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Some remarks on wave solutions in general relativity theory

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Abstract This paper reviews the concept of pp-waves and plane waves in classical general relativity theory. The first four sections give discussions of some algebraic constructions and symmetry concepts which will be needed in what is to follow. The final sections deal with the definitions of such wave solutions, their associated geometrical tensors in space–time and their Killing, homothetic, conformal and wave surface symmetries. Some unusual geometrical features of these solutions (compared with standard positive definite geometry) are described.

Mathematics Subject Classification 83C20 · 83C35

1 Introduction

This paper gives a brief review of the study of classical pp-waves and plane waves as used in general relativity theory. The intention is to provide simple definitions of these features together with alternative equivalent descriptions which may help in forming intuitive ideas about them.

To establish notation M is a 4-dimensional, smooth, connected, Hausdorff manifold admitting a smooth metric g of Lorentz signature so that the pair (M, g) is a space–time. For $m \in M$, T_mM denotes the tangent space to M at m and Λ_m the 6-dimensional vector space of 2-forms (bivectors) at m. For $m \in M$ a *real null tetrad* $l, n, x, y \in T_mM$ is often useful where l and n are null and (using a dot to denote an inner product with $g(m)$) $l.n = 1$ and $x.x = y.y = 1$ with all other inner products between tetrad members zero. A 2-dimensional subspace (2-space) V of T_mM is called *spacelike* if each non-zero member of V is spacelike, *timelike* if V contains exactly two, null 1-dimensional subspaces (*directions*) and *null* if V contains exactly one null direction. A member $F \in \Lambda_m$ has even matrix rank since it is skew self-adjoint. If its rank is 2 it is called *simple* and this occurs if and only if there exists a non-zero $k \in T_mM$ such that $F^a{}_bk^b = 0$ (in fact, exactly two such independent vectors exist). Otherwise it is called *non-simple*. If F is simple it may be written as $F^{ab} = 2p^{[a}q^{b]}$ for $p, q \in T_mM$ (where square brackets denote the usual anti-symmetrisation of the enclosed indices) and then the 2-space determined by p and q is unique and called the *blade* of F. (In this case the bivector and its blade are sometimes denoted by $p \wedge q$). Such an F is then called, respectively, *spacelike*, *timelike* or *null* if its blade is a spacelike, timelike or null 2-space. For $F \in \Lambda_m$ its Hodge dual is denoted $\overset{*}{F}$ and $\overset{*}{F}$ is independent of F and simple if and only if F is.

The Levi-Civita connection arising from g is denoted by ∇ and its associated curvature tensor $Riem$ has components $R^a{}_{bcd}$. From this arise the Ricci tensor $Ricc$ with components $R_{ab} \equiv R^c{}_{acb}$, the Ricci scalar $R = R_{ab}g^{ab}$ and the Weyl tensor C with components $C^a{}_{bcd}$. (The distinction between contravariant and

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covariant indices will sometimes be ignored because of the existence of the metric.) These tensors are related by the equations

$$R_{abcd} = C_{abcd} + E_{abcd} + \frac{R}{6} G_{abcd}, \tag{1}$$

where

$$E_{abcd} = \tilde{R}_{a[c} g_{d]b} + \tilde{R}_{b[d} g_{c]a}, \qquad G_{abcd} = g_{a[c} g_{d]b} \tag{2}$$

and $\tilde{Ricc}$ is the *trace-free* Ricci tensor with components $\tilde{R}_{ab} = R_{ab} - \frac{R}{4} g_{ab}$. The condition $E(m) = 0$ is equivalent to the Einstein space condition $\tilde{R}_{ab} = R_{ab} - \frac{R}{4} g_{ab} = 0$ at m. Einstein's field equations (with the cosmological constant set to zero) are written as

$$G_{ab} \equiv R_{ab} - \frac{R}{2} g_{ab} = 8\pi T_{ab}, \tag{3}$$

where G is the Einstein tensor and T the energy–momentum tensor. For a fixed metric g on M the tensor E determines and is determined by $\tilde{Ricc}$ since $E^c{}_{acb} = \tilde{R}_{ab}$ (and $E = 0 \Leftrightarrow \tilde{Ricc} = 0$). The tensor T is forced to satisfy the energy conditions given at any $m \in M$ by (i) $T_{ab} u^a u^b \geq 0$ and (ii) $T^a{}_b u^b$ is not spacelike, each for any timelike or null $u \in T_m M$.

