Research Article

On a calculation of definite integrals by using of the calculation of indefinite integrals

Galina Mehdiyeva[1] · Vagif Ibrahimov[1] · Mehriban Imanova[1]

Received: 19 July 2019 / Accepted: 17 October 2019 / Published online: 25 October 2019

Abstract
It is known that in the construction of the numerical methods for solving of the initial-value problem of ODE in basically used the methods which have applied to the calculation of the definite integrals. Here for the computing of definite integrals propose to use the methods which have used in solving of the initial-value problem for the ODEs. The definite integrals express by the indefinite integrals which are the solutions of the above-mentioned problems. For the construction of more exact methods for calculation of the definite integrals here propose to use forward-jumping (advanced) methods and the hybrid methods. Here establishes some connection between the Gauss and hybrid methods. And also have determined some necessary conditions for which the coefficients of the proposed methods have to satisfy. Constructed stable methods with the degree $p \leq 8$. Shown that, how received here results can be applied to the computing of the double integrals. For this aim, determines some connection between double integrals and single definite integrals. By using this relation have constructed methods which are applied to calculate the double integrals. Advantages of this method illustrated by calculation of model double integral by the constructed here methods.

Keywords Definite integrals · Indefinite integrals · Multistep hybrid methods · Forward-jumping methods · Initial-value problem for ODE

1 Introduction

As is known by using the definite integrals the scientists have investigated many practical problems as the computation of the area bounded by some functions or by the direct lines, the volume of some different figures, the volumes of rotation bodies, distances between objects, energy of signals, earthquakes and others (see for example [1, pp. 169–222, 2, 3]). The investigation of the definite integrals is narrowly connected with the determined of the solution of the initial-value problem for the ODEs. Therefore to construct the methods for the computation of the definite integrals were engaged many known scientists as the Newton, Kottes, Gauss, Chebyshev, Simpson, Adams and etc. (see for example [4, 5]). The scientists are constructed different formulas for the computation of definite integrals having the various exactness.

And also is known that for the receiving more exact results in computing of definite integrals there used making smaller of the step size which reduces to increasing of the volume of calculation works. And is known that the first direct method for solving of the initial-value problem for ODE has been constructed by Euler using the simplest form of quadrature formula. By taking into account that the order of accuracy for Euler method equal to 1 (one) therefore for the construction of more accurate methods than Euler's, the specialists have proposed to use interpolation polynomials with the high exactness in the construction of the methods to compute the definite integrals (see for example [6, 7]). But here for the construction of more exact methods have proposed to use the method of unknown coefficients. And

✉ Galina Mehdiyeva, imn_bsu@mail.ru; Vagif Ibrahimov, ibvag47@mail.ru; Mehriban Imanova, imn_bsu@yahoo.com | [1]Department of Computational Mathematics, Baku State University, Z.Khalilov 23, 1148 Baku, Azerbaijan.

SN Applied Sciences (2019) 1:1489 | https://doi.org/10.1007/s42452-019-1519-8

for investigation of definite integrals using the subinterval method by using that compared the results received here with the known.

Now let us to calculation of definite integrals written as:

$$I = \int_{x_0}^{b} f(x)dx, \tag{1}$$

here sufficiently smooth function *f*(*x*) is defined in the interval $[x_0, b]$. For the construction of the methods to calculation of definite integral (1) let us denote by f_i the values of the function of *f*(*x*) at the node points $x_i = x_0 + ih$ $(i = 0, 1, \ldots, N)$. Here $0 < h$-step size by the help of which the interval $[x_0, b]$ to divided to *N* equal parts.

For the calculation of the integral (1) let us consider the following function:

$$y(x) = \int_{x_0}^{b} f(s)ds, \quad x_0 \le x \le b. \tag{2}$$

From here receive that $I = y(b)$.

It is evident that $y'(x) = f(x), y(x_0) = 0$, which calls as the initial-value problem for the ODE of the first order. This problem in the subinterval $[x_i, x_{i+1}]$ can be written as:

$$y'(x) = f(x), y(x_i) = y_i, x \in [x_i, x_{i+1}]. \tag{3}$$

Let the problem to be presented as the following:

$$y(x_{i+1}) = y(x_i) + \int_{x_i}^{x_{i+1}} f(s)ds, (i = 0, 1, \ldots, N-1). \tag{4}$$

By using this equality scientists have constructed the methods to solve the problem (3) with the different orders of accuracy. Here we proposed to use finite-difference methods and applied them to solve the problems (3) and (4) and have constructed the methods of forward-jumping (advanced) and hybrid types to calculate the definite integral (1).

