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On classical n-absorbing submodules

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Abstract Let R a commutative ring with identity and M be a unitary R-module. In this paper, we investigate some properties of n-absorbing submodules of M as a generalization of 2-absorbing submodules. We also define the classical n-absorbing submodule, a proper submodule N of an R-module M is called a classical n-absorbing submodule if whenever $a_1a_2\ldots a_{n+1}m \in N$ for $a_1, a_2, \ldots, a_{n+1} \in R$ and $m \in M$, there are n of a_i's whose product with m is in N. Furthermore, we give some characterizations of n-absorbing and classical n-absorbing submodules under some conditions.

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1 Introduction

Throughout this paper, we assume that all rings are commutative with $1 \neq 0$. Let R be a commutative ring. An ideal I of R is said to be proper if $I \neq R$. Let M a unitary module over R and N be a submodule of M. The residual of N by M, $(N :_R M)$ or simply $(N : M)$, denotes the ideal $\{r \in R : rM \subseteq N\}$. For any element x of M, the ideal $(N : x)$ is defined by $(N : x) = \{r \in R : rx \in N\}$. Let $a \in R$. Then, $N_a = \{x : x \in M \ \ and \ \ ax \in N\}$ is a submodule of the R-module M. Let $m \in M$, a cyclic submodule that is generated by m is a submodule of M has the form $Rm = \{rm : r \in R\}$. A proper submodule N of M is said to be irreducible if N is not an intersection of two submodules of M that properly contain it. The set of zero divisors of M, denoted by $Zd(M)$ is defined by $Zd(M) = \{r \in R : \ \ for \ \ some \ \ x \in M \ \ and \ \ x \neq 0, rx = 0\}$. An R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R. Prime ideals play a crucial role in ring theory, since they interfere with many branches of algebra and they represent an important role in understanding the structure of ring. A proper ideal I of a ring R is called a prime ideal if, whenever $ab \in I$ for $a, b \in R$, then $a \in I$ or $b \in I$. A proper submodule N of an R-module M is said to be a prime submodule if, whenever $a \in R$, $m \in M$, and $am \in N$, then $m \in N$ or $a \in (N : M)$.

In [5], Badawi introduced a new generalization of prime ideals in a commutative ring R. He defined a nonzero proper ideal I of R to be a 2-absorbing ideal of R if, whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. The concept of 2-absorbing ideal has been transferred to modules. A proper submodule N of an R-module M is a 2-absorbing submodule of M [6] if, whenever $abm \in N$ for $a, b \in R$ and $m \in M$, then $am \in N$ or $bm \in N$ or $ab \in (N : M)$. The class of 2-absorbing submodules of modules was introduced as a generalization of the class of 2-absorbing ideals of rings. Then, many generalizations of 2-absorbing submodules were studied such as primary 2-absorbing [8], almost 2-absorbing [3], almost 2-absorbing primary [2], and classical 2-absorbing [9]. In this article, we investigate some properties of n-absorbing submodules of M as a generalization of 2-absorbing submodules. We also define the classical n-absorbing submodule.

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Furthermore, we give some characterizations of n-absorbing and classical n-absorbing submodules under some conditions. In addition, we investigate the sufficient and necessary conditions for a submodule N to be classical n-absorbing submodule of M.

2 n-Absorbing submodules

The concept of 2-absorbing has been extended to n-absorbing in ideals and submodules, where n is any positive integer. In this section, we investigate some properties of n-absorbing submodules.

Definition 2.1 [1] A proper ideal I of a ring R is said to be an n-absorbing ideal if, whenever $a_1 \dots a_{n+1} \in I$ for $a_1, \dots, a_{n+1} \in R$, then there are n of $a_i's$ whose product is in I.

Definition 2.2 [7] A proper submodule N of an R-module M is called an n-absorbing submodule if, whenever $a_1 \dots a_n m \in N$ for $a_1, \dots, a_n \in R$ and $m \in M$, then either $a_1 \dots a_n \in (N : M)$ or there are $n-1$ of $a_i's$ whose product with m is in N.

