T_0,H_0)$ is the reference magnetization, and $H_0\hat{\mathbf{k}}$ is the applied uniform vertical magnetic field.

We note from Eq. $(7)_1$ that we can write $\mathbf{H}=\nabla\phi$, where ϕ is the magnetic potential.

2.2 Boundary conditions

We consider the boundaries to be such that the tangential stress vanishes. To maintain a static temperature difference across the layer, we keep the lower boundary ($z=0$) at a temperature $T_0+\Delta T(\Delta T>0)$ and the upper boundary ($z=d$) at a temperature T_0. We, therefore, assume stress-free and isothermal boundary conditions:

$$\left.\begin{aligned} w=\frac{\partial^2 w}{\partial z^2}=0,\ T=T_0+\Delta T \text{ at } z=0\\ w=\frac{\partial^2 w}{\partial z^2}=0,\ T=T_0 \text{ at } z=d\end{aligned}\right\}, \tag{10}$$

where w is the z-components of velocity vector, $\mathbf{q}$.

To specify the magnetic potential boundary condition, we assume continuity of the tangential component of the magnetic field and the normal component of the magnetic induction across the boundary. To derive magnetic potential boundary condition, we considered general boundary condition for the perturbed magnetic potential, ϕ:

$$\left.\begin{aligned} D\phi+\frac{k\phi}{1+\chi}-T=0 \text{ at } z=0\\ D\phi-\frac{k\phi}{1+\chi}-T=0 \text{ at } z=d\end{aligned}\right\}, \tag{11}$$

where $D=\frac{d}{dz}$ and k is dimensionless wave number. Since isothermal boundary condition is assumed for temperature and when $\chi\to\infty$ at the boundary, we obtain $D\phi=0$ (see Finlayson [16]) at $z=0$ and d.

There are umpteen number of instances in which idealized boundary conditions have been considered with the sole purpose of seeking qualitative results (see Finlayson [16], Suslov [51], Laroze et al. [38], Rahman and Suslov [39], and certain references therein). The present paper is one of them. The analysis is exact and trustworthy in this case. We, however, note that the problem is best done with realistic boundary condition but the price we pay for this is that the analysis, especially the nonlinear one, is fraught with errors due to mandatory numerical procedure imposed by it. This study is at the present time is contemplated to be completed in the nearest future.

2.3 Basic state solution

Using boundary conditions (10) and (11), the conductive basic quiescent state solution may be written as

$$\mathbf{q}_b= \quad \mathbf{0}, \tag{12}$$

$$T_b(z)= \quad T_0-\beta(z-d), \tag{13}$$

$$\mathbf{H}_b(z)=\left(H_0-\frac{K\beta(z-d)}{1+\chi}\right)\hat{\mathbf{k}}, \tag{14}$$

$$\mathbf{M}_b(z)=\left(M_0+\frac{K\beta(z-d)}{1+\chi}\right)\hat{\mathbf{k}}, \tag{15}$$

where $\beta=\Delta T/d$ and the subscript b represents the basic state.

3 Stability analysis

To perform stability analysis, we introduce perturbations to the basic state in the form

$$\left.\begin{array}{l}\mathbf{q} = \mathbf{0} + \mathbf{q}', \quad \mathbf{H} = \mathbf{H}_b + \mathbf{H}', \quad \mathbf{M} = \mathbf{M}_b + \mathbf{M}', \\ \quad \rho = \rho_0 + \rho', \quad P = P_b + P', \quad T = T_b + T', \end{array}\right\} \tag{16}$$

where the prime denotes perturbed quantities.

3.1 Perturbation equations

Using Eqs. (9) and (16) in Eq. (8), we get

$$\left.\begin{array}{l} H'_x + M'_x = \left(1 + \dfrac{M_0}{H_0}\right) H'_x, \\ H'_z + M'_z = (1+\chi) H'_z - KT' \end{array}\right\}. \tag{17}$$

Imposing perturbation on governing equations (3)–(5), we get the perturbed governing equations in the following form:

$$\nabla \cdot \mathbf{q}' = 0, \tag{18}$$

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{q}'}{\partial t} &= -\nabla P_{eff} + \rho_0 \alpha T' g \hat{\mathbf{k}} + \mu \nabla^2 \mathbf{q}' + 2\rho_0 \left(\mathbf{q}' \times \boldsymbol{\Omega}\right) \\ &+ \mu_0 \left[\nabla\left(\phi'_z M_b\right) - \frac{K\beta\left\{(1+\chi)\phi'_z - KT'\right\}\hat{\mathbf{k}}}{1+\chi} - KT' \nabla \phi'_z \right], \end{aligned} \tag{19}$$

$$\frac{\partial T'}{\partial t} + (\mathbf{q}' \cdot \nabla) T' = \kappa \nabla^2 T' + \beta w'. \tag{20}$$

where $P_{eff} = P - \frac{\mu_0 M_0}{2H_0}\phi_x'^2 - \frac{\mu_0 \chi}{2}\phi_z'^2$, w' is the perturbed velocity in z-direction, and ϕ_z is the differentiation of ϕ with respect to z.