2 Construction of finite-difference methods

As is known one of the popular methods for solving of the problem (3) is the finite difference or multistep method which can be presented as the following:

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} \tag{5}$$

$(n = 0, 1, \ldots, N - l; l = \max(s, k))$.

This method was investigated by some authors (see for example [8–14]). Note that method (5) in the case $s = k$ is fundamentally investigated by Dahlquist [8] and have shown that if the method of (5) stable in the case $s = k$ and have the degree *p* then $p \le 2[k/2] + 2$ and for all the values of the order of *k*, there are stable methods with the maximal degrees. Here we use the known definition of the conception degree and stability has been proposed by Dahlquist [8].

Definition 1 The method (5) is stable if the roots of the characteristic polynomials $\rho(\lambda)$ of the method (5) lie in the unit circle on the boundary of which there is no multiply root. Here $\rho(\lambda)$ can be presented as:

$$\rho(\lambda) = \alpha_s \lambda^s + \alpha_{s-1} \lambda^{s-1} + \cdots + \alpha_1 \lambda + \alpha_0. \tag{6}$$

Definition 2 The method has the degree of *p*, if the following holds:

$$\sum_{i=0}^{s} \alpha_i y(x + ih) - h \sum_{i=0}^{k} \beta_i y'(x + ih) = O(h^{p+1}), \quad h \to 0.$$

Here *p* is integer.

It follows to remark that all known methods of type (5) are subjected to Dahlquist's law. But in the work [13] have proved the existence of the stable methods of type (5) with the degree $p = k + m + 1$ (for the values $k \ge 3m, m = k - s$). The methods of the same property have constructed by other authors (see for example [14–16]). By taking into account this property of the forward-jumping methods here consider the application of the forward-jumping methods to solving of the problem (3). Therefore let us suppose that $s < k$. Note that if in this case, the function of *f*(*x*) depends on the function of *y*(*x*), it is to say that $f(x) \equiv \varphi(x, y)$ then application of the forward-jumping methods accompanied by some difficulties. By taking into account this receive that the application of the forward-jumping methods to calculation of the definite integrals prefers than the known implicit or explicit methods. But for the calculation of $y(x_N)$ by forward-jumping methods we are need to use some values of function *f*(*x*) in outside of the considering segment.

By using the bounders $p \le k + m + 1$ $(m \le [k/3])$, receive that for the construction of more exact stable methods the value of *k* must be chosen greater. For example, if $k \ge 9$ then one can construct the stable method of type (5) with the degree $p \le k + 4$. As is known for increasing exactness the method (5) the increasing of the exactness of the of initial-values y_j $(j = 0, 1, \ldots, k - 1)$ is necessary. Therefore the scientists have discussed to use the other schema for the construction of the stable methods with the high degrees. For this aim, proposed to use the Gauss, Chebyshev, Labbotto, and others interpolation polynomials. In the results of which appeared the new methods. These methods remained of the hybrid methods and in simple form can be written as following (see for example [17–20]):

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i+v_i}. \tag{7}$$

here $v_i\,(i = 0, 1, \ldots, k)$ are the hybrid points which can be defined as the solution of the some nonlinear system of algebraic equations.

To construct more accurate methods here have proposed to use the following method, which have been investigated and applied to the solving some similarly problems (see for example [21–24]):

$$\sum_{i=0}^{k-m} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h \sum_{i=0}^{k} \gamma_i f_{n+i+v_i} \tag{8}$$

$(|v_i| < 1, i = 0, 1, \ldots, k).$

This method can be received by using some of interpolation polynomials in the computing of definite integrals. But here we want to construct the methods of type (8) by help of the method of unknown coefficients. For this aim, let us use the equality $y'(x) = f(x)$ and to consider following Taylor expansions:

$$y(x + ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!} y''(x) + \cdots + \frac{(ih)^p}{p!} y^{(p)}(x) + O(h^{p+1}),$$

$$y'(x + l_i h) = y'(x) + l_i hy''(x) + \frac{(l_i h)^2}{2!} y'''(x) + \cdots + \frac{(l_i h)^{p-1}}{(p-1)!} y^{(p)}(x) + O(h^p).$$