Proposition 2.3 *If N is an n-absorbing submodule of an R-module M, then $(N : m)$ is an n-absorbing ideal in R for all $m \in M - N$.*

Proof For $m \in M-N$, $(N : m)$ is a proper ideal of R. Assume that $a_1 \dots a_{n+1} \in (N : m)$ for $a_1, \dots, a_{n+1} \in R$. Then, $a_1 \dots a_{n+1} m = a_1 \dots a_n(a_{n+1} m) \in N$. Since N is an n-absorbing submodule, then $a_1 \dots a_n \in (N : M) \subseteq (N : m)$ or there are $n-1$ of the $a_i's$, $1 \le i \le n$ whose product with $a_{n+1}m$ in N, the latter case means that there are $n-1$ of the $a_i's$, $1 \le i \le n$ whose product with a_{n+1} belongs to $(N : m)$. Thus, $(N : m)$ is an n-absorbing ideal in R. □

Proposition 2.4 [4] *Let M an R-module and N be a proper submodule of M. Then, $Zd(M/N) = \bigcup_{x \in M-N} (N : x)$.*

Proposition 2.5 *Let N be an n-absorbing submodule of M. If the set of all zero divisors of M/N, $Zd(M/N)$, forms an ideal in R, then it is an n-absorbing ideal of R.*

Proof Let $a_1 \dots a_{n+1} \in Zd(M/N)$ for $a_1, \dots, a_{n+1} \in R$, and then, by *Proposition* 2.4, $a_1 \dots a_{n+1} \in (N : m')$ for some $m' \in M - N$. Since N is an n-absorbing submodule, then $(N : m')$ is an n-absorbing ideal of R. Therefore, there are n of $a_i's$ whose product belongs to $(N : m')$, and hence, there are n of $a_i's$ whose product belongs to $Zd(M/N)$. □

Remark 2.6 The set of all zero divisors may not be an ideal. For example, consider the $\mathbb{Z}$-module $M = \mathbb{Z}_6$, we have $2, 3 \in Zd(M)$ but $2 + 3 \notin Zd(M)$.

The following theorem characterizes n-absorbing submodule in terms of submodules.

Theorem 2.7 *Let N be a submodule of an R-module M. The following are equivalent:*

(1) *N is an n-absorbing submodule.*
(2) *For $a_1, \dots, a_n \in R$, such that $a_1 \dots a_n \notin (N : M)$, $N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a}_i}$, where $\hat{a}_i = a_1 \dots a_{i-1} a_{i+1} \dots a_n$.*

Proof (1) $\Rightarrow$ (2) Let $m \in N_{a_1 \dots a_n}$ and assume that $a_1 \dots a_n \notin (N : M)$, and then, $a_1 \dots a_n m \in N$. Since N is an n-absorbing submodule, then there are $n-1$ of $a_i's$, $1 \le i \le n$, such that $\hat{a}_i m \in N$, $\hat{a}_i = a_1 \dots a_{i-1} a_{i+1} \dots a_n$, and hence, $m \in N_{\hat{a}_i}$. For the other containment, let $m \in \bigcup_{i=1}^{n} N_{\hat{a}_i}$, then $\hat{a}_j m \in N$ for some $j \in \{1, \dots, n\}$, then $a_j \hat{a}_j m = a_1 \dots a_n m \in N$, so $m \in N_{a_1 \dots a_n}$.
(2) $\Leftarrow$ (1) Let $a_1, \dots, a_n \in R$ and $m \in M$ such that $a_1 \dots a_n m \in N$. Assume that $a_1 \dots a_n \notin (N : M)$, then $m \in N_{a_1 \dots a_n} = \bigcup_{i=1}^{n} N_{\hat{a}_i}$ then $m \in N_{\hat{a}_j}$ for some $j \in \{1, \dots, n\}$, implies that $\hat{a}_j m = a_1 \dots a_{j-1} a_{j+1} \dots a_n m \in N$. Thus, N is an n-absorbing submodule. □

The following example shows that if N is not an n-absorbing submodule of M, then the second statement in the previous theorem does not hold.