3.2 Nondimensionalization

To convert the governing equations to dimensionless form, we introduce the characteristic scales d for length, d^2/κ for time, κ/d for velocity, βd for temperature, $\beta d^2 K/(1+\chi)$ for magnetic scalar potential, and $\mu\kappa/d^2$ for the pressure and the stress tensor. Using these scales in Eqs. (18)–(20) and simplifying, we get the dimensionless governing equations for the perturbations in the following form:

$$\nabla \cdot \mathbf{q} = 0, \tag{21}$$

$$\begin{aligned} \frac{1}{\mathrm{Pr}} \frac{\partial \mathbf{q}}{\partial t} &= -\nabla P + \mathrm{Ra}(1 + M_1)\theta \hat{k} - \mathrm{Ra} M_1 \phi_z \hat{\mathbf{k}} \\ &+ \nabla^2 \mathbf{q} + \sqrt{\mathrm{Ta}}\left(\mathbf{q} \times \hat{\mathbf{k}}\right) - \mathrm{Ra} M_1 \theta \nabla \phi_z, \end{aligned} \tag{22}$$

$$\frac{\partial \theta}{\partial t} + (\mathbf{q} \cdot \nabla)\theta = \nabla^2 \theta + w, \tag{23}$$

$$M_3 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{\partial \theta}{\partial z} = 0. \tag{24}$$

The nondimensional parameters in Eqs. (22) and (24) are $\mathrm{Pr} = \dfrac{\mu}{\rho_0 \kappa}$, Prandtl number, $\mathrm{Ra} = \dfrac{\rho_0 \alpha \Delta T d^3 g}{\mu \kappa}$, Rayleigh number, $\mathrm{Ta} = \left(\dfrac{2d^2 \rho_0 \Omega}{\mu}\right)^2$, Taylor number, $M_1 = \dfrac{\mu_0 K^2 \Delta T}{g \rho_0 (1+\chi) d}$, ratio of the magnetic and buoyancy forces, and $M_3 = \dfrac{1 + M_0/H_0}{1+\chi}$, non-buoyancy magnetization parameter. If we take $M_1 = 0$, then that corresponds to the non-magnetic Rayleigh–Bénard convection problem. The term M_3 measures the departure of linearity in the magnetic equation of state defined in Eq. (9) and $M_3 = 1$corresponds to linear magnetization.

We operate *curl* on Eq. (22) to get the vorticity equation:

$$\frac{1}{\mathrm{Pr}} \frac{\partial \zeta}{\partial t} = \nabla^2 \zeta + \sqrt{\mathrm{Ta}} \frac{\partial w}{\partial z} - \mathrm{Ra} M_1 \frac{\partial(\theta, \phi_z)}{\partial(x,z)}, \tag{25}$$

where $\zeta = [\nabla \times \mathbf{q}]_z$ is the z-component of vorticity which describes the local spinning motion of the continuum point. The term, $\frac{\partial(\theta,\phi_z)}{\partial(x,z)}$, represents the Jacobian term and is defined as

$$\frac{\partial(\theta, \phi_z)}{\partial(x,z)} = \frac{\partial \theta}{\partial x}\frac{\partial \phi_z}{\partial z} - \frac{\partial \theta}{\partial z}\frac{\partial \phi_z}{\partial x}.$$

Operating $\hat{\mathbf{k}} \cdot$ *curl curl* on Eq. (22) results in the following equation:

$$\begin{aligned} \frac{1}{\mathrm{Pr}} \frac{\partial}{\partial t}\left(\nabla^2 w\right) &= \nabla^4 w - \sqrt{\mathrm{Ta}} \frac{\partial \zeta}{\partial z} + \mathrm{Ra}(1 + M_1)\nabla_1^2 \theta - \mathrm{Ra} M_1 \nabla_1^2 \phi_z \\ &- \mathrm{Ra} M_1 \left[\frac{\partial}{\partial x} \frac{\partial\left(\theta, \phi_z\right)}{\partial(x,z)} \right] \end{aligned} \tag{26}$$

where $\nabla_1^2 = \dfrac{\partial^2}{\partial x^2}$. The system of Eqs. (23)–(26) are solved subject to the boundary condition in nondimensional form:

$$w = D^2 w = \theta = D\zeta = D\phi = 0 \text{ at } z = 0, 1. \tag{27}$$

This boundary condition, though unrealistic, facilitates an analytical solution that gives qualitatively similar results to that of realistic boundaries.