By using these equalities in the asymptotic equality (4), receive that to hold the asymptotic equality (4) the satisfying of the following conditions is necessary and sufficient:

$$\sum_{i=0}^{k-m} \alpha_i = 0;$$
$$\sum_{i=0}^{k} \beta_i + \sum_{i=0}^{k} \gamma_i = \sum_{i=0}^{k-m} i\alpha_i; \tag{9}$$
$$\sum_{i=0}^{k} \left(\frac{i^l}{l!}\beta_i + \frac{(i+v_i)^l}{l!}\gamma_i\right) = \sum_{i=0}^{k-m} \frac{i^{l+1}}{(l+1)!}\alpha_i$$
$$l = 1, 2, \ldots, p-1.$$

By using the solution of the nonlinear system (9) one can construct the stable methods of type (8). Note that in the system of (9) there are $p + 1$ equations and $4k + 3 - m$ unknowns. If the system (9) has the solution in the case $p + 1 = 4k + 3 - m$, then receive that there exist the methods of type (8) with the degree $p = 4k + 2 - m$. In usually, the methods with the degree $p = 4k + 2 - m$ are unusable. Here prove that there are exist the stable methods for $k > 2$ with the degree $p \le 3k + 2 + m$. But it not follows from here that $P_{max} = 3k + 2 + m\,(k \ge 3m)$. The system (8) is nonlinear therefore to find some conditions for the existence of the unique solution is not easy.

Consequently, the existence of stable methods of type (9) with maximal degrees is not known for us. These properties of the nonlinear system of algebraic equations have been met in investigation of the Gauss and Chebyshev methods. Therefore, to find of the Gauss node points the authors have used the known polynomials (see for example [4, pp. 189–199, 25, pp. 463–469]). But here we for the finding of the solution of the system (9) have used the Mathcad program and in some cases receive the existence of the solution of the system (9) in the case $p > 3k + 2 + m$. Let us note that these solutions are approximate, therefore to assert about the existence of the stable methods with the degree $p > 3k + 2 + m$ is not correct. But to say that it is not stable method with the degree $p > 3k + 2 + m$ also is not correct. If to put $v_i = 0\,(i = 0, 1, \ldots, k)$ in the equality of (8) then receive the fundamentally investigated method which coincides with the method of (5). In this case, for the finding the values of the coefficients of this method one can be used the following way (see for example [26, 27]):

$$\alpha_0 = -\sigma_0^{(1)} + \sigma_1^{(1)} - \sigma_2^{(1)} + \cdots + (-1)^{k-1}\sigma_{k-2}^{(1)} + (-1)^k \sigma_{k-1}^{(1)}$$
$$\alpha_i = \sum_{j=i-1}^{k-1} (-1)^{j-i+1}(j+1)j(j-1)\ldots(j-i+2)\sigma_j^{(1)}/i!$$
$$(i = 1, 2, \ldots, k), \tag{10}$$

$$\beta_0 = \sigma_0^{(2)} - \sigma_1^{(2)} + \sigma_2^{(2)} + \cdots + (-1)^{k-1}\sigma_{k-1}^{(2)} + (-1)^k \sigma_k^{(2)},$$
$$\beta_i = \sum_{j=i-1}^{k-1} (-1)^{j-i} j(j-1)\ldots(j-i+1)\sigma_j^{(2)}/i!$$
$$(i = 1, 2, \ldots, k).$$

For the determined of the values of the constant $\sigma_j^{(1)}$ and $\sigma_j^{(2)}$ $(i = 0, 1, \ldots, k)$ one can be used the following system:

$$\sum_{i=0}^{j} C_i \sigma_{j-i}^{(1)} = \sigma_j^{(2)} (j = 0, 1, \ldots, k; \sigma_k^{(1)} = 0),$$

here

$$C_m = \sum_{\nu=1}^{m} (-1)^{\nu-1} C_{m-i}/(i+1); (C_0 = 1; m = 1, 2, 3, \ldots)$$
$$\sum_{i=j-k+1}^{j} C_i \sigma_{j-i}^{(1)} = 0 (j = k+1, k+2, \ldots, p-1). \quad (11)$$

As it follows from here the determination of the solution of the systems (10) and (11) is simpler than the determination of the solution of the system (9). Because in solving (11) on the first step we find the values of constants $\sigma_0^{(1)}, \sigma_1^{(1)}, \sigma_{k-1}^{(1)}$ and after them on the second step of the values of the constant $\sigma_j^{(2)}$ $(i = 0, 1, \ldots, k)$. But in solving of the system we must find the values of $2k$-unknowns as the solution of the system (9). Let us remark that this idea is more effective in application of it to the construction of the multistep second derivative methods.

3 Application of the multistep second derivative methods to calculation of the definite integral

The multistep second derivative methods in more general form can be written as:

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i y''_{n+i}, \quad (12)$$

here k, s, l are integer variable and independent on each other. By choosing these variables from the method (12) can be received the explicit, implicit and forward-jumping methods. Therefore method (12) called as more general form.

Let us consider application of the method (12) to calculation of the integral (2). In this case, receive the following:

$$\sum_{i=0}^{s} \alpha_i y_{n+i} = h \sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{l} \gamma_i f'_{n+i}. \quad (13)$$

By taking into account that the function of $f(x)$ is known then receive that by using the method (13) one can calculate the value of definite integral (1). The method (13) has been fundamentally investigated by some authors (see for example [28–31]). If the method of (13) is stable and has the degree of p then there exist the methods with the degree $p \le 2k + 2$ (in the case $s = k = l$) and there exists the methods with the degree $p > 2k + 2$ in other cases.

It follows to note that the conception of stability for the method (13) depends on the values of the coefficients β_i $(i = 0, 1, \ldots, k)$. If the conditions $|\beta_0| + |\beta_1| + \cdots + |\beta_s| \ne 0$ are hold then the conception of stability for the method (13) is define as the stability of the method (5). But in the case $|\beta_0| + |\beta_1| + \cdots + |\beta_s| = 0$, the method (13) is said to be stable, if the roots of the polynomial $\rho(\lambda)$ lie in the unit circle on the boundary of which there are no multiple roots without the double root $\lambda = 1$. It is evident that in this case the degree of the method (13) will be same with the degree of the method (5). For the simplicity, let us in the equality of (13) put $s = k = l$. As was noted above in this case there are stable methods with the degree $p = 2k + 2$ if $|\beta_0| + |\beta_1| + \cdots + |\beta_s| \ne 0$ holds. It is not difficult to understand that by using of the method (13) one can determine the value of y_{n+k}, if the values y_j $(j = 0, 1, \ldots, k-1)$ are known. For solving of this, here proposed to use the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{s} h^j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)}. \quad (14)$$

From this formula one can be received the Taylor expansion for the value $k = 1$. In this case, receive the one-step method which can be always applied to calculation of the definite integral (1), so that $y_0 = 0$. Thus by using the described way we can construct the predictor-corrector methods with the different degrees to compute the values y_j $(j = 0, 1, \ldots, k-1)$ and the values y_{n+k} $(n = 0, 1, \ldots, N-k; x_N = x_0 + Nh)$.

For the construction of stable methods of type (13) (in the case $s = k = l$), let us consider for determined the values of the coefficients of the method (13).

As was noted above to determine the values of the coefficients of the method (13) one can be used the method of unknown coefficients. And in this case, receive the system of algebraic equations which is similar to the system (9). But here we want to present the way for the finding of the values of the coefficients for the method (13) in using of which the volume of the computational work less than the computational work receiving in solving of the system which is similar to the system (9).

To find of the values of the coefficients $\alpha_i, \beta_i, \gamma_i$ $(i = 0, 1, \ldots, k)$ here proposed to use the following sequence linear systems of the algebraic equations and the system of (10):

$$\gamma_0 = \sigma_0^{(3)} - \sigma_1^{(3)} + \sigma_2^{(3)} + \dots + (-1)^{k-1}\sigma_{k-1}^{(3)} + (-1)^k\sigma_k^{(3)},$$
$$\gamma_i = \sum_{j=i-1}^{k-1} (-1)^{j-i} j(j-1)\dots(j-i+1)\sigma_j^{(3)}/i!, i = 1,2,\dots,k. \tag{15}$$

But for the finding of the values of coefficients $\sigma_m^{(l)}$ $(l = 1,2,3; m =, 1,\dots,k)$ here present to use the following system of algebraic equations:

$$\sum_{i=0}^{j} C_i\sigma_{j-i}^{(1)} + \sum_{i=0}^{j} \frac{(-1)^{j-i+1}}{j-i+1}\sigma_{j-i}^{(3)} = \sigma_j^{(2)}, \quad j = 0,1,\dots,k, \quad \sigma_k^{(1)} = 0;$$
$$\sum_{i=j+1}^{j+k} C_i\sigma_{j+k-i}^{(1)} + \sum_{i=j}^{j+k} \frac{(-1)^i}{i}\sigma_{j+k-i}^{(3)} = 0, \quad j = 1,\dots,k; \tag{16}$$

$$\sum_{i=j+k+1}^{j+2k} C_i\sigma_{j+2k-i}^{(1)} + \sum_{i=j+k}^{j+2k} \frac{(-1)^i}{i}\sigma_{j+2k-i}^{(3)} = 0, \quad j = 1,\dots,k;$$
$$\sum_{i=2k+1}^{3k+1} C_i\sigma_{3k+1-i}^{(1)} + \sum_{i=2k+1}^{3k+1} \frac{(-1)^i}{i}\sigma_{3k+1-i}^{(3)} = C.$$

Here, the constant C is the coefficient of the principal part in the expansion of the remainder term of the constructed methods. The equivalents of the system (9) and (16) one can be found in the work [25].

Let us note that in the case $k = 2$ linear part of the method (13) can be proposed as: $y_{n+2} - y_n$. In this case, by replace h to $h/2$ from the mentioned linear part receive: $y_{n+1} - y_n$. Thus by this way have constructed the one-step method which is similar to hybrid methods. This scheme can be generalized by the following way:

$$y_{n+k} - y_n = h\sum_{i=0}^{k} \beta_i f_{n+i} + h^2 \sum_{i=0}^{l} \gamma_i f'_{n+i}. \tag{17}$$

For the receiving of the one-step method from the method (17) it is enough to choose step size h in the form h/k. It follows to note that the method (17) can be unstable for the values $k > 2$. But the receiving of the one-step form that will be stable.

4 The construction of some simple methods and their application to the calculation of definite integrals

In first let us consider constructing of stable methods of type (5). As is known for the case $s = k$ one can construct stable methods with the degree $p \le 8$ having the explicit and implicit type for the $k \le 6$. Therefore let us consider the case when $s < k$ and put $k = 2$. In this case, the integer m can receive the value $m = 1$ and the maximal value of the degree for this method will be equal to $p = 3$, which can be written as:

$$y_{n+1} = y_n + h(5f_n + 8f_{n+1} - f_{n+2})/12. \tag{18}$$

And now let us put $k = 3$ and $m = 2$ then receive:

$$y_{n+1} = y_n + h(9f_n + 19f_{n+1} - 5f_{n+2} + f_{n+3})/24. \tag{19}$$

But for the case $k = 4$ and $m = 3$ the stable method can be written as:

$$y_{n+1} = y_n + h(251f_n + 646f_{n+1} - 264f_{n+2} + 106f_{n+3} - 19f_{n+4})/720. \tag{20}$$

The method (19) has the degree $p = 4$ but the method (20) has the degree $p = 5$. In [11] has proved that for the coefficients $|\beta_{k-m+i}| > |\beta_{k-m+i+1}|$ and $\beta_{k-m} > 0$ (for $\alpha_{k-m} > 0$), $\beta_{k-m+i}\beta_{k-m+i+1} < 0$ are hold if the coefficients satisfies the condition: $\beta_{k-m+i} \ne 0$, $\beta_{k-m+i+1} \ne 0$. The above described methods satisfy these conditions.

The application of the methods (18) and (20) to the calculation of the integral (1) is very simple because y_0 is known. Now let us consider the case $k = 3$ and $m = 1$. In this case, the method with the maximal degree can be written as:

$$y_{n+2} = (8y_{n+1} + 11y_n)/19 + h(10f_n + 57f_{n+1} + 24f_{n+2} - f_{n+3})/57. \tag{21}$$

This method stable and has the degree $p = 5$. If in this method the step size h replace by the $h/2$, then receive:

$$y_{n+2} = (8y_{n+1} + 11y_n)/19 + h(10f_n + 57f_{n+1/2} + 24f_{n+1} - f_{n+3/2})/114. \tag{22}$$

The using the method (22) is more difficult than the method of (20) so that in the method of (22) is participate the term of $y_{n1/2}$. To find this value, one can use the method (19) or the method of (20). But if consider the case $m = 0$ and $k = 2$ then receive the following stable method with the degree $p = 4$:

$$y_{n+2} = y_n + h(f_{n+2} + 4f_{n+1} + f_n)/3,$$

and in this method after replacing of h by the $h/2$ receive:

$$y_{n+1} = y_n + h(f_{n+1} + 4f_{n+1/2} + f_n)/6, \tag{23}$$

which can be applied to the computing of the integral (1) in separate form. Now let us consider construction of the stable method of type (8) and put $k = 1$. In this case, from the method (8) one can receive the stable methods having the different order of accuracy. For example the method with the degree $p = 6$ can be written as:

$$y_{n+1} = y_n + h(f_n + f_{n+1})/12 + 5h(f_{n+\alpha} + f_{n+1-\alpha})/12,$$
$$\alpha = 1/2 - \sqrt{5}/10. \tag{24}$$

This method is implicit. In this case, the explicit method with the degree $p = 5$ can be written as:

$$y_{n+1} = y_n + hf_n/9 + h((16+\sqrt{6})f_{n+3/5-\alpha} + (16-\sqrt{6})f_{n+3/5+\alpha})/36,\quad \alpha = \sqrt{6}/10. \tag{25}$$

If suppose that $\beta_i = 0\,(i = 0, 1, \ldots, k)$, then can be prove that there are the stable methods with the degree $p = 2k + 2$. In the case $k = 1$, the method with the degree $p = 4$ has the following form:

$$y_{n+1} = y_n + h(f_{n+l} + f_{n+1-l})/2,\quad l = 1/2 - \sqrt{3}/6. \tag{26}$$

And now let us consider construction of the multistep second derivative methods. As was noted above if these methods are stable then the degree of these methods satisfies the condition $p \le 2k + 2$. In first let us put $s = k = l$ and $k = 1$. Then from the method of (18) one can receive the following methods (see for example [27–30]):

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 + h^2(-f'_{n+1} + f'_n)/12,\quad R_n = h^5 f^{(IV)}_{(\xi)}/720, \tag{27}$$

$$y_{n+1} = y_n + h(f_{n+1} + 2f_n)/2 + h^2 f'_n/6,\quad R_n = -h^4 f^{(3)}_{(\xi)}/72, \tag{28}$$

$$y_{n+1} = y_n + hf_n + h^2 f'_n/2,\quad R_n = h^3 f''_{(\xi)}/6. \tag{29}$$

Note that the method of (27) is implicit, the method (29) is explicit, but the method (28) depends on its application can be explicit or implicit. In our case, the method (28) is explicit because function $f(x)$ independed on the function of $y(x)$.

And now let us consider the case $k = 2$. In this case, from the (14) receive (see for example [11, p. 288]):

$$y_{n+2} = y_{n+1} + h(-f_{n+1} + 3f_n)/2 + h^2(17f'_{n+1} + 7f'_n)/12, \tag{30}$$

$$y_{n+2} = y_{n+1} + h(101f_{n+2} + 128f_{n+1} + 11f_n)/240 + h^2(-13f'_{n+2} + 40f'_{n+1} + 3f'_n)/240. \tag{31}$$

These methods are stable and have the degree $p = 4$ and $p = 6$ respectively. Now let us consider construction of the stable forward-jumping second derivative methods. It is clear that if $m = 1$ then $k \ge 2$. Therefore consider the case $m = 1$ and $k = 2$. In this case the stable method with the degree $p = 6$ can be written as (see for example [11, p. 288]):

$$y_{n+1} = y_n + h(11f_{n+2} + 128f_{n+1} + 101f_n)/240 + h^2(-3f'_{n+2} - 40f'_{n+1} + 3f'_n)/240. \tag{32}$$

For the construction of more exact methods let us put $k = 3$ and $m = 1$. In this case the stable method with the degree $p = 9$ can be written as following:

$$y_{n+2} = (416y_{n+1} - 103y_n)/313 + h(157f_{n+3} + 1123f_{n+2} + 8451f_{n+1} - 2830f_n)/25353 + h^2(-11f'_{n+3} - 630f'_{n+2} + 1557f'_{n+1} - 92f'_n)/8451. \tag{33}$$

If in the case $k = 3$, put $m = 2$, then receive the following one-step method with the degree $p = 9$ (see for example [11, p. 289]):

$$y_{n+1} = y_n + h(1985f_{n+3} + 12015f_{n+2} + 42255f_{n+1} + 34465f_n)/90720 + h^2(-163f'_{n+3} - 2421f'_{n+2} - 7659f'_{n+1} + 1283f'_n)/30240. \tag{34}$$

The stable methods (32)–(34) have maximal degrees. As was noted above by using the high derivatives of the solution of the considering problems in the construction of the numerical methods, the values of the degrees for these methods can make greater. For example the following methods:

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 - h^2(f'_{n+1} - f'_n)/10 + h^3(f''_{n+1} - f''_n)/120, \tag{35}$$

$$y_{n+1} = y_n + h(f_{n+1} + f_n)/2 - 3h^2(f'_{n+1} - f'_n)/28 + h^3(f''_{n+1} - f''_n)/84 - h^4(f'''_{n+1} - f'''_n)/1680. \tag{36}$$

The methods (35) and (36) are stable and have the degrees $p = 6$ and $p = 8$, respectively. Note that the above described methods don't allow some difficulties in application of them to calculate the definite integrals. For the illustration of the received here results let us apply the methods (18)–(20) and (24)–(26) to calculate the following definite integral:

$$I = \lambda \int_0^1 exp(\lambda s)ds. \tag{37}$$

This problem has been reduced to solve the problem: $y'(x) = exp(\lambda x)\; y(0) = 0$ the exact solution for which can be written as: $y(x) = exp(\lambda x) - 1$. This problem has been solved by using of forward-jumping methods (17)–(20), hybrid methods (24)–(26) and multistep multiderivative methods (27), (35) and (36). For the comparison of the received results have been solved the considering problem for the different values λ and $h = 0, 1$.

The results of the calculation of given definite integral have been tabulated in Tables 1 and 2.

Table 1 Results for solving example in the case $\lambda = \pm 1$

	$\lambda = 1$	$\lambda = -1$
Method 27	2.39E−7	8.78E−8
Method 35	1.70E−11	6.27E−12
Method 36	2.22E−16	3.33E−16
Method 18	7.43E−5	2.54E−5
Method 19	4.91E−6	1.54E−6
Method 20	3.65E−7	1.05E−7
Method 26	3.98E−8	1.46E−8
Method 25	2.38E−10	8.79E−11
Method 24	1.14E−12	4.18E−13

Table 2 Results for solving example in the case $\lambda = \pm 5$

	$\lambda = 5$	$\lambda = -5$
Method 27	0.36624	8.57E−5
Method 35	2.27E−5	1.53E−7
Method 36	2.25E−8	1.52E−10
Method 18	0.926199	4.32E−3
Method 19	0.366244	1.12E−5
Method 20	0.16476	3.21E−4
Method 26	2.11E−3	1.43E−5
Method 25	6.30E−5	4.31E−7
Method 24	1.51E−6	1.02E−8

For construction of the stable methods with high degrees here proposed the three ways. And have given some of the concrete methods of forward-jumping, hybrid and multistep second derivative types which have been compared by application of them to solve the example (37).

Remark. As in known in solving of many natural science problems arises the necessity for the computation of double integrals which in simple form can be written as:

$$ID = \int_a^b \int_c^d f(x,y)dxdy. \tag{38}$$

Let us transfer above described way to calculation of the integral (38). For this aim let us consider the following function:

$$u(x,y) = \int_a^x \int_c^y f(s,t)dsdt.$$

Calculation of this integral can be reduced to solving of the following initial-value problem:

$$F'(x_i,y) = f(x_i,y), \quad F(x_i,c) = 0, \quad (i = 0,1,\ldots,n),$$
$$u'(x,d) = F(x,d), \quad u(a,d) = 0,$$

here

$$F(x,y) = \int_c^y f(x,t)dt.$$

5 Conclusion

As was noted above here have described some ways to calculate the definite integral and their application to the computing of the double integrals. Given some comparison of the methods which have been recommended for calculation of the definite integrals. Have shown that the behavior of the errors for using methods depends on its type. For this aim, have used the results received for the step-size $h = 0,1$ which have tabulated in Tables 1 and 2. And also have shown that the errors basically depend on the properties of the solution of considering problems. As is seen from the Table 2 the results received by forward-jumping methods for the $h = 0,1$ and $m = 5$ are not satisfying results and cannot be considered as normal. But results received for the step-size $h = 0,1$ and $m = 1$ can be considered as normal. We hope that the above-described method will find its wide application in solving scientific and engineering problems. To show the advantages of described here method has considered the application of the above-mentioned method to calculate double definite integrals. And to calculate the values of concrete double integrals were applied to the proposed here methods. Taking into account the received here results have obtained that both methods to allow as theoretical and practical interest. This method is a new direction in the computation of definite integrals and we believe that it will find its followers.

Acknowledgements The authors express their thanks to the academicians Telman Aliyev for their suggestion to investigate the computational aspects of our problem. This work was supported by the Science Development Foundation under the President of the Republic of Azerbaijan—Grant No. EIF-KETPL-2-20151(25)-56/07/1.

Compliance with ethical standards

Conflict of interest The author(s) declare that they have no competing interests.

References

1. Fichtengolts GM (1970) Course of differential and integral calculus. Nauka, Moscow
2. Michael P (2001) Fitz analog communication theory. The Ohio State University
3. Aliev TA, Rzaeva NE (2017) Techniques for determining robust estimates of correlation functions for random noisy signals. Meas Tech 60(4):343–349
4. Berezin IS, Jidkov NP (1966) Computational methods. Nauka, Moscow
5. Quarteroni A, Saccar R, Saleri F (2007) Numerical mathematics. Springer, Berlin
6. Euler L (1957) Integral calculus, vol II. Gostehus-Date, Moscow
7. Krylov AN (1950) Lectures on approximate calculations. Gocteh-izdat, Moscow
8. Dahlquist G (1956) Convergence and stability in the numerical integration of ordinary differential equations. Math Scand 4:33–53
9. Henrici P (1962) Discrete variable methods in ordinary differential equation. Wiley, New York
10. Lambert RJ (1973) Numerical methods for ordinary differential systems: the initial value problem. Wiley, Chichester
11. Mehdiyeva G, Ibrahimov V (2013) On the research of multistep methods with constant coefficients. LAP LAMBERT Academic Publishing, Saarbrücken
12. Brunner H (1970) Marginal stability and stabilization in the numerical integration of ordinary differential equations. Math Comput 24(111):635–646
13. Ibrahimov VR (1982) One non-linear method for the numerical solution of the Cauchy problem for ordinary differential equations. Bulgarian, pp 310–319
14. Ibrahimov VR (1990) On some properties of Richardson extrapolation. Differ Equ 26(12):2170–2173
15. Williams J, de Hoog F (1974) A class of A-stable advanced multistep methods. Math Comput 28(125):163–177
16. Ibrahimov VR (1990) On a relation between order and degree for stable forward jumping formula. Zh Vychis Mat 7:1045–1056
17. Butcher JC (2008) Numerical methods for ordinary differential equation. The University of Auckland, Auckland
18. Gear CW (1965) Hybrid methods for initial value problems in ordinary differential equations. SIAM J Numer Anal 2:69–86
19. Gupta GK (1979) A polynomial representation of hybrid methods for solving ordinary differential equations. Math Comput 33(148):1251–1256
20. Areo EA, Ademiluyi RA, Babatola PO (2008) Accurate collocation multistep method for integration of first order ordinary differential equations. J Mod Math Stat 2:1–6
21. Mehdiyeva G, Ibrahimov V, Imanova M (2011) On a way for constructing numerical methods on the joint of multistep and hybrid methods. World Academy of Science, Engineering and Technology, Paris, pp 240–243
22. Mehdiyeva GYu, Imanova MN, Ibrahimov VR (2013) On one application of forward jumping methods. Appl Numer Math 72:234–245
23. Mehdiyeva GYu, Imanova MN, Ibrahimov VR (2016) Some refinement of the notion of symmetry for the Volterra integral equations and the construction of symmetrical methods to solve them. J Comput Appl Math 306:1–6
24. Makroglou A (1982) Hybrid methods in the numerical solution of Volterra integro-differential equations. J Numer Anal 2:21–35
25. Krylov VI, Bobkov VV, Monastyrniy PI (1982) Origin of theory or calculation methods, differential equations. Nauka i technika, Minsk
26. Urabe M (1970) An implicit one-step method of high-order accuracy for the numerical integration of ordinary differential equations. Numer Math 2:151–164
27. Mirzoyev RR, Mehdiyeva GY, Ibrahimov VR (2010) On an application of a multistep method for solving Volterra integral equations of the second kind. In: Proceeding the 2010 international conference on theoretical and mathematical foundations of computer science, Orlando, USA, pp 46–50
28. Dahlquist G (1959) Stability and error bounds in the numerical integration of ordinary differential equation. Transactions of the Royal Institute of Technology, Stockholm, Sweden, Nr. 130
29. Iserles A, Norsett SP (1987) Two-step methods and Bi-orthogonality. Math Comput 180:543–552
30. Kobza J (1975) Second derivative methods of Adams type. Appl Math 20:389–405
31. Hairer E, Norsett SP, Wanner G (1990) Solving ordinary differential equations. Mir, Moscow