Example 2.8 Take $n = 2$. Let $M = \mathbb{Z}$ be a module over itself, and let $N = 8\mathbb{Z}$, N is not a 2-absorbing submodule of M and $N_{2.2} = 2\mathbb{Z} \neq N_2 = 4\mathbb{Z}$.

Now, we give a necessary and sufficient condition for capability of reducing (by 1) the index of the residual $(N : M)$ of the proper submodule N of M.

Theorem 2.9 *Let N be an n-absorbing submodule of an R-module M. Then, $(N : M)$ is an $(n-1)$-absorbing ideal of R if and only if $(N : m)$ is an $(n-1)$-absorbing ideal of R for all $m \in M - N$.*

Proof ($\Rightarrow$) Let $a_1, \ldots, a_n \in R$, $m \in M - N$ and $a_1 \ldots a_n \in (N : m)$. Then, $a_1 \ldots a_n m \in N$. Since N is an n-absorbing submodule of M, then $a_1 \ldots a_n \in (N : M)$ or there are $n-1$ of the $a_i's$ whose product with m is in N. If $a_1 \ldots a_n \in (N : M)$, then, by assumption, there are $n-1$ of the $a_i's$, $1 \le i \le n$, whose product belongs to $(N : M)$, and hence, there are $n-1$ of the $a_i's$, $1 \le i \le n$, whose product belongs to $(N : m)$. In the other case, if there are $n-1$ of the $a_i's$ whose product with m is in N, and hence, there are $n-1$ of the $a_i's$, $1 \le i \le n$, whose product belongs to $(N : m)$ and we are done.
($\Leftarrow$) Suppose that $a_1 \ldots a_n \in (N : M)$ for some $a_1, \ldots, a_n \in R$ and assume that, for every i, $1 \le i \le n$, there exists $m_i \in M$, such that $\hat{a}_i m_i \notin N$, where $\hat{a}_i = a_1 \ldots a_{i-1} a_{i+1} \ldots a_n$. By $a_1 \ldots a_n m_i \in N$, it follows that $\hat{a}_j m_i \in N$, where $j \neq i$ and $\hat{a}_j = a_1 \ldots a_{j-1} a_{j+1} \ldots a_n$, since $(N : m_i)$ is $(n-1)$-absorbing ideal. If $\sum_{i=1}^{n} m_i \in N$, then $\hat{a}_j m_j \in N$, since $\hat{a}_j m_i \in N, \forall i \neq j$, which is a contradiction. Thus, $\sum_{i=1}^{n} m_i \notin N$. Now, by $a_1 \ldots a_n \sum_{i=1}^{n} m_i \in N$, we have $a_1 \ldots a_n \in (N : \sum_{i=1}^{n} m_i)$, and then, there are $n-1$ of the $a_i's$ whose product is in $(N : \sum_{i=1}^{n} m_i)$, and hence, there are $n-1$ of the $a_i's$ whose product with $\sum_{i=1}^{n} m_i$ belongs to N, and then, we must have $\hat{a}_k m_k \in N$, for some $k \in \{1, \ldots, n\}$, which is a contradiction. Thus, there are $n-1$ of the $a_i's$ whose product with M is contained in N. Therefore, $(N : M)$ is $(n-1)$-absorbing ideal of R. □

Proposition 2.10 *Let N be an n-absorbing submodule of an R-module M, $y \in M$, and $a_1, \ldots, a_n \in R$. If $a_1 \ldots a_n \notin (N : M)$, then*

$$(N : a_1 \ldots a_n y) = \bigcup_{i=1}^{n} (N : \hat{a}_i y),$$

where $\hat{a}_i = a_1 \ldots a_{i-1} a_{i+1} \ldots a_n$.