4 Lorenz model

We assume the following truncated Fourier series solutions for u, v, w, θ, and ϕ:

$$\left.\begin{aligned} u(x,z,t) &= -A_1(t)\pi \sin(kx)\cos(\pi z)\\ w(x,z,t) &= kA_1(t)\cos(kx)\sin(\pi z)\\ v(x,z,t) &= B_1(t)\cos(kx)\cos(\pi z) + B_2(t)\cos(2kx)\\ \theta(x,z,t) &= C_1(t)\cos(kx)\sin(\pi z) + C_2(t)\sin(2\pi z)\\ \phi(x,z,t) &= D_1(t)\cos(kx)\cos(\pi z) + D_2(t)\cos(2\pi z) \end{aligned}\right\}, \tag{28}$$

where $A_1(t)$, $B_1(t)$, $B_2(t)$, $C_1(t)$, $C_2(t)$, $D_1(t)$ and $D_2(t)$ are amplitudes. Substituting Eq. (28) in Eq. (24) yields D_1 and D_2 in terms of C_1 and C_2, respectively:

$$D_1(t) = -\frac{\pi}{\delta_M^2}C_1(t) \quad D_2(t) = -\frac{1}{2\pi}C_2(t), \tag{29}$$

where $\delta_M^2 = \left(\pi^2 + M_3k^2\right)$. On substituting Eq. (28) in Eqs. (25) and (26) and using the slave relationship (29) in one of the equations and the orthogonality conditions, we get the following system of equations

$$\left.\begin{aligned} \dot{A}_1(t) &= -\delta^2 \mathrm{Pr} A_1(t) - \frac{\pi \mathrm{Pr}\sqrt{\mathrm{Ta}}}{\delta^2}B_1(t)\\ &\quad + k\mathrm{Pr}\ \mathrm{Ra}C_1(t)\left[\xi_1 - \xi_2 C_2(t)\right],\\ \dot{B}_1(t) &= A_1(t)\pi\left(\mathrm{Pr}\sqrt{\mathrm{Ta}} - kB_2(t)\right) - \delta^2\mathrm{Pr}B_1(t),\\ \dot{B}_2(t) &= -4k^2\mathrm{Pr}B_2(t) - \frac{\pi k}{2}A_1(t)B_1(t),\\ \dot{C}_1(t) &= A_1(t)(\pi k C_2(t) + k) - \delta^2 C_1(t),\\ \dot{C}_2(t) &= -\frac{\pi k}{2}A_1(t)C_1(t) - 4\pi^2 C_2(t), \end{aligned}\right\} \tag{30}$$

where an over dot on amplitudes denotes derivative with respect to t, $\delta^2 = k^2 + \pi^2$, $\xi_1 = \dfrac{\delta_M^2 + k^2M_1M_3}{\delta^2\delta_M^2}$, and $\xi_2 = \dfrac{\pi k^2 M_1 M_3}{\delta^2\delta_M^2}$.

5 Steady finite-amplitude convection and subcritical motions

Assuming the amplitudes to be independent of time in the Lorenz system (30), we get the governing equations for the study of steady two-dimensional finite-amplitude convection in the form of a coupled system of algebraic equations which is then reduced to a cubic equation in A_1^2:

$$a_1\left(A_1^2\right)^3 + a_2\left(A_1^2\right)^2 + a_3A_1^2 + a_4 = 0. \tag{31}$$

The expressions for a_i's, $i = 1, 2, 3, 4$, are quite lengthy and have been omitted for brevity. The minimum value of Ra for which the discriminant of Eq. (31) vanishes gives us the steady, finite-amplitude Rayleigh number ($\mathrm{Ra_f}$). For the present problem, a polynomial of degree nine in Ra was obtained by setting the discriminant of the above equation to be zero and thus $\mathrm{Ra_f}$ had to be evaluated numerically together with the wave number, k, as a function of Ta, M_1, M_3, and Pr.

With the intention of comparing the results obtained in the present study with those of the linear stability and the energy stability ones, we recollect that for the present problem the linear stability Rayleigh number is given by Gupta and Gupta [40]

$$\mathrm{Ra_{sF}} = \frac{\delta_M^2\left(\delta^6 + \pi^2\mathrm{Ta}\right)}{k^2\left(k^2M_1M_3 + \delta_M^2\right)} \tag{32}$$

and the energy stability Rayleigh number (as reported in [35]) is given by

$$\mathrm{Ra_{eF}} = \frac{\left[4\delta_M^2 - \dfrac{\pi^2M_1}{(1+M_1)}\right]\left(\delta^6 + \pi^2\mathrm{Ta}\right)}{4k^2(1+M_1)\delta_M^2 - 2\pi^2M_1}. \tag{33}$$

Equation (33) is Eq. (75) in Sunil and Mahajan [43] but with the notations used in the present paper. In the next section, we reduce the fifth-order Lorenz system into a first-order Landau equation by using the method of multi-scales.

