Research Article

On connectivity of basis graph of splitting matroids

M. Pourbaba[1] · H. Azanchiler[1]

Abstract

For a set $\mathcal{C}$ of circuits of a matroid M, $G(B(M), \mathcal{C})$ is defined by the graph with one vertex for each basis of M, in which two basis B_1 and B_2 are adjacent if $B_1 \cup B_2$ contains exactly one circuit and this circuit lies in $\mathcal{C}$. For two elements of a and b of ground set of a binary matroid M a splitting matroid $M_{a,b}$ is constructed. It is specified by two collections of circuits $\mathcal{C}_0$ and $\mathcal{C}_1$ dependent with collections of circuits of M. We want to study connectivity of $G(B(M_{a,b}), \mathcal{C}_0)$ and $G(B(M_{a,b}), \mathcal{C}_1)$.

Keywords Basis graph · Splitting matroid · Connected matroid

1 Introduction

We assume that reader is familiar with the basic concepts of matroid theory and graph theory. For more details one can see [6] for matroid theory and [3] for graph theory. By considering bases of a matroid M, the basis graph of a matroid M, $G(B(M))$, is a graph in which each vertex is labeled as a basis of M and two bases (vertices) B_1 and B_2 are adjacent if $|B_1 \triangle B_2| = 2$. In other words, two bases are adjacent if they differ in only one element. In [4] Holzmann and Harary proved that $G(B(M))$ is hamiltonian and therefore connected.

Many different variations of the basis graph have been studied, for instance, Li et al. [5] defined the basis graph of a matroid M that related to a set $\mathcal{C}$ of circuits. It is a spanning subgraph of $G(B(M))$ such that two bases B_1 and B_2 are adjacent if they are adjacent in $G(B(M))$ and the unique circuit of M contained in $B_1 \cup B_2$ is a circuit of $\mathcal{C}$. In an special case when M is a binary matroid, they found a sufficient condition and a necessary condition for this graph to be connected and they proved that given an element e of a binary matroid M and by supposing $\mathcal{C}_e$ is the family of all the circuits that contains e, $G(B(M), \mathcal{C}_e)$ is connected. Figueroa et al. [1] generalized this result and they proved that it is true for every matroid of M with given $\mathcal{C}_e$.

The splitting operation is defined for both graph and binary matroid. Fleischner [2] defined splitting operation for graph by the following way; let G be a connected graph and suppose that v is a vertex with degree at least 3. Let $a = vv_1$ and $b = vv_2$ be two edges incident at v, then splitting away of a, b from v results in a new graph $G_{a,b}$ obtained from G by deleting the edges a and b, and adding a new vertex v' adjacent to v_1 and v_2. The transition from G to $G_{a,b}$ is called the *splitting operation* on G. We also denote the new edges $v'v_1$ and $v'v_2$ in $G_{a,b}$ by a and b, respectively.

The notion of the splitting operation extends to binary matroid in the following way [7]. Let $M = (E, \mathcal{C})$ be a binary matroid and a, b be two elements of E. Let

$$\begin{aligned}
\mathcal{C}_0 = &\{C \in \mathcal{C} : a, b \in C \text{ or } a, b \notin C\} \\
\mathcal{C}_1 = &\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C},\ C_1 \cap C_2 = \emptyset, \\
&a \in C_1,\ b \in C_2 \text{ and } C_1 \cup C_2 \\
&\text{contains no member of } \mathcal{C}_0\}.
\end{aligned}$$

Let $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1$, then $M_{a,b} = (E, \mathcal{C}')$ is a binary matroid. As the collection of cycles of splitting graph is the same with the collection of circuits of splitting binary matroid defined above, we used the same notation. In [7], the authors showed that $M_{a,b}$ obtains from M by adding an extra row to

✉ M. Pourbaba, m.pourbaba@urmia.ac.ir; H. Azanchiler, h.azanchiler@urmia.ac.ir | [1]Department of Mathematics, Faculty of Science, Urmia University, Urmia, Iran.

SN Applied Sciences (2019) 1:444 | https://doi.org/10.1007/s42452-019-0470-z

Received: 11 January 2019 / Accepted: 8 April 2019 / Published online: 11 April 2019

binary matrix representation of M, in which arrays respect to a, b are 1 and remind arrays are 0.

Shikare and Azadi [8] characterized the collection of bases of a splitting binary matroid as the following theorem.

Theorem 1.1 *Let $M=(E,\mathcal{C})$ be a binary matroid and $a, b \in E$. Let B be the set of all bases of M. Let $\mathcal{B}_{a,b} = \{B \cup \{\alpha\} : B \in \mathcal{B}, \alpha \in E - B$ and the unique circuit contained in $B \cup \{\alpha\}$ contains either a or $b\}$. Then $\mathcal{B}_{a,b}$ is a set of bases of $M_{a,b}$.*

2 Main results

We shall want to consider two cases of $G(B(M_{a,b}), \mathcal{C})$ respect the two collections of circuits of $M_{a,b}$. First, we start with a helpful lemma.

Lemma 2.1 *Let M be a binary matroid and $T = \{a, b, c\}$ be a triangle of M. Let B_1 be a basis of M contains two elements of T. Let $B_2 \subseteq E(M)$ such that $|B_2| = |B_1|$ and $B_2 - B_1 = \{e\}$ where $e \in T$, then B_2 is a basis of M.*

Proof Let $B_1 = \{a, b, 1, 2, \dots, n\}$ be a basis of M and let $B_2 = \{a, t, 1, 2, \dots, n\}$ be the assumed subset of the lemma. It is clear that $X = \{a, 1, 2, \dots, n\}$ is independent, we prove $X \cup \{t\} = B_2$ is too. Assume the contrary and let $X \cup \{t\}$ contains a circuit C. Clearly $t \in C$. If $a \in C$, as M is binary and $\{a, b, t\}$ is a triangle, then $a + b = t$. So $C' = b \cup C - \{t, a\}$ is a circuit and $C' \subseteq B_1$, a contradiction. If $a \notin C$, then $C'' = \{a, b\} \cup C - \{t\}$ is a circuit and $C'' \subseteq B_1$, a contradiction. Thus $X \cup \{t\}$ is independent and therefore B_2 is a basis of M. □

The following theorem is our main result.

Theorem 2.2 *Let M be a binary matroid and $T = \{a, b\}$, where $a, b \in E(M)$. Let $\mathcal{C}_0$ be the collection of circuits in M in which meet T at even elements, then a sufficient condition for $G(B(M_{a,b}), \mathcal{C}_0)$ to be connected is that T lies in a triangle in M.*

Proof If $\mathcal{C}_1$ at the collection of circuits of splitting matroids is empty set, then $M_{a,b} = M$ and hence $\mathcal{C}_0 = \mathcal{C}(M)$. Therefore $G(B(M_{a,b}), \mathcal{C}_0) = G(B(M))$ and $G(B(M))$ always is connected. Thus, suppose $\mathcal{C}_1$ is non-empty set. Let $\{a, b, t\}$ be the assumed triangle. We consider four cases about vertices of $G(B(M_{a,b}), \mathcal{C}_0)$. In these cases B_i are bases of $M_{a,b}$ as characterized in Theorem 1.1 and $x, y \in E(M_{a,b}) - T$.

Case 1 $B_1 = \{a, 1, 2, \dots, x\}$ and $B_2 = \{a, 1, 2, \dots, y\}$.

Since $M_{a,b}$ does not have a circuit with just an element of T, then $B_1 \cup B_2$ contains a circuit C that avoids a. Hence $C \cap T = \emptyset$, so $C \in \mathcal{C}_0$. We conclude B_1 and B_2 are adjacent in $G(B(M_{a,b}), \mathcal{C}_0)$.

Case 2 $B_1 = \{a, 1, 2, \dots, x\}$ and $B_2 = \{a, 1, 2, \dots, b\}$.

Suppose B_1 and B_2 are not adjacent. Then $t \notin B_1 \cup B_2$; otherwise the triangle $\{a, b, t\}$ contained in $B_1 \cup B_2$, contradicting the fact that B_1 and B_2 are not adjacent. By Lemma 2.1 $B_3 = \{a, 1, 2, \dots, t\}$ is a basis of $M_{a,b}$. Now $B_2 \cup B_3$ contains a unique circuit in which t and b belong to it and since a belongs to that union, the circuit is the triangle. Thus, it is on $\mathcal{C}_0$. Then B_2 and B_3 are adjacent. By the first case B_1 and B_3 are adjacent. Then there is a path from B_1 to B_2.

Case 3 $B_1 = \{a, b, 1, 2, \dots, x\}$ and $B_2 = \{a, b, 1, 2, \dots, y\}$.

If a circuit X in $B_1 \cup B_2$ is a member of $\mathcal{C}_0$, the result is trivial and this two vertices are adjacent in $G(B(M_{a,b}), \mathcal{C}_0)$. By the Lemma 2.1, $B_3 = \{a, t, 1, 2, \dots, x\}$ and $B_4 = \{a, t, 1, 2, \dots, y\}$ are two bases of $M_{a,b}$. By the second case $B_1 = \{a, b, 1, 2, \dots, x\}$ and $B_3 = \{a, t, 1, 2, \dots, x\}$ are adjacent and similarly two bases of $B_2 = \{a, b, 1, 2, \dots, y\}$ and $B_4 = \{a, t, 1, 2, \dots, y\}$ are adjacent by the second case. In the other hand B_3 and B_4 are adjacent by the first case. Therefore there is a path between B_1 and B_2.

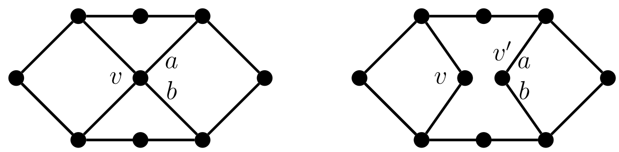
Case 4 $B_1 = \{1, 2, \dots, a\}$ and $B_2 = \{1, 2, \dots, b\}$.

Suppose B_1 and B_2 are not adjacent in $G(B(M_{a,b}), \mathcal{C}_0)$. The uniqe circuit X in $B_1 \cup B_2$ is disjoint union of two circuits C_1 and C_2 of M such that each of them meets T precisely in one element. In fact $C_1 = \{a, a_1, a_2, \dots, a_m\} \subseteq B_1$ and $C_2 = \{b, b_1, b_2, \dots, b_n\} \subseteq B_2$. It is clear that $t \notin B_1 \cup B_2$. We construct B_3 by the following way; we delete a member of C_1 like a_1 and add t in it. Without loss of generality, we can assume that $a_1 = 1$. Hence $B_3 = \{t, 2, \dots, a\}$. Since $a + b = t$ and $B_3 - \{a, t\} \subseteq B_2$ it is clear that B_3 is a basis of $M_{a,b}$. Now by the second case B_1 and B_3 are connected by a path. Suppose $B_4 = \{b, 2, \dots, a\}$ is constructed by deleting t and adding b. By the Lemma 2.1, B_4 is a basis of $M_{a,b}$. By the second case B_4 and B_3 are connected by a path too. Now if we apply the same procedure for the basis B_2 and without loss of generality by considering $b_1 = 2$, we get bases $B_5 = \{1, t, \dots, b\}$ and $B_6 = \{1, a, \dots, b\}$ that by the second case there is a path between B_2 to B_5 and a path between B_5 to B_6. Now notice that $B_4 \triangle B_6 = \{1, 2\}$ and B_4 and B_6 are connected by a path by the second case. Hence there is a path between B_1 and B_2.

Now suppose that B_1 and B_2 are two arbitrary bases of $M_{a,b}$. As $G(B(M_{a,b}))$ is connected, there is a path from B_1 to B_2 in $G(B(M_{a,b}))$. Then B_1 has an adjacent vertex in $G(B(M_{a,b}))$. This two vertices are connected with a path by using four cases mentioned above. Thus we conclude B_1 and B_2 in $G(B(M_{a,b}), \mathcal{C}_0)$ are connected by a path, then the graph $G(B(M_{a,b}), \mathcal{C})$ is connected. □

Evidently the sufficient condition in the theorem is not necessary. Consider the following example.

Example 2.3 For the graph G in the following Figure (a) $G_{a,b}$ is shown in Figure (b). There is no triangle in G but one can show $G(B(M(G)_{a,b}), \mathcal{C}_0)$ is connected.



$(a)\ G$ $(b)\ G_{a,b}$

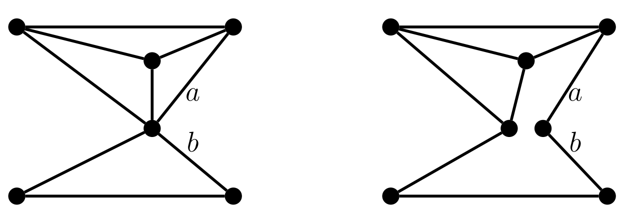
Corollary 2.4 *Let M be a matroid and $\mathcal{C}_0$ be specified collection of circuits in the last Theorem. Let S be a triangle of M. Suppose $T = \{a, b\}$ and T, S are disjoint. If $M_{x,y} = M_{a,b}$, where $x, y \in S$, then $G(B(M), \mathcal{C}_0)$ is connected.*

Theorem 2.5 *Let a and b be two elements of a binary matroid M. If $\mathcal{C}_0$ is empty, then $G(B(M_{a,b}), \mathcal{C}_1)$ is connected.*

Proof Suppose $\mathcal{C}_0$ is empty. If $\mathcal{C}_1$ is empty, then $M_{a,b}$ is a free matroid and clearly $G(B(M_{a,b}), \mathcal{C}_1)$ is connected. Thus, we can assume that $\mathcal{C}_1$ is non-empty. Now suppose C and C' are two circuit of $M_{a,b}$. As $\mathcal{C}_0$ is empty, so a and b belong to both of C and C'. Since $M_{a,b}$ is binary then $C \triangle C'$ contains a circuit and none of a or b belongs to this circuit, hence $\mathcal{C}_0$ is non-empty, a contradiction. Thus $M_{a,b}$ just has one circuit. Therefore $G(B(M_{a,b}), \mathcal{C}_1)$ is a complete graph and hence it is connected. $\square$

Notice that the converse of the last theorem is not true generally, for instance consider following example.

Example 2.6 Consider the graph G and its splitting graph $G_{a,b}$ in following figure. One can easily check that there is a path between every two vertices of $B(M(G)_{a,b}, \mathcal{C}_1)$. Hence $B(M(G)_{a,b}, \mathcal{C}_1)$ is connected while $\mathcal{C}_0$ is non-empty.



G $G_{a,b}$

The converse of Theorem 2.5 can be true with a special condition.

Theorem 2.7 *Let a and b be two elements of a binary matroid M. Let $\mathcal{C}_0$ be non-empty and $\mathcal{C}_1$ has only one circuit. Then $G(B(M_{a,b}), \mathcal{C}_1)$ is disconnected.*

Proof Let C_1 be the only circuit of $\mathcal{C}_1$. Since $\mathcal{C}_0$ is non-empty, there is a basis of $M_{a,b}$ called B_1 such that

$$|C_1 - B_1| \geq 2. \tag{1}$$

Let B_2 be a basis of $M_{a,b}$ in which contains $C_1 - e$, where $e \in C_1$. We claim that vertices B_1 and B_2 of $G(B(M_{a,b}), \mathcal{C}_1)$ are not adjacent. Assume the contrary, let $B_1 \cup B_2$ contains a circuit of $\mathcal{C}_1$, that is C_1. As $B_2 = (B_1 - f) \cup g$, where $f \in B_1$ and $g \in B_2$, by (1), $|C_1 - (B_1 \cup B_2)| \geq 1$. Hence C_1 is not contained in $B_1 \cup B_2$, a contradiction. In fact B_1 has no adjacent vertex, then $G(B(M_{a,b}), \mathcal{C}_1)$ is disconnected. $\square$

Acknowledgements The authors gratefully acknowledge the Faculty of Science of Urmia University for helpfull support given.

Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Figueroa AP, Campo E (2012) The basis graph of a bicolored matroid. Discret Appl Math 160:2694–2697
2. Fleischner H (1990) Eulerian graphs and related topics: part 1, vol 1. North-Holland, Amsterdam
3. Harary F (1969) Graph theory. Addison-Wesley, Boston
4. Holzmann CA, Harary F (1972) On the tree graph of a matroid. SIAM J Appl Math 22:187–193
5. Li X, Neumann-Lara V, Rivera-Campo E (2003) The tree graph defined by a set of cycles. Discret Math 271:303–310
6. Oxely JG (2011) Matroid theory, 2nd edn. Oxford University Press, New York
7. Raghunathan TT, Shikare MM, Waphare BN (1998) Splitting in a binary matroid. Discret Math 184:267–271
8. Shikare MM, Azadi Gh (2003) Determination of the bases of a splitting matroid. Eur J Comb 24:45–52