Proof Let $r \in (N : a_1 \ldots a_n y)$, and then, $r a_1 \ldots a_n y = a_1 \ldots a_n (ry) \in N$. Since N is an n-absorbing submodule and $a_1 \ldots a_n \notin (N : M)$, then $\hat{a}_i (ry) \in N$, where $\hat{a}_i = a_1 \ldots a_{i-1} a_{i+1} \ldots a_n$, for some i, and hence, $r \in (N : \hat{a}_i y)$. For the reverse inclusion, let $r \in \bigcup_{i=1}^{n} (N : \hat{a}_i y)$, and then, $r \in (N : \hat{a}_j y)$ for some $j \in \{1, \ldots, n\}$. Then, $r a_j \hat{a}_j y = r a_1 \ldots a_n y \in N$ implies $r \in (N : a_1 \ldots a_n y)$. □

In the following two propositions, we study the absorbing property under the homomorphism and localization.

Proposition 2.11 *Let $f : M \to M'$ be an epimorphism of R-modules.*

1. *If N' is an n-absorbing submodule of M', then $f^{-1}(N')$ is an n-absorbing submodule of M.*
2. *If N is an n-absorbing submodule of M containing $ker(f)$, then $f(N)$ is an n-absorbing submodule of M'.*

Proof (1) Let $a_1, \ldots, a_n \in R$ and $m \in M$, such that $a_1 \ldots a_n m \in f^{-1}(N')$ then $a_1 \ldots a_n f(m) \in N'$, but N' is n-absorbing submodule of M', so $a_1 \ldots a_n \in (N' : M')$ or $\hat{a}_i f(m) \in N'$, where $\hat{a}_i = a_1 \ldots a_{i-1} a_{i+1} \ldots a_n$. If $a_1 \ldots a_n \in (N' : M')$, then $a_1 \ldots a_n M' \subseteq N'$, then $a_1 \ldots a_n M \subseteq f^{-1}(N')$, so $a_1 \ldots a_n \in (f^{-1}(N') : M)$. If $\hat{a}_i f(m) \in N'$, then $f(\hat{a}_i m) \in N'$ so $\hat{a}_i m \in f^{-1}(N')$. Thus, $f^{-1}(N')$ is an n-absorbing submodule of M.
(2) Let $a_1, \ldots, a_n \in R$, $m' \in M'$, and $a_1 \ldots a_n m' \in f(N)$. Then, there exists $t \in N$, such that $a_1 \ldots a_n m' = f(t)$. Since f is an epimorphism therefore for some $m \in M$, we have $f(m) = m'$. Thus, $a_1 \ldots a_n f(m) = f(t)$. This implies that $f(a_1 \ldots a_n m - t) = 0$, so $a_1 \ldots a_n m - t \in ker(f) \subseteq N$. Thus, $a_1 \ldots a_n m \in N$. Now, since N is an n-absorbing, therefore, $\hat{a}_i m \in N$ or $a_1 \ldots a_n \in (N : M)$. Thus, $\hat{a}_i m' \in f(N)$ or $a_1 \ldots a_n \in (f(N) : M')$. Hence, $f(N)$ is an n-absorbing submodule of M'. □

Proposition 2.12 *Let S be a multiplicatively closed subset of R and $S^{-1}M$ be the module of fraction of M. Then, the following statements hold.*

1. *If N is an n-absorbing submodule of M, then $S^{-1}N$ is an n-absorbing submodule of $S^{-1}M$.*
2. *If $S^{-1}N$ is an n-absorbing submodule of $S^{-1}M$ such that $Zd(M/N) \cap S = \phi$, then N is an n-absorbing submodule of M.*

Proof (1) Assume that $a_1, \ldots, a_n \in R$, $s_1, \ldots, s_n, l \in S$, $m \in M$ and $\frac{a_1 \ldots a_n m}{s_1 \ldots s_n l} \in S^{-1}N$. Then, there exists $s' \in S$, such that $s'a_1 \ldots a_n m = a_1 \ldots a_n(s'm) \in N$. By assumption, N is an n-absorbing submodule of M, and thus, $a_1 \ldots a_n \in (N : M)$ or $\hat{a}_i s'm \in N$, where $\hat{a}_i = a_1 \ldots a_{i-1}a_{i+1} \ldots a_n$ for some $1 \leq i \leq n$. If $\hat{a}_i s'm \in N$, then $\frac{\hat{a}_i s'm}{s_1 \ldots s_{i-1}s_{i+1} \ldots s_n s'l} = \frac{\hat{a}_i m}{\hat{s}_i l} \in S^{-1}N$, and if $a_1 \ldots a_n \in (N : M)$, then $\frac{a_1 \ldots a_n}{s_1 \ldots s_n} \in S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$. Therefore, $S^{-1}N$ is an n-absorbing submodule of $S^{-1}M$.
(2) Let $a_1, \ldots, a_n \in R$ and $m \in M$ be such that $a_1 \ldots a_n m \in N$. Then, $\frac{a_1 \ldots a_n m}{1} \in S^{-1}N$. Since $S^{-1}N$ is an n-absorbing submodule of $S^{-1}M$, either $\frac{a_1 \ldots a_n}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$ or $\frac{\hat{a}_i m}{1} \in S^{-1}N$, where $\hat{a}_i = a_1 \ldots a_{i-1}a_{a+1} .. a_n$ for some $1 \leq i \leq n$. Therefore, there exists $s \in S$, such that $s\hat{a}_i m \in N$. This implies $\hat{a}_i m \in N$, since $S \cap Zd(M/N) = \phi$. Now, consider the case when $\frac{a_1 \ldots a_n}{1} \in (S^{-1}N :_{S^{-1}R} S^{-1}M)$, then $a_1 \ldots a_n S^{-1}M \subseteq S^{-1}N$. Now, we have to show $a_1 \ldots a_n M \subseteq N$. Assume that $m' \in M$, and then, $\frac{a_1 \ldots a_n m'}{1} \in a_1 \ldots a_n S^{-1}M \subseteq S^{-1}N$, so there exists $t \in S$, such that $ta_1 \ldots a_n m \in N$. Since $S \cap Zd(M/N) = \phi$, then $a_1 \ldots a_n m' \in N$, and therefore, $a_1 \ldots a_n M \subseteq N$. Hence, N is an n-absorbing submodule of M. □

3 Classical n-absorbing submodules

In this section, we introduce and study the concept of classical n-absorbing submodules as a generalization of n-absorbing submodules.

Definition 3.1 A proper submodule N of an R-module M is called a classical n-absorbing submodule if, whenever $a_1 a_2 \ldots a_{n+1} m \in N$ for $a_1, a_2, \ldots, a_{n+1} \in R$ and $m \in M$, there are n of a_i's whose product with m is in N.

Example 3.2 (1) Let $R = \mathbb{Z}$ and $M = R \times R$. The submodule $N = \{(k, k) : k \in R\}$ is a classical n-absorbing submodule of M.
(2) Let $R = \mathbb{Z}$ and $M = \mathbb{Z}_3 \oplus \mathbb{Q} \oplus \mathbb{Z}$. Take $n = 2$, the submodule $N = \bar{0} \oplus \{0\} \oplus \mathbb{Z}$ is a classical 2-absorbing submodule of M. To see this, let $a, b, c, z \in \mathbb{Z}$, $w \in \mathbb{Q}$ and $\bar{x} \in \mathbb{Z}_3$ such that $abc(\bar{x}, w, z) \in N$. Hence, $\overline{abcx} = \bar{0}$ and $abcw = 0$. If $abcz \neq 0$, then $w = 0$. We have $3|abcx$, then $3|ab$ or $3|cx$, if $3|ab$, then $ab(\bar{x}, w, z) = (\overline{abx}, 0, abz) = (0, 0, abz) \in N$. Similarly if $3|cx$, then $c(\bar{x}, w, z) = (\overline{cx}, 0, cz) = (0, 0, cz) \in N$. Now, if $abcz = 0$, then one of a, b, c, z is zero; first, we take one of the scalars which is zero, say a, then $a(\bar{x}, w, z) = (\bar{0}, 0, 0) \in N$, and hence $ab(\bar{x}, w, z) \in N$. if $a, b, c \neq 0$ and $z = 0$, since $abcw = 0$, then $w = 0$ (this was a previous case). If $a, b, c \neq 0$, $z = 0$ and $w \neq 0$, then $abcw \neq 0$ so $abc(\bar{x}, w, z) \notin N$, a contradiction. Thus, N is a classical 2-absorbing submodule of M.

Proposition 3.3 *Let N be a proper submodule of an R-module M.*

(i) *If N is an n-absorbing submodule of M, then N is a classical n-absorbing submodule of M.*
(ii) *If N is an n-absorbing submodule of M and $(N : M)$ is an $(n-1)$-absorbing ideal of R, then N is a classical $(n-1)$-absorbing submodule of M.*

Proof (i) Assume that N is an n-absorbing submodule of M. Let $a_1, a_2, \ldots, a_{n+1} \in R$ and $m \in M$, such that $a_1 a_2 \ldots a_n a_{n+1} m = a_1 a_2 \ldots a_n (a_{n+1} m) \in N$. Then, either there are $n-1$ of a_i's whose product with $a_{n+1} m$ is in N or $a_1 a_2 \ldots a_n \in (N : M)$. The first case leads us to the claim. In the second case, we have that $a_1 a_2 \ldots a_n m \in N$. Consequently, N is a classical n-absorbing submodule.
(ii) Assume that N is an n-absorbing submodule of M and $(N : M)$ is an $(n-1)$-absorbing ideal of R. Let $a_1 a_2 \ldots a_n m \in N$ for some $a_1, a_2, \ldots, a_n \in R$ and $m \in M$, such that there are no $n-1$ of a_i's whose product with m is in N. Then, $a_1 a_2 \ldots a_n \in (N : M)$, and so, there are $n-1$ of a_i's whose product is in $(N : M)$, which is a contradiction. Hence, N is a classical $(n-1)$-absorbing submodule of M. □

Remark 3.4 The following example shows that the converse of Proposition 3.3(i) is not true. Take $n = 2$, and let $R = \mathbb{Z}$ and $M = \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}$. The zero submodule of M is a classical 2-absorbing submodule, but is not 2-absorbing, since $3.5(1, 1, 0) = (0, 0, 0)$, but $3(1, 1, 0) \neq (0, 0, 0)$, $5(1, 1, 0) \neq (0, 0, 0)$, and $3.5 \notin (0 : \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}) = 0$.

The following theorem characterizes classical n-absorbing submodule in terms of n-absorbing ideals.

Theorem 3.5 *Let M an R-module and N be a proper submodule of M. Then, the followings are equivalent:*