6 Landau amplitude equation for small amplitude convective motion

We assume a slow time scale $\tau = \epsilon^2 t$ and use the following regular perturbation expansion as done by Siddheshwar and Kanchana [48] for the amplitudes and also for Ra appearing in the system of Eq. (30):

$$\begin{bmatrix} A_1\\ B_1\\ B_2\\ C_1\\ C_2\\ \mathrm{Ra}\end{bmatrix} = \begin{bmatrix} 0\\0\\0\\0\\0\\ \mathrm{Ra}_0\end{bmatrix} + \epsilon\begin{bmatrix} A_{11}\\ B_{11}\\ B_{21}\\ C_{11}\\ C_{21}\\ 0\end{bmatrix} + \epsilon^2\begin{bmatrix} A_{12}\\ B_{12}\\ B_{22}\\ C_{12}\\ C_{22}\\ \mathrm{Ra}_2\end{bmatrix} + \epsilon^3\begin{bmatrix} A_{13}\\ B_{13}\\ B_{23}\\ C_{13}\\ C_{23}\\ 0\end{bmatrix} + \cdots \tag{34}$$

Substituting (34) in Eq. (30) and comparing like powers of ϵ on either side of the resulting equations, we get systems of different orders.

6.1 First-order system

$$\mathcal{L}\mathbf{v_1} = 0, \tag{35}$$

where

$$\mathbf{v_1} = \left[A_{11} \quad B_{11} \quad B_{21} \quad C_{11} \quad C_{21}\right]^{Tr}. \tag{36}$$

$$\mathcal{L} = \begin{bmatrix} -\delta^2\mathrm{Pr} & -\dfrac{\pi\mathrm{Pr}\sqrt{\mathrm{Ta}}}{\delta} & 0 & k\mathrm{PrRa}_0\xi_1 & 0 \\ \pi\mathrm{Pr}\sqrt{\mathrm{Ta}} & \delta^2\mathrm{Pr} & 0 & 0 & 0 \\ 0 & 0 & -4k^2\mathrm{Pr} & 0 & 0 \\ k & 0 & 0 & -\delta^2 & 0 \\ 0 & 0 & 0 & 0 & -4\pi^2 \end{bmatrix}. \tag{37}$$

The condition for nontrivial solution of the homogeneous linear system (35) yields

$$\mathrm{Ra}_0 = \mathrm{Ra}_s \tag{38}$$

and the solution to the system (35) is given by

$$\mathbf{v_1} = \left[A_{11} \quad \frac{\pi\sqrt{\mathrm{Ta}}}{\delta^2}A_{11} \quad 0 \quad \frac{k}{\delta^2}A_{11} \quad 0\right]^{Tr}. \tag{39}$$

6.2 Second-order system

$$\mathcal{L}\mathbf{v_2} = \mathcal{N}_2, \tag{40}$$

where

$$\mathbf{v_2} = \left[A_{12} \quad B_{12} \quad B_{22} \quad C_{12} \quad C_{22}\right]^{Tr}, \tag{41}$$

and

$$\mathcal{N}_2 = \left[0 \quad 0 \quad 0 \quad 0 \quad \tfrac{1}{2}\pi k A_{11}C_{11}\right]^{Tr}. \tag{42}$$

The solution of Eq. (40) is

$$\mathbf{v_2} = \left[0 \quad 0 \quad 0 \quad 0 \quad -\frac{k^2}{8\pi\delta^2}A_{11}^2\right]^{Tr}. \tag{43}$$

6.3 Third-order system

$$\mathcal{L}\mathbf{v_3} = \mathcal{N}_3, \tag{44}$$

where

$$\mathbf{v_3} = \left[A_{13} \quad B_{13} \quad B_{23} \quad C_{13} \quad C_{23}\right]^{Tr}, \tag{45}$$

$$\mathcal{N}_3 = \begin{bmatrix} \dfrac{dA_{11}}{d\tau} + k\mathrm{PrRa}_0\xi_2 C_{11}C_{22} - k\mathrm{PrRa}_2\xi_1 C_{11} \\ \dfrac{dB_{11}}{d\tau} + \pi k A_{11}B_{22} \\ \dfrac{dB_{21}}{d\tau} \\ \dfrac{dC_{11}}{d\tau} - \pi k A_{11}C_{22} \\ \dfrac{dC_{21}}{d\tau} + \frac{1}{2}\pi k\left(A_{11}C_{12} + A_{12}C_{11}\right) \end{bmatrix}. \tag{46}$$

We used the Fredholm alternative condition to get the Landau amplitude equation in the following form :

$$Q_1\frac{dA_{11}}{d\tau} = Q_2A_{11} - Q_3A_{11}^3, \tag{47}$$

where

$$\left.\begin{aligned} Q_1 &= 1 - \frac{\pi^2\mathrm{Ta}}{\delta^6} + \frac{k^2\xi_1\mathrm{PrRa}_0}{\delta^4}, \quad Q_2 = \frac{k^2\xi_1\mathrm{PrRa}_2}{\delta^2}, \\ Q_3 &= \frac{k^4\xi_1\mathrm{PrRa}_0}{8\delta^4} - \frac{k^4\xi_2\mathrm{PrRa}_0}{8\pi\delta^4} - \frac{\pi^4\mathrm{Ta}}{8\delta^6\mathrm{Pr}}. \end{aligned}\right\} \tag{48}$$