A space–time (M, g) is called *non-flat* if $Riem$ does not vanish on any non-empty open subset of M, *vacuum* if $T \equiv 0$ on M and *conformally flat* if $C \equiv 0$ on M.

2 Wave surfaces

Let $m \in M$ and consider a null direction spanned by a null vector $l \in T_m M$. A *wave surface* to l (more precisely to the null direction spanned by l) is a *spacelike* 2-space $W \subset T_m M$ every member of which is orthogonal to l. Of course there are infinitely many wave surfaces to l at m and, choosing independent space-like members $x, y \in T_m M$ which span a particular such wave surface W, and writing $W \equiv\ < x, y >$, one can check that any other such wave surface can be written in the form $< x + al, y + bl >$ where $a, b \in \mathbb{R}$ and that different ordered pairs give rise to different wave surfaces. Thus the family of wave surfaces to l at m is in a one-to-one correspondence with $\mathbb{R}^2$ and is denoted by $W_l(m)$. Further, if $u \in T_m M$ is unit timelike, then u, together with l, spans a timelike 2-space whose orthogonal complement is a wave surface to l, say, $W =\ < x, y >$. In this sense u determines W uniquely and W is called the *instantaneous* wave surface to l for u at m. If u' is another unit timelike member of $T_m M$ it will similarly determine its corresponding instantaneous wave surface W' and $W = W'$ if and only if u, u' and l are coplanar (that is, they are confined to a (timelike) 2-space). Thus, given l, one may define an equivalence relation $\sim$ on the set of unit timelike vectors in $T_m M$ by $u \sim u'$ if u, u' and l are coplanar and then, since the orthogonal complement of a wave surface to l is timelike and contains l, the associated quotient space is in a one-to-one correspondence with $W_l(m)$. Then, given l, one may interpret this physically by associating unit timelike vectors with observers and then $u \sim u'$ means that in the local frame of u, u' and l determine the same spatial direction.

The concept of a wave surface W at m was introduced in [7].

3 Algebraic classification of the Weyl and energy–momentum tensors

An algebraic classification of the Weyl and energy–momentum tensors is rather convenient for space–times. For the Weyl tensor the classification required is the usual Petrov classification [23] (as extended by Pirani [24] and Bel [3,4]) and described in terms of the *Petrov types* **I**, **D**, **II**, **N**, **III** and **O**. These types are, of course, algebraic and apply at a point only with type **O** at m meaning that $C(m) = 0$. For the symmetric tensor T the Jordan-Segre classification of the linear map $T_m M \to T_m M$ given by $k^a \to T^a{}_b k^b$ is used. This produces precisely the types given in Segre symbols by $\{1, 111\}$ (diagonalisable over $\mathbb{R}$), $\{11z\bar{z}\}$ (diagonalisable over $\mathbb{C}$), $\{211\}$, and $\{31\}$, together with their degeneracies (denoted by enclosing the degenerate eigenvalues inside round brackets) where, in the first of these Segre symbols, the comma separates off the single eigenvalue corresponding to a timelike eigenvector (and this particular type is characterised by admitting a timelike eigenvector). At any $m \in M$ only the types $\{1, 111\}$ and $\{211\}$ may satisfy the energy conditions (and then only with restrictions on the actual eigenvalues at m) and so the other algebraic types are not needed. [It is remarked that this classification applies to *any* second order symmetric tensor for this dimension and signature and, of course, the two discarded Segre types may then be relevant]. Full details of these two classifications may be found in [14,28]. The Segre types of $Ricc$, $\tilde{Ricc}$ and T at $m \in M$ are identical (including degeneracies) differing only in their eigenvalues (and then only by adding a fixed real number to each eigenvalue). The classification of $\tilde{Ricc}$ may be achieved by a classification of E, the latter being accomplished along similar lines to that used in the Petrov classification of C [14].