(i) *N is a classical n-absorbing submodule of M.*

(ii) *$(N : m)$ is a n-absorbing ideal of R for every $m \in M - N$.*

Proof $(i) \Rightarrow (ii)$ Assume that N is a classical n-absorbing submodule. $(N : m)$ is a proper ideal, since $m \in M - N$. Let $a_1a_2 \dots a_{n+1} \in (N : m)$ for some $a_1, a_2, \dots, a_{n+1} \in R$. Since N is a classical n-absorbing submodule and $a_1a_2 \dots a_{n+1}m \in N$, then there are n of a_i's whose product with m is in N, and hence, there are n of a_i's whose product is in $(N : m)$. Thus, $(N : m)$ is n-absorbing ideal.
$(ii) \Leftarrow (i)$ Assume that $(N : m)$ is a n-absorbing ideal of R for every $m \in M - N$. let $a_1, a_2, \dots, a_{n+1} \in R$ and $m \in M$ with $a_1a_2 \dots a_{n+1}m \in N$. If $m \in N$, we are done. Assume that $m \notin N$, since $(N : m)$ is a n-absorbing ideal and $a_1a_2 \dots a_{n+1} \in (N : m)$, then there are n of a_i's whose product is in $(N : m)$, and hence, there are n of a_i's whose product with m is in N. Therefore, N is a classical n-absorbing submodule of M. □

Theorem 3.6 *Let M a cyclic R-module and N be a submodule of M. If N is a classical n-absorbing submodule, then N is an n-absorbing submodule of M.*

Proof Let $M = Rm$ for some $m \in M$. Suppose that $a_1a_2 \dots a_nx \in N$ for some $a_1, a_2, \dots, a_n \in R$ and $x \in M$. Then, there exists an element $a_{n+1} \in R$, such that $x = a_{n+1}m$. Therefore, $a_1a_2 \dots a_nx = a_1a_2 \dots a_na_{n+1}m \in N$, and since N is a classical n-absorbing submodule, then there are n of a_i's whose product with m is in N. Since M is cyclic, $(N : m) = (N : M)$; hence, there are n of a_i's whose product with m is in N or $a_1a_2 \dots a_n \in (N : M)$. Thus, N is an n-absorbing submodule of M. □

Now, in the following two corollaries, we characterize the classical n-absorbing submodules in terms of n-absorbing submodules and n-absorbing ideal.

Corollary 3.7 *Let M a cyclic R-module and N be a submodule of M. Then, the followings are equivalent:*

(i) *N is a classical n-absorbing submodule of M.*
(ii) *N is an n-absorbing submodule of M.*

Corollary 3.8 *Let M a cyclic multiplication R-module and N be a submodule of M. Then, the followings are equivalent:*

(i) *N is a classical n-absorbing submodule of M.*
(ii) *$(N : M)$ is an n-absorbing ideal of R.*

Proof Directly by Corollary 3.7 and Proposition 2.4 in [7]. □

Here, in the next theorem, we investigate a submodule to be classical n-absorbing under some conditions.

Theorem 3.9 *Let M an R-module and N be a proper irreducible submodule of M, such that $N_r = N_{r^n}$ for all $r \in R$, and then, N is a classical n-absorbing submodule of M.*

Proof Let $r_1, r_2, \dots, r_{n+1} \in R$ and $m \in N$ with $r_1r_2 \dots r_{n+1}m \in N$, and assume that N is not a classical n-absorbing submodule of M, and so, there are no n of a_i's whose product with m is in N. We have $N \subseteq \bigcap_{i=1}^n (N + R\hat{r}_im)$, where $\hat{r}_i = r_1r_2 \dots r_{i-1}r_{i+1} \dots r_n$. Let $x \in \bigcap_{i=1}^n (N + R\hat{r}_im)$, then $x = x_1 + s_1\hat{r}_nm = x_2 + s_2\hat{r}_{n-1}m = \dots = x_n + s_n\hat{r}_1m$ where $x_i \in N$ and $s_i \in R$ for every i, then $r_1^{n-1}x = r_1^{n-1}x_1 + s_1r_1^{n-1}\hat{r}_nm = r_1^{n-1}x_2 + s_2r_1^{n-1}\hat{r}_{n-1}m = \dots = r_1^{n-1}x_n + s_nr_1^{n-1}\hat{r}_1m$, since ${r_1}^{n-1}x_n, s_n{r_1}^{n-1}\hat{r}_1m \in N$, so $s_1{r_1}^{n-1}\hat{r}_nm \in N$ which implies that $s_1(r_2r_3 \dots r_{n-1})m \in N_{{r_1}^n}$, but $N_{{r_1}^n} = N_{r_1}$, and hence, $s_1\hat{r}_nm \in N$, and so, $x \in N$. Therefore, $\bigcap_{i=1}^n (N + R\hat{r}_im) \subseteq N$; consequently, $\bigcap_{i=1}^n (N + R\hat{r}_im) = N$, a contradiction, because N is an irreducible. Hence, N is a classical n-absorbing submodule of M. □

Theorem 3.10 *Let M an R-module and N be a classical n-absorbing submodule of M, such that $(N : y)$ is a prime ideal of R for $y \in M - N$. For $x \in M$, if $(N : x) - \bigcup_{x_i \in M-N}(N : x_i) \neq \phi$, then $N = (N + Rx) \cap \bigcap_{x_i \in M-N}(N + Rx_i)$.*

Proof Suppose that N is a classical n-absorbing submodule of M. Let $a_1a_2 \dots a_n \in (N : x) - \bigcup_{x_i \in M-N}(N : x_i)$, where $a_1, a_2, \dots, a_n \in R$, then $a_1a_2 \dots a_nx \in N$ and $a_1a_2 \dots a_nx_i \notin N$ for every $x_i \in M - N$. It is Clear that $N \subseteq (N + Rx) \cap \bigcap_{x_i \in M-N}(N + Rx_i)$. For the reverse inclusion, let $n \in (N + Rx) \cap \bigcap_{x_i \in M-N}(N + Rx_i)$, then $n = n' + r'x = n_i + r_ix_i$ for every $x_i \in M - N$, where $n', n_i \in N$ and $r', r_i \in R$. Now, $a_1a_2 \dots a_nn = a_1a_2 \dots a_nn' + a_1a_2 \dots a_nr'x = a_1a_2 \dots a_nn_i + a_1a_2 \dots a_nr_ix_i$ and $a_1a_2 \dots a_nr'x, a_1a_2 \dots a_nn', a_1a_2 \dots a_nn_i \in N$, so $a_1a_2 \dots a_nr_ix_i \in N$. Since N is a classical n-absorbing submodule and $a_1a_2 \dots a_nx_i \notin N$, then there are $n-1$ of a_i's whose product with r_ix_i is in N. Hence, there are

$n-1$ of a_i's whose product with r_i is in $(N : x_i)$. If $x_i \in N$, then $r_i x_i \in N$, and so $n = n_i + r_i x_i \in N$. Assume that $x_i \notin N$, so, by hypothesis, $(N : x_i)$ is a prime, and hence, either there are $n-1$ of a_i's whose product is in $(N : x_i)$ or $r_i \in (N : x_i)$. From the first case, we have $a_1 a_2 \dots a_n x_i \in N$ which is a contradiction. Therefore, $r_i \in (N : x_i)$, and hence, $r_i x_i \in N$. Thus, we have $n = n_i + r_i x_i \in N$, so $(N+Rx) \cap \bigcap_{x_i \in M-N}(N+Rx_i) \subseteq N$. Hence, $N = (N+Rx) \cap \bigcap_{x_i \in M-N}(N+Rx_i)$. □

Corollary 3.11 *Let M an R-module and N be a classical n-absorbing submodule of M, such that $(N : y)$ is a prime ideal of R for $y \in M - N$. For $x \in M - N$, if $(N : x) - \bigcup_{x_i \in M-N}(N : x_i) \neq \phi$, then N is not irreducible.*

Proof By Theorem 3.10, $N = (N+Rx) \cap \bigcap_{x_i \in M-N}(N+Rx_i)$. Since $x \in M - N$, we have $N \subset (N+Rx)$ and $N \subset \bigcap_{x_i \in M-N}(N+Rx_i)$. Thus, N is not irreducible. □

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