It is now well known that there is a clear analogy between electro-convection and ferro-convection (see Siddheshwar [20]). To reiterate this in the context of the present problem, we first write down the dimensionless governing equations of rotating electro-convection by following Siddheshwar and Radhakrishna [46] in applying an AC electric field and by incorporating Coriolis force as done in Sect. 3. We get the dimensionless governing equations in the form:

$$\nabla\cdot\mathbf{q} = 0, \tag{49}$$

$$\begin{aligned} \frac{1}{\mathrm{Pr}}\left[\frac{\partial\mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla\mathbf{q})\right] &= -\nabla P + \mathrm{Ra}(1+L)\theta\hat{\mathbf{k}} \\ &- \mathrm{Ra}L\phi_z\hat{\mathbf{k}} + \nabla^2\mathbf{q} + \sqrt{\mathrm{Ta}}\left(\mathbf{q}\times\hat{\mathbf{k}}\right) - \mathrm{Ra}L\theta\nabla\phi_z, \end{aligned} \tag{50}$$

$$\frac{\partial\theta}{\partial t} + (\mathbf{q}\cdot\nabla)\theta = \nabla^2\theta + w, \tag{51}$$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} - \frac{\partial\theta}{\partial z} = 0. \tag{52}$$

In Eqs. (49)–(52), the quantity ϕ is now the electric potential and L is the electric number. It is quite clear that Eqs. (49)–(52) can be obtained from Eqs. (21)–(24) by replacing M_1 and M_3 by L and unity, respectively. Using this one-to-one correspondence between the ferro-convection and the electro-convection problems, we can easily obtain the Rayleigh numbers of the linear stability, weakly nonlinear stability, and energy stability analyses of rotating electro-convection by using Eqs. (32), (31), and (33), respectively.

With the necessary groundwork prepared thus far on the study pertaining to rotating free and