It will suffice here to note that, if $m \in M$ and $C(m) \neq 0$, there exists $k \in T_mM$ such that $C_{abcd}k^d = 0$ if and only if the Petrov type of $C(m)$ is **N**. The vector k is unique up to a scaling, necessarily null and called the (repeated) principal null direction for $C(m)$. Similarly, a symmetric second-order tensor $T(m)$ has Segre type $\{(211)\}$ with zero eigenvalue if and only if $T_{ab}(m)$ is a non-zero multiple of l_al_b for some null vector $l \in T_mM$. If T is the energy–momentum tensor and this type holds over some open subset U of M the situation on U is referred to as a "null fluid".

If an electromagnetic field exists on M it is described by the Maxwell (-Minkowski) tensor (bivector) F_{ab} which satisfies Maxwell's equations $F^{ab}{}_{;b} = j^a$ and $F_{[ab;c]} = 0$ where j is the current density vector field and a semi-colon denotes a covariant derivative with respect to ∇. This contributes a term $Q_{ab} \equiv \frac{1}{4\pi}(F_{ac}F_b{}^c - \frac{1}{4}F_{cd}F^{cd}g_{ab})$ to the energy–momentum tensor T and $Q^a{}_a = 0$. If $F(m)$ is a null bivector one may write $F_{ab}(m) = 2\nu l_{[a}x_{b]}$ in some null tetrad at m ($\nu \in \mathbb{R}$) and the electromagnetic field is called an (algebraic) *null (Maxwell) field* at m. Then, at m, Q takes the form $Q_{ab} = \frac{\nu^2}{4\pi}l_al_b$ (with Segre type $\{(211)\}$ and eigenvalue zero). Otherwise $F(m)$ may be written, again in terms of some null tetrad, as $F_{ab} = 2\alpha l_{[a}n_{b]} + 2\beta x_{[a}y_{b]}$ ($\alpha, \beta \in \mathbb{R}$) and then Q takes the form $Q_{ab} = -\frac{1}{8\pi}(\alpha^2+\beta^2)(l_an_b+n_al_b-x_ax_b-y_ay_b)$ which is of Segre type $\{(1,1)(11)\}$ and admits two independent timelike eigenvectors in the $l \wedge n$ $2-space$. This is called an (algebraic) *non-null (Maxwell) field* at m. Thus for a null Maxwell field the so-called Maxwell invariants $F^{ab}F_{ab}$ and $\overset{*}{F}{}^{ab}F_{ab}$ each vanish, whereas for a non-null field they are given by $F^{ab}F_{ab} = 2(\beta^2-\alpha^2)$, $\overset{*}{F}{}^{ab}F_{ab} = 4\alpha\beta$. In the null case and since the Segre type of Q is $\{(211)\}$ there is a unique null direction for Q spanned by l and in *any* (pseudo-)orthonormal frame at m where the components of l are taken to be $(1, n_1, n_2, n_3)$ for a unit 3-vector $\mathbf{n} = (n_1, n_2, n_3)$ one may consult the usual (Minkowski) relations between the components of F and the components of the electric and magnetic fields **E** and **B** in that frame to see that, in 3-dimensional language, $\mathbf{E} = -c(\mathbf{n} \times \mathbf{B})$ and $c\mathbf{B} = \mathbf{n} \times \mathbf{E}$ where c is the speed of light. Thus **E**, **B** and **n** are mutually perpendicular and **E** and **B** span the 2-dimensional "wave front" which is, in the 3-dimensional space geometry for this frame, perpendicular to the spatial direction **n** (cf wave surfaces in Sect. 2). Then F is determined, up to a scaling, by l and an angle of "polarisation" of the wave. Thus null Maxwell fields are taken to represent electromagnetic radiation.

This can be seen in another way due to a result in [24]. Suppose, at some $m \in M$, O is an observer with timelike 4-velocity u. The (spacelike) Poynting vector P of energy–momentum flow for O is given in terms of u and the energy–momentum tensor T by $P_a = (\delta_a^b + u_au^b)T_{bc}u^c$ ($\Rightarrow P_au^a = 0$) and for any instantaneous wave surface carried along by O with normal n orthogonal to the wave surface and to $O's$ worldline the Poynting flow across it is $P_an^a = T_{ab}u^an^b$. If this is zero for all such wave surfaces (that is, for all space-like n satisfying $n.u = 0$, so that O is "following the field" [24]) one finds that u is a (timelike) eigenvector of T. However, for a null electromagnetic field (where one would expect no such observer to exist) the algebraic Segre type of T is $\{(211)\}$ with zero eigenvalue and has *no* timelike eigenvectors and this expectation is confirmed.

One can view this in yet another way. In [8] an asymptotic expansion of a bounded charge/current distribution reveals that the surviving Maxwell bivector at large distances is that of a null Maxwell field. In the gravitational case the so-called "peeling theorem" of Sachs reveals a similar predominance of type **N** fields at large distances [26] (see also [25]). Alternatively, one may consider discontinuities and characteristic surfaces for the Maxwell and Einstein equations to show that null electromagnetic fields and Petrov type **N** gravitational fields appear to play the role of idealised electromagnetic and gravitational radiation fields, respectively [31] (see also [25]).

4 Killing and homothetic symmetry in (M,g)

A global, smooth vector field X on M with local flows ϕ_t is called a *Killing vector field* if the *pull backs* ϕ_t^* satisfy $\phi_t^*g = g$ for each ϕ_t. Then Killing's equations $\mathcal{L}_Xg = 0$ hold on M, where $\mathcal{L}_X$ denotes the Lie derivative along X. For the components of X these give

Table 1 Subalgebras of $o(1, 3)$

Type	Dimension	Basis	Type	Dimension	Basis
R_2	1	$l \wedge n$	R_9	3	$l \wedge n, l \wedge x, l \wedge y$
R_3	1	$l \wedge x$	R_{10}	3	$l \wedge n, l \wedge x, n \wedge x$
R_4	1	$x \wedge y$	R_{11}	3	$l \wedge x, l \wedge y, x \wedge y$
R_5	1	$l \wedge n + \omega x \wedge y$	R_{12}	3	$l \wedge x, l \wedge y, l \wedge n + \omega(x \wedge y)$
R_6	2	$l \wedge n, l \wedge x$	R_{13}	3	$x \wedge y, y \wedge z, x \wedge z$
R_7	2	$l \wedge n, x \wedge y$	R_{14}	4	$l \wedge n, l \wedge x, l \wedge y, x \wedge y$
R_8	2	$l \wedge x, l \wedge y$	R_{15}	6	L

$$X_{a;b} + X_{b;a} = 0 \Leftrightarrow X_{a;b} \equiv F_{ab} = -F_{ba}, \quad F^a{}_{b;c} = R^a{}_{bcd} X^d. \tag{4}$$

Here, F is the *Killing bivector field* of X. The set of all such Killing vector fields on M is a *finite-dimensional* Lie algebra (under Lie bracket) called the *Killing algebra* and is denoted by $K(M)$ ($\dim K(M) \leq 10$). For each $m \in M$ consider the map $m \to D_m$, where $D_m \equiv \{X(m) : X \in K(M)\} \subset T_m M$. Then D is a *generalised distribution on M* (generalised in the sense that $\dim D_m$ may vary with m). Since $K(M)$ is finite-dimensional, D is integrable in the sense that it admits a unique, connected, maximal integrable manifold through each $m \in M$ whose tangent space at m equals D_m [19]. For $m \in M$ one can also consider the set of points that m can reach by continued application of maps ϕ_t (where defined) for any finite collection $\phi^1_{t_1}, \phi^2_{t_2}, \ldots \phi^k_{t_k}$ of local flows of members of $K(M)$, for example,

$$m \to \phi^1_{t_1}\left(\phi^2_{t_2}\left(\ldots \phi^k_{t_k}(m)\ldots\right)\right). \tag{5}$$

Such a set is the connected, maximal, integral manifold above [30] and each is called a (*Killing*) *orbit* for $K(M)$. Although the orbits are submanifolds they are not completely well-behaved but are well-behaved enough for the purposes required here [13,14,29].

An orbit O is called *proper* if $1 \leq \dim O \leq 3$ and *dimensionally stable* if O is proper and given $m \in O$ there exists an open neighbourhood U of m in M such that the dimension of any orbit through any point of U equals that of O [13,14]. Let $V_i \equiv \{m \in M : \dim D_m = i\}$ for $0 \leq i \leq 4$. Then one may disjointly decompose M in the following ways;

$$M = \bigcup_{i=0}^{4} V_i = \bigcup_{i=0}^{4} \operatorname{int} V_i \cup Z = V_4 \cup \bigcup_{i=0}^{3} \operatorname{int} V_i \cup Z, \tag{6}$$

where int denotes the interior in the manifold topology on M (and so, by rank, V_4 is open, hence $\operatorname{int} V_4 = V_4$, and $\operatorname{int} V_0 = \emptyset$ if $K(M)$ is not trivial since, if a Killing vector field vanishes over some non-empty open subset of M, it vanishes on M). It can be shown that the closed set Z, which is defined by the disjointness of the decomposition, satisfies $\operatorname{int} Z = \emptyset$. Thus, for $1 \leq i \leq 3$, $\operatorname{int} V_i$ consists of the totality of i-dimensional, dimensionally stable orbits. It can be shown that any 3-dimensional, spacelike or timelike orbit is necessarily dimensionally stable [13,14].

The set $I_m \equiv \{X \in K(M) : X(m) = 0\}$ is a Lie subalgebra of $K(M)$ called the *isotropy algebra at m arising from* $K(M)$. It is isomorphic to the Lie algebra $\{F^a{}_b(m)\}$ under matrix commutation where $F(m)$ is the Killing bivector of some member of I_m at m, and is a subalgebra of the Lorentz algebra $o(1, 3)$. For this purpose Table 1 lists the non-trivial subalgebras of the Lorentz algebra in bivector form following the notation of [27] and in which ω is a non-zero real number in types R_5 and R_{12}. Thus m is a *zero* of X, $X(m) = 0$, and a *fixed point* of any local flow ϕ_t of X whose domain contains m, $\phi_t(m) = m$. From linear algebra we have, from a consideration of the linear map $K(M) \to D_m$ given by $X \to X(m)$ for each $m \in M$, the result

$$\dim K(M) = \dim D_m + \dim I_m = \dim O_m + \dim I_m, \tag{7}$$

where O_m is the orbit containing m. It can be shown [13] that if $K(M)$ admits a (proper) dimensionally stable orbit O and $m \in O$ the non-zero Killing bivectors in I_m have a common annihilator (that is, there exists $k \in T_m M$ such that $F_{ab} k^b = 0$ for *each* $F \in I_m$ and hence each such F is simple). Any such annihilator k is normal to the orbit. Such subalgebras I_m are necessarily of dimension ≤ 3 and will be called *special* [18].

If $X \in K(M)$ then a consideration of the local flows of X easily shows that $\mathcal{L}_X Riem = \mathcal{L}_X Ricc = \mathcal{L}_X C = \mathcal{L}_X R = 0$.

A global vector field X on M is called a *conformal vector field* if each local flow of X is a conformal transformation, that is, $\phi^* g = \lambda g$ for each ϕ_t, where λ is a smooth function on the domain of ϕ_t. Then one has the conformal Killing equation $\mathcal{L}_X g = \mu g$ for some smooth function μ on M. In the case where μ is a constant function on M, X is called *homothetic* and, of course, it is Killing if $\mu \equiv 0$ on M. A conformal vector field which is not homothetic is called *proper conformal* