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EDITORIAL

Edge volume, part II

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William Edge was born in Stockport on 8th November 1904 to parents who were both schoolteachers. After a secondary education at the local Stockport Grammar School, in 1923 he went up to Cambridge to study mathematics at Trinity College. After graduation, he became a Ph.D. student of Henry Baker, where his dissertation generalised Luigi Cremona's results about ruled surfaces in the real projective space. His appointment as a research fellow in Trinity College ensued in 1928. After 4 years he secured a lectureship (assistant professorship) at the University of Edinburgh, where he remained for the rest of his career. Elected a Fellow of the Royal Society of Edinburgh 2 years later in 1932, Edge was later promoted to reader (associate professor) in 1949. However, it took until 1969 for him to be appointed to a full professorship, 6 years prior his retirement in 1975. Edge never married, he had no children, he never drove a car, he was reluctant to travel, and he disdained radio and television. Apart from mathematics, Edge loved hill walking, singing and playing the piano. He spent his final years in the retirement house in Bonnyrigg near Edinburgh, where he died on 27 September 1997.

William Edge's scientific results influenced the research of many mathematicians worldwide. His papers, even from 50 years ago, continue to attract attention. This volume is dedicated to the memory of William Edge.

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Edge wrote nearly 100 research papers. Most of them are about classical algebraic varieties like the Veronese surface, Klein's quartic curve, Maschke's quartic surfaces, Kummer's quartic surface, Weddle surface, Fricke's curve etc. A typical example is the paper *Humbert's plane sextics of genus* 5, where he studied basic properties of Humbert's curves. These are smooth curves of genus 5 that admit a faithful action of the group $\mathbb{Z}_2^4$. In 1894, Humbert constructed some of them as curves of degree 7 in $\mathbb{P}^3$. Later, Baker constructed another examples as curves of contact of Weddle surfaces with the tangent cone to one of its six singular points. Edge proved that both constructions essentially give the same curves. Moreover, he proved that each Humbert's curve is not hyperelliptic, and its canonical model in $\mathbb{P}^4$ is given by

$$\begin{cases} a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0, \\ b_0x_0^2 + b_1x_1^2 + b_2x_2^2 + b_3x_3^2 + b_4x_4^2 = 0, \\ c_0x_0^2 + c_1x_1^2 + c_2x_2^2 + c_3x_3^2 + c_4x_4^2 = 0 \end{cases}$$

for appropriate complex numbers $a_0, a_1, a_2, a_3, a_4, b_0, b_1, b_2, b_3, b_4, c_0, c_1, c_2, c_3$ and c_4. Some of Humbert's curves have larger automorphism groups than $\mathbb{Z}_2^4$. For example, McKelvey proved that $\mathbb{Z}_2^4 \rtimes \mathrm{D}_{10}$ is the full automorphism group of the curve

$$\begin{cases} x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \\ x_0^2 + \xi_5x_1^2 + \xi_5^2x_2^2 + \xi_5^3x_3^2 + \xi_5^4x_4^2 = 0, \\ \xi_5^4x_0^2 + \xi_5^3x_1^2 + \xi_5^2x_2^2 + \xi_5x_3^2 + x_4^2 = 0 \end{cases}$$

where ξ_5 is a primitive fifth root of unity. In 2017, this partucular Humbert's curve has been used by Cheltsov and Shramov to prove that $\mathbb{P}^3$ is $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_5$-birationally rigid.

To carry Edinburgh's tradition in geometry into the future, Ahmadinezhad, Cheltsov, Guletskiĭ, Kaloghiros, Logvinenko and Testa organised five *Edge Days* workshops at Edinburgh in 2013–2017. Among their participants were most of contributors to this volume as well as Caucher Birkar, Alexey Bondal, Alessio Corti, Ruadhai Dervan, Yoshinori Gongyo, Liana Heuberger, Ilia Itenberg, Dmitry Kaledin, Masayuki Kawakita, Alvaro Liendo, Angelo Lopez, Daniel Loughran, Frédéric Mangolte, Takuzo Okada, John Ottem, Dmitri Panov, Elisa Postinghel, Francesco Russo, Edoardo Sernesi, Nicholas Shepherd-Barron, Alexei Skorobogatov and Yuri Tschinkel.

The second part of the *Edge Volume* contains 23 research papers, whose authors are Florin Ambro, Asher Auel, Artem Avilov, Edoardo Ballico, Christian Böhning, Hans-Christian Graf von Bothmer, Harry Braden, Aiden Bruen, Ivan Cheltsov, Giulio Codogni, Igor Dolgachev, David Eklund, Philippe Ellia, Andrea Fanelli, Benson Farb, Ivan Fesenko, Claudio Fontanari, Juan Frías-Medina Patricio Gallardo, Marat Gizatullin, Vladimir Guletskiĭ, David Holmes, Martin Kalck, Joseph Karmazyn, Jesse Leo Kass, Ludmil Katzarkov, Eduard Looijenga, Lisa Marquand, Jesus Martinez-Garcia, James McQuillan, Nicola Pagani, Jihun Park, Alena Pirutka, Yuri Prokhorov, Costya Shramov, Leonardo Soriani, Roberto Svaldi, Luca Tasin, Alessandro Verra, Sergei Vostokov, Jonas Wolter, Joonyeong Won, Seok Ho Yoon, Misha Zaidenberg, Alexis Zamora, Zheng Zhang, and a postface by James Hirschfeld. Most of them were participants of Edge Days, while the others are mathematicians who personally knew William Edge or used his work in their research.

Let us briefly describe the scientific content of the papers in the second part of the volume.

In the paper *On toric face rings I*, Florin Ambro constructs a Deligne–Du Bois complex for algebraic varieties which are locally isomorphic to the spectrum of a toric face ring.

In the paper *Stable rationality of quadric and cubic surface bundle fourfolds*, Asher Auel, Christian Böhning and Alena Pirutka study the stable rationality problem for quadric and cubic surface bundles over surfaces. Their main result is the following

Theorem *A very general hypersurface of bidegree* $(2, 3)$ *in* $\mathbb{P}^2 \times \mathbb{P}^3$ *is not stably rational.*

This result provides another example of a smooth family of rationally connected fourfolds with rational and nonrational fibers. The paper also contains examples of a quadric surface bundle over $\mathbb{P}^2$ with discriminant curve of even degree (greater than 7) that have nontrivial unramified Brauer groups and universally CH_0-trivial resolutions.

In the paper *Automorphisms of singular three-dimensional cubic hypersurfaces*, Artem Avilov studies singular three-dimensional cubic hypersurfaces in $\mathbb{P}^4$ faithfully acted on by a finite group. He proves the following

Theorem *Let* X *be a rational cubic threefold in* $\mathbb{P}^4$*, let* G *be a finite subgroup in* $\mathrm{Aut}(X)$*. Suppose that* $\mathrm{Cl}^G(X)$ *is of rank* 1*, and* X *is* G*-birational to none of the following:* $\mathbb{P}^3$*, a conic bundle, a fibration into del Pezzo surfaces. Then* X *and* G *can be described as follows:*

- *the threefold X is the Segre cubic,* $\mathrm{Aut}(X) \cong \mathfrak{S}_6$*, and G is one of the following subgroups:* $\mathfrak{A}_5$ (*standard subgroup*), $\mathfrak{S}_5$ (*standard subgroup*), $\mathfrak{A}_6$, $\mathfrak{S}_6$*;*
- *up to projective equivalence, the threefold X is given by*

$$x_0x_1x_2 + x_3x_4(x_0 + x_1 + x_2 + x_3 + x_4) = 0,$$

  $\mathrm{Aut}(X) \cong \mathfrak{S}_3^2 \rtimes \mathbb{Z}_2$*, and G is one of the following subgroups:* $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2$, $\mathfrak{S}_3^2$, $\mathfrak{S}_3^2 \rtimes \mathbb{Z}_2$*;*
- *up to projective equivalence, the threefold X is given by*

$$\begin{aligned} x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_0 + x_4x_0x_1 \\ = x_0x_1x_3 + x_1x_2x_4 + x_2x_3x_0 + x_3x_4x_1 + x_4x_0x_2, \end{aligned}$$

  and $G = \mathrm{Aut}(X) \cong \mathfrak{S}_5$*;*
- *up to projective equivalence, the threefold X is given by*

$$\begin{aligned} x_0x_1x_2 - x_0x_1x_3 + x_0x_1x_4 + x_0x_2x_3 - 3x_0x_2x_4 \\ + x_0x_3x_4 - x_1x_2x_3 + x_1x_2x_4 - x_1x_3x_4 + x_2x_3x_4 = 0, \end{aligned}$$

  $\mathrm{Aut}(X) \cong \mathfrak{S}_3^2 \rtimes \mathbb{Z}_2$*, and either* $G \cong \mathfrak{S}_3^2$ *or* $G \cong \mathfrak{S}_3^2 \rtimes \mathbb{Z}_2$*;*
- *up to projective equivalence, the threefold X is given by*

$$\sum_{0 \leqslant i < j < k \leqslant 4} x_i x_j x_k = 0,$$

  $\mathrm{Aut}(X) \cong \mathfrak{S}_5$*, and G is one of the following subgroups:* $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$, $\mathfrak{A}_5$, $\mathfrak{S}_5$*;*
- *up to projective equivalence, the threefold X is given by*

$$\begin{aligned} x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_0 + x_4x_0x_1 \\ = \omega\big(x_0x_1x_3 + x_1x_2x_4 + x_2x_3x_0 + x_3x_4x_1 + x_4x_0x_2\big), \end{aligned}$$

  where ω is a primitive cubic root of unity, and $G = \mathrm{Aut}(X) \cong \mathfrak{A}_5$.

A smooth connected curve C in $\mathbb{P}^3$ is said to be of maximal rank if the natural restriction maps $H^0(\mathcal{O}_{\mathbb{P}^3}(m)) \to H^0(\mathcal{O}_C(m))$ are either injective or surjective for every m. In the paper *Maximal rank of space curves in the range A*, Edoardo Ballico, Claudio Fontanari and Philippe Ellia prove the following

Theorem *There exists a constant K such that for all natural numbers d and g with $g \leqslant Kd^{3/2}$ there exists an irreducible component of the Hilbert scheme of $\mathbb{P}^3$ whose general element is a smooth connected curve of degree d and genus g of maximal rank.*

To describe the paper *Degenerations of Gushel–Mukai fourfolds, with a view towards irrationality proofs* by Christian Böhning and Hans-Christian Graf von

Bothmer, recall that a Gushel–Mukai fourfold is a smooth dimensionally transverse intersection

$$\mathrm{Gr}(2, W) \cap Q \cap H,$$

where W is a five-dimensional vector space, $\mathrm{Gr}(2, W)$ is the Grassmannian of lines in $\mathbb{P}(W)$ naturally embedded in $\mathbb{P}(\Lambda^2 W)$, the variety Q is a quadric in $\mathbb{P}(\Lambda^2 W)$, and H is a hyperplane in this projective space. One may ask

Question *Is very general Gushel–Mukai fourfold irrational?*

To study this question, Böhning and Bothmer study degenerations of Gushel–Mukai fourfolds that satisfy certain (strong) conditions, which are natural if one wants to apply degeneration technique of Voisin, Colliot–Thélène, Pirutka and Totaro to Gushel–Mukai fourfolds. However, they prove that such degenerations do not exist.

In the paper *A canonical form for a symplectic involution*, Harry Braden presents a canonical form for an involution in $\mathrm{Sp}(2g, \mathbb{Z})$ and applies his construction to Riemann surfaces.

In a projective plane over a field $\mathbb{F}$, the diagonal points of a quadrangle are collinear if and only if $\mathbb{F}$ has characteristic 2. Such a quadrangle together with its diagonal points and the lines connecting these points form the subplane of order 2, called a Fano plane. In the paper *Desargues configurations with four self-conjugate points*, Aiden Bruen and James McQuillan provide a similar type of synthetic criterion and construction for characteristic 3 fields. This fits perfectly the research interests of Edge, since he moved towards finite geometry after studying classical geometry.

To describe the paper *Alpha-invariants and purely log terminal blow-ups* by Ivan Cheltsov, Jihun Park and Constantin Shramov, fix a germ of a Kawamata log terminal singularity $U \ni P$. Then there is a birational morphism $\phi\colon X \to U$ such that its exceptional locus consists of a single prime divisor E_X such that $\phi(E_X) = P$, the log pair (X, E_X) has purely log terminal singularities, and the divisor $-(K_X + E_X)$ is ϕ-ample. The birational morphism $\phi\colon X \to U$ is a purely log terminal blow-up of the singularity $U \ni P$. One can show that E_X is normal, and there exists a (naturally defined) $\mathbb{Q}$-divisor $\mathrm{Diff}_{E_X}(0)$ on the variety E_X, which is usually called the different of the pair (X, E_X), such that

$$-(K_X + E_X)|_{E_X} \sim_{\mathbb{Q}} -\big(K_{E_X} + \mathrm{Diff}_{E_X}(0)\big),$$

and $(E_X, \mathrm{Diff}_{E_X}(0))$ is Kawamata log terminal. Then $(E_X, \mathrm{Diff}_{E_X}(0))$ is a log Fano variety with Kawamata log terminal singularities. Note that the germ $U \ni P$ can have many purely log terminal blow-ups, so that we may consider a commutative diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\rho"] \arrow[dr, "\phi"'] & & Y \arrow[dl, "\psi"] \\ & U & \end{tikzcd}$$

where ψ is (another) purely log terminal blow-up of the germ $U \ni P$, and ρ is a birational map. Let E_Y be the ψ-exceptional divisor. Cheltsov, Park and Shramov prove the following

Theorem *The birational map ρ is an isomorphism provided that*

$$\alpha(E_X, \mathrm{Diff}_{E_X}(0)) + \alpha(E_Y, \mathrm{Diff}_{E_Y}(0)) \geqslant 1.$$

Here, the numbers $\alpha(E_X, \mathrm{Diff}_{E_X}(0))$ and $\alpha(E_Y, \mathrm{Diff}_{E_Y}(0))$ are α-invariants of Tian of the log Fano varieties $(E_X, \mathrm{Diff}_{E_X}(0))$ and $(E_Y, \mathrm{Diff}_{E_Y}(0))$, respectively.

In the paper *A note on the fibres of Mori fibre spaces*, Giulio Codogni, Andrea Fanelli, Roberto Svaldi and Luca Tasin study Fano varieties that can be realised as fibres of a Mori fibre space.

In the paper *Geometry of the Wiman–Edge pencil, I: algebro-geometric aspects*, Igor Dolgachev, Benson Farb and Eduard Looijenga study basic properties of the Wiman–Edge pencil from modern point of view. To describe this pencil, denote by S the (unique) smooth del Pezzo surface of degree 5. Then

$$\mathrm{Aut}(S) \cong \mathfrak{S}_5,$$

and this group leaves invariant two curves in the linear system $|-2K_S|$. One of them is the union of 10 lines, and another one is a a smooth curve of genus 6, whose automorphism group is the group $\mathfrak{S}_5$. These two curves generates a $\mathfrak{S}_5$-invariant pencil in $|-2K_S|$, which contains all $\mathfrak{A}_5$-invariant curves in $|-2K_S|$. This pencil is the Wiman–Edge pencil. If we identify S with the blow-up of $\mathbb{P}^2$ at the points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$ and $[1:1:1]$, then the image in $\mathbb{P}^2$ of the unique smooth $\mathfrak{S}_5$-invariant curve in $|-2K_S|$ is given by

$$x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2 = 0.$$

This curve has ordinary nodes at the points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$ and $[1:1:1]$. It was discovered by Wiman in 1896, and now it is known as the Wiman sextic curve, which should not be confused with smooth plane sextic curve also studied by Wiman. Moreover, the image in $\mathbb{P}^2$ of the union of 10 lines in S is the union of six lines in $\mathbb{P}^2$ that is given by

$$(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0.$$

Thus, the image of the Wiman–Edge pencil on $\mathbb{P}^2$ is the pencil given by

$$\lambda(y^2 - z^2)(z^2 - x^2)(x^2 - y^2) \\ = \mu\big(x^6 + y^6 + z^6 + (x^2 + y^2 + z^2)(x^4 + y^4 + z^4) - 12x^2y^2z^2\big),$$

where $[\lambda:\mu] \in \mathbb{P}^1$. In 1981, Edge studied properties of this pencil in the paper *A pencil of four-nodal plane sextics*. He explicitly described all curves in this pencil that are singular away from the points $[0:0:1]$, $[0:1:0]$, $[1:0:0]$ and $[1:1:1]$. In 2017, Edge's description has been used by Cheltsov, Kuznetsov and Shramov to investigate the rationality problem for $\mathfrak{S}_6$-symmetric quartic threefolds in $\mathbb{P}^4$.

In the paper *Curves on Heisenberg invariant quartic surfaces in projective 3-space*, David Eklund studies quartic surfaces in $\mathbb{P}^3$ that are G-invariant, where G is the Heisenberg subgroup in $\mathrm{PGL}_4(\mathbb{C})$ of order 16. Recall that $G \cong \mathbb{Z}_2^4$, and the group G is generated by the projective transformations that correspond to the following four matrices:

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Eklund proves many interesting results about G-invariant smooth quartic surfaces. For instance, he proves that a very general such surface contains exactly 320 smooth conics, which generate the Picard group (that has rank 16).

In the paper *Generalised Kawada–Satake method for Mackey functors in class field theory*, Ivan Fesenko, Sergei Vostokov and Seok Ho Yoon generalise Kawada–Satake method for Mackey functors in the class field theory of positive characteristic.

In the paper *Some remarks on Humbert–Edge's curves*, Juan Frías-Medina and Alexis Zamora explain Edge's approach to the study of Humbert's curves. They reprove some of his results and consider their generalisations.

In the paper *Compactifications of the moduli space of plane quartics and two lines*, Patricio Gallardo, Jesus Martinez-Garcia and Zheng Zhang study the moduli space of triples (C, L_1, L_2) that consists of a quartic curve C and two lines L_1 and L_2 in $\mathbb{P}^2$. They construct and compactify this moduli space in two ways: via geometric invariant theory and by using the period map of certain lattice polarized $K3$ surfaces.

In the paper *Two examples of affine homogeneous varieties*, Marat Gizatullin studies two explicit flexible affine homogeneous varieties that have infinite-dimensional groups of automorphisms. In both cases, Gizatullin proves the existence of automorphisms that do not belong to the connected components of identity.

In the paper *Chow motives of abelian type over a base*, Vladimir Guletskiĭ generalises the theorem of Kimura about motives of smooth projective curves to the relative setting.

In the paper *Extending the double ramification cycle using Jacobians*, David Holmes, Jesse Leo Kass and Nicola Pagani prove that the extension of the double ramification cycle defined earlier by Holmes coincides with the extension of the double ramification cycle defined earlier by Kass and Pagani.

In the paper *Ringel duality for certain strongly quasi-hereditary algebras*, Martin Kalck and Joseph Karmazyn introduce quasi-hereditary endomorphism algebras defined over a new class of finite dimensional monomial algebras with a special ideal structure, and present a uniform formula describing the Ringel duals of these quasi-hereditary algebras.

In the paper *Homological mirror symmetry, coisotropic branes and $P = W$*, Ludmil Katzarkov and Leonardo Soriani discuss a possible approach to the famous conjecture of de Cataldo, Hausel and Migliorini via the theory of coisotropic brains. For Fano manifolds and their Landau–Ginsburg models, this conjecture is known as Katzarkov–Kontsevich–Pantev conjecture.