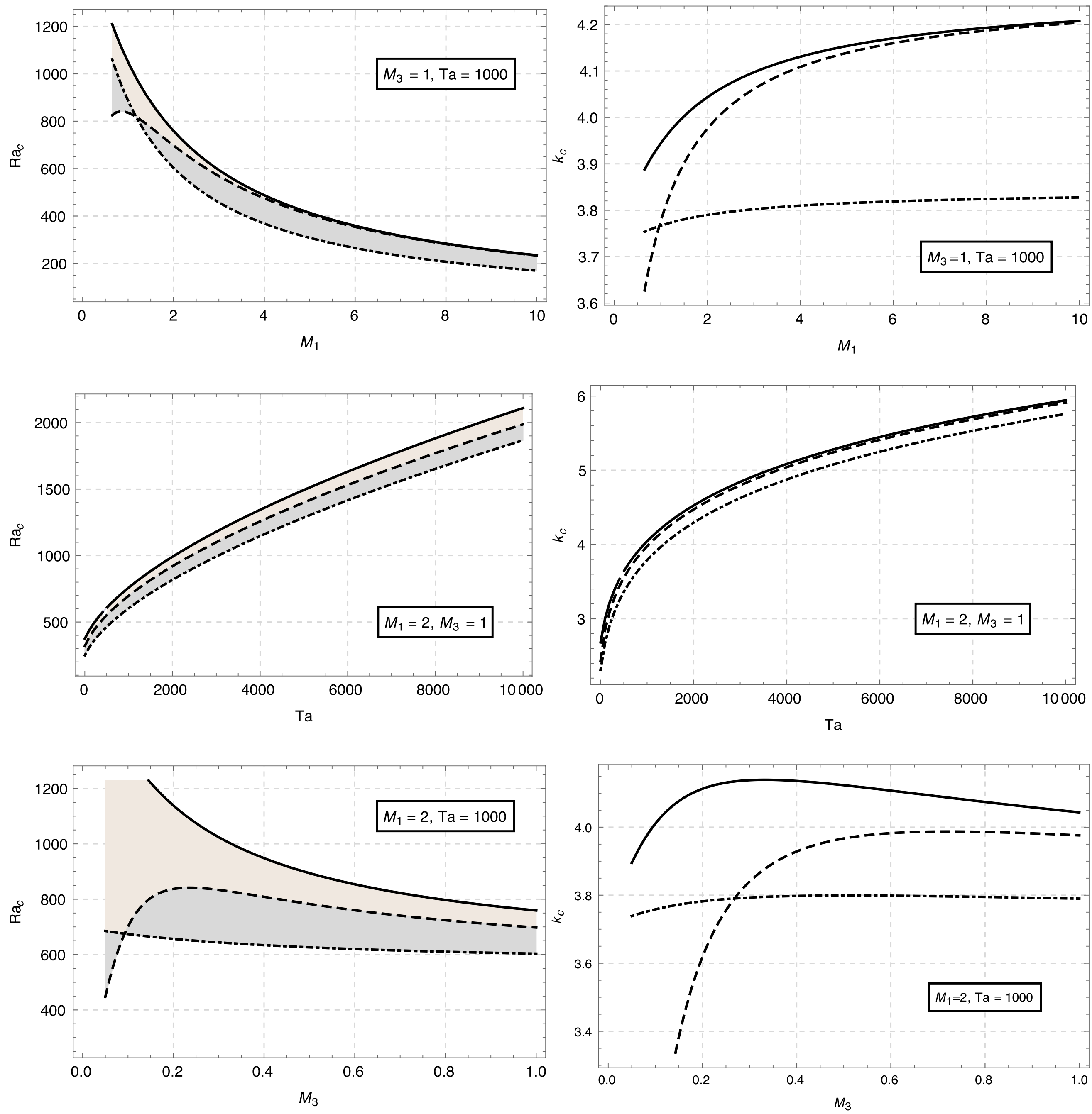

Fig. 2 Comparison of the critical Rayleigh and wave numbers for the rotating ferro-convection problem. The solid lines (——), dashed lines (– –), and the dot-dashed lines (–·–) correspond to the results obtained by linear stability, weakly nonlinear stability, and energy stability, respectively. The shaded region between the solid lines (——) and dashed lines (– –) in the critical Rayleigh number plots represents the subcritical region due to weakly nonlinear stability analysis and the shaded region between the dashed lines (– –) and the dot-dashed lines (–·–) in the critical Rayleigh number plots represents the subcritical region due to energy stability

electro-convection, we move on to the results of the study and the discussion of it after noting that the results on rotating ferro-convection are available in the following cases:

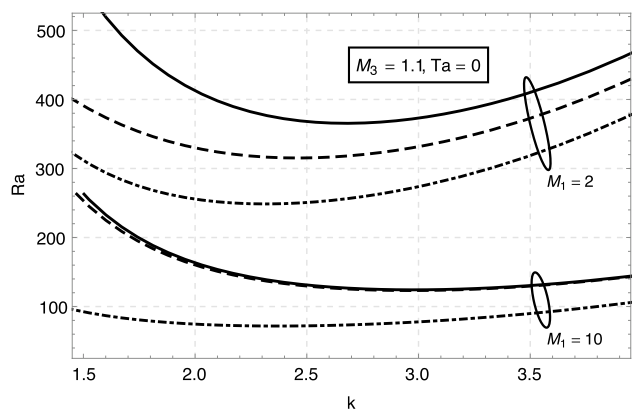

Fig. 3 Comparison of the Rayleigh numbers vs. wave number plots obtained by (——) linear stability [16], (— —) weakly nonlinear (present study), and (– – –) energy stability [43] for Ta = 0

1. linear stability analysis [40] and
2. energy stability analysis [43].

The corresponding works on rotating electro-convection are not available.

7 Results and discussion

In the first part of the paper, we present a weakly nonlinear stability analysis and this yields a critical Rayleigh number which throws light on the possibility of subcritical motions. Comparison of the critical Rayleigh number and the wave number of the linear, weakly nonlinear, and energy stability theories is made for different values of the

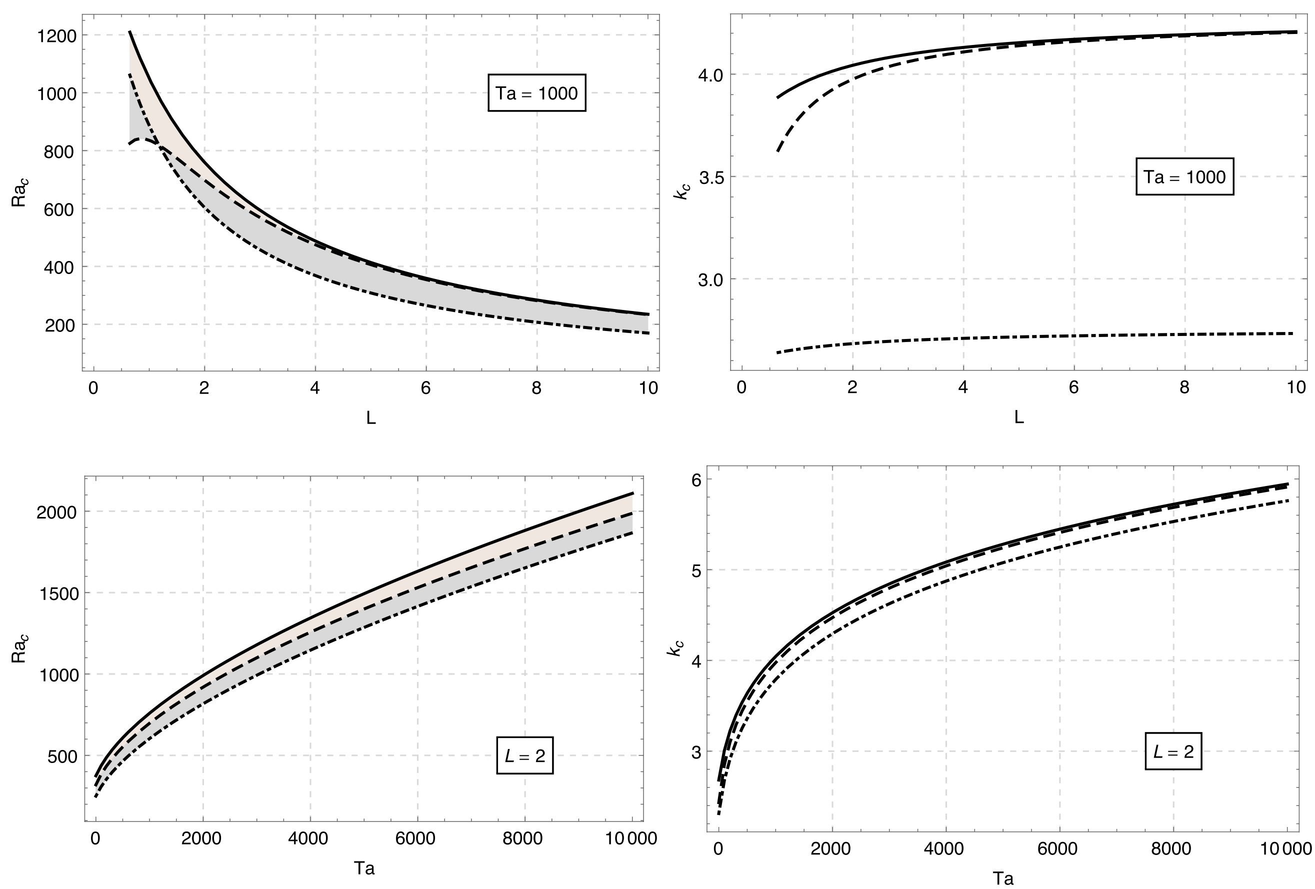

Fig. 4 Comparison of the critical Rayleigh and wave numbers for the rotating electro-convection problem. The solid lines (——), dashed lines (— —), and the dot-dashed lines (– – –) correspond to the results obtained by linear stability, weakly nonlinear stability, and energy stability, respectively. The shaded region between the solid lines (——) and dashed lines (— —) in the critical Rayleigh number plots represents subcritical region due to weakly nonlinear stability analysis and the shaded region between the dashed lines (— —) and the dot-dashed lines (– – –) in the critical Rayleigh number plots represents the subcritical region due to energy stability

parameters of the problem. These results are presented in Fig. 2.

From the figure, it is obvious that for the present problem subcritical motions are possible. Further, it is apparent that the energy stability analysis yields the least critical Rayleigh number. A look at the critical wave number plot in Fig. 2 reveals that for the considered range of parameter values, the critical wave number varies in tandem with its corresponding Rayleigh number.

The comparison of the present results for the case Ta = 0, with the linear stability analysis of Finlayson [16] and with the energy stability analysis of Sunil and Mahajan [43], is depicted in Fig. 3. The possibility of subcritical motions can be observed using the finite-amplitude analysis for smaller values of the buoyancy magnetization parameter M_1, whereas for larger values of M_1 the linear and finite-amplitude analysis predicts approximately the same critical Rayleigh number.

In the second part of the paper, the governing equations for the rotating electro-convection problem are shown to be obtainable from the corresponding problem of rotating ferro-convection. For most values in the considered range of parameters' values, the result of the two problems is qualitatively similar. This result is depicted in Fig. 4.

The analytically intractable fifth-order Lorenz model with two quadratic nonlinearities for the rotating ferro/electro-convection problems is reduced to the analytical tractable Landau model with cubic nonlinearity using the method of multi-scales. The analytical solution of the Landau equation in the case of a representative rotating ferro-convection problem is presented for the following two cases:

1. Linearized Landau equation and
2. The cubic Landau equation.

We observed from Fig. 5 that in the absence of the cubic nonlinearity the solution of the Landau equation blows up in time while in its presence the solution remains bounded. This is a feature of the solution of the original Lorenz model and is preserved by the reduced Landau

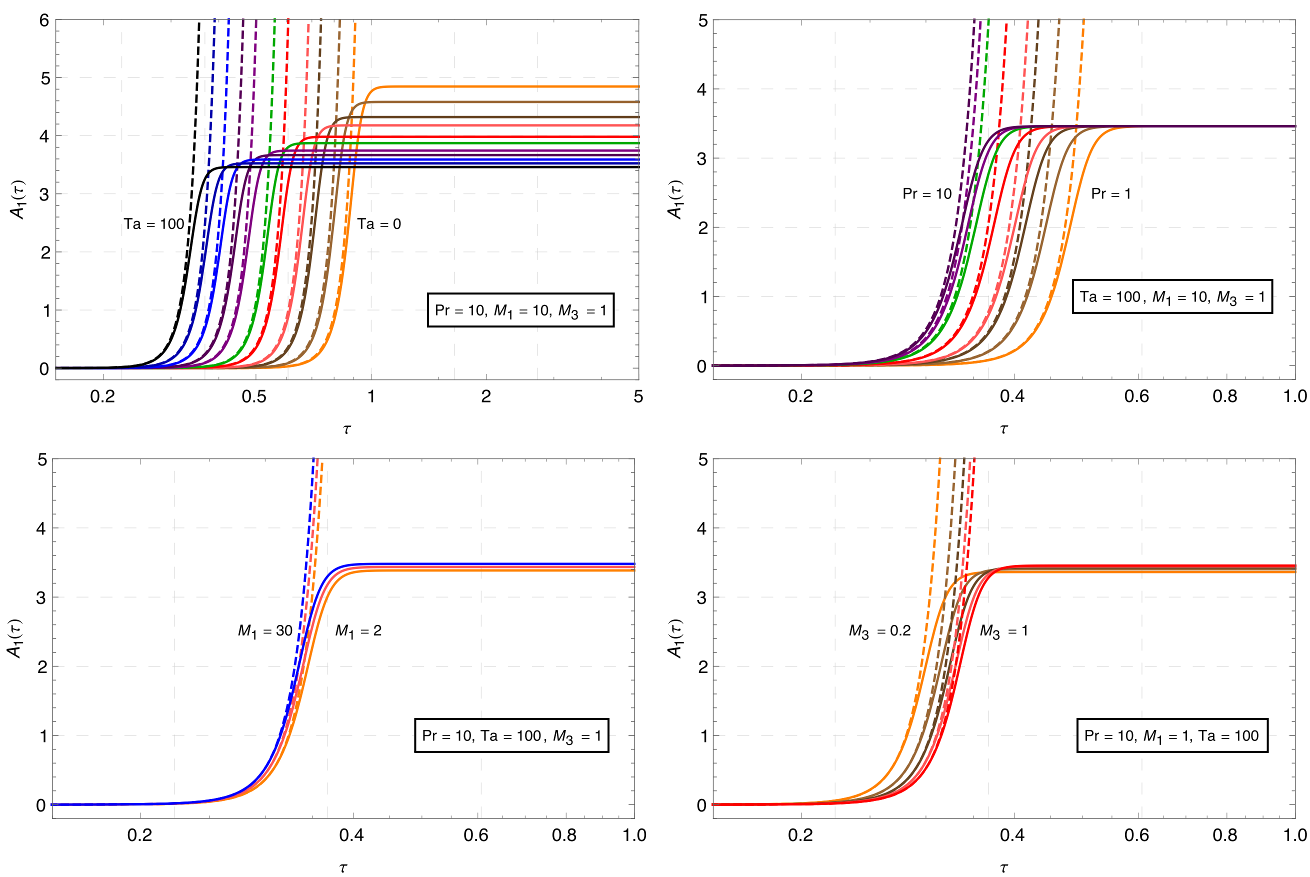

Fig. 5 The time-series plot of the amplitude of convection. The dashed lines (— —) correspond to the exact solutions of the linearized form of Landau equation and solid lines (——) correspond to the exact solutions of Landau equation

equation. Computation reveals that the nature of the solution of the rotating ferro-convection problem is also seen in the rotating electro-convection problem.

8 Conclusion

1 The present study leads to the conclusion that the critical value of the Rayleigh number of infinitesimal amplitude perturbation and those by the energy stability and weakly nonlinear stability analyses are not the same. The latter two instabilities are, in fact, subcritical in nature. These results clearly point to the fact that in so far as rotating Rayleigh–Bénard convection in ferromagnetic liquids or dielectric liquids is concerned, the linear stability analysis is inadequate and so is the weakly nonlinear stability analysis.

2 The present paper is a good example of treating rotating ferro-convection and electro-convection in a unified way. This would mean that there is really no need to study the electro-convection in isolation as the results can be got from the ferro-convection problem by considering the electric number in place of the buoyancy magnetization parameter and by taking a value of unity for the non-buoyancy magnetization parameter.

3 The weakly nonlinear stability analysis leads to the analytically intractable five-dimensional Lorenz model (with two quadratic nonlinearities) and this can be reduced to the one-dimensional Landau model (with a cubic nonlinearity) that is analytically tractable. The latter arises due to a local nonlinear stability analysis. This procedure of reduction gives us a physically realistic bounded solution of the Landau equation and thus we thereby come to know that some results of a weakly nonlinear stability analysis can, in fact, be obtained from a local nonlinear stability analysis without the need to pursue a numerical method.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

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