In the paper *Cylinders in rational surfaces*, Lisa Marquand and Joonyeong Won prove the following

Theorem *Let S be a smooth rational surface such that $K_S^2 \geqslant 3$, and let A be an ample $\mathbb{Q}$-divisor on S. Then there exists an A-polar cylinder in S except the case when S is a smooth cubic surface and $A \in \mathbb{Q}_{>0}[-K_S]$.*

To explain the geometrical meaning of this result, fix a smooth rational surface S, and fix an ample $\mathbb{Q}$-divisor A on it. An A-polar cylinder in S is a Zariski open subset U in S such that

(C) $U \cong \mathbb{C}^1 \times Z$ for some affine curve Z,
(P) there is an effective $\mathbb{Q}$-divisor D on S such that $D \sim_{\mathbb{Q}} A$ and $U = S \setminus \mathrm{Supp}(D)$.

Then the theorem of Marquand and Won is the most natural generalisation of the following

Theorem (Cheltsov, Kishimoto, Park, Prokhorov, Won, Zaidenberg) *Let S be a smooth del Pezzo surface such that $K_S^2 \geqslant 3$, and let A be an ample $\mathbb{Q}$-divisor on S. Then there exists an A-polar cylinder in S except the case when S is a smooth cubic surface and $A \in \mathbb{Q}_{>0}[-K_S]$.*

Note also that if S is a smooth cubic surface and $A \in \mathbb{Q}_{>0}[-K_S]$, then S does not contain A-polar cylinders by a result of Cheltsov, Park and Won.

In the paper *Fano–Mukai fourfolds of genus* 10 *as compactifications of* $\mathbb{C}^4$, Yuri Prokhorov and Mikhail Zaidenberg study smooth Fano–Mukai fourfolds in $\mathbb{P}^{12}$ of degree 18. For each such smooth fourfold V, one has

$$-K_V \sim 2H$$

for some ample Cartier divisor H such that $H^3 = 18$, and H generates the whole Picard group of the fourfold V. Up to isomorphism, such foufolds form a one-parameter family. Prokhorov and Zaidenberg prove that all of them are compactifications of $\mathbb{C}^4$. In fact, they prove the following (more precise) result:

Theorem *Let V be a smooth Fano–Mukai fourfold in $\mathbb{P}^{12}$ of degree* 18. *Then V contains a hyperplane section X such that X is singular along a cone over a rational twisted cubic curve, and $V \setminus X \cong \mathbb{C}^4$.*

In the paper *Edge and Fano on nets of quadrics*, Alessandro Verra revisits and reconstructs classical results of Edge and Fano about the family of scrolls of degree 8 in the complex projective space, whose plane sections are projected bicanonical models of a genus 3 curve. Verra shows that this beautiful subject is related to the moduli of semistable rank two vector bundles on genus 3 curves with bicanonical determinants.

In the paper *Equivariant birational geometry of quintic del Pezzo surface*, Jonas Wolter (explicitly) describes all G-equivariant birational transformations of the (unique) smooth del Pezzo surface of degree 5 into G-Mori fibre spaces, where $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. In particular, he proves that there are exactly two such G-Mori fibre spaces:

$\mathbb{P}^1 \times \mathbb{P}^1$ and the del Pezzo surface of degree 5 itself. In particular, the smooth del Pezzo surface of degree 5 is not G-birational to a conic bundle, and it is not G-birational to $\mathbb{P}^2$.

The second part of Edge Volume is concluded by a postface *William Leonard Edge* by James Hirschfeld, who was the only research student of William Edge. James opened the first Edge Days back in 2013. His short note contains some personal memories about Edge.

William Edge stayed research active throughout his entire life. His very last paper 28 *real bitangents* has been published in 1994 when he was 90 years old. Geometry gave him energy, kept his spirits high and probably prolonged his life. We hope that *Edge Volume* will help to keep Edge's legacy alive. The diversity of its contributions reflects the vitality of algebraic geometry in the directions impulsed by William Edge. Their authors have the same passion about mathematics as Edge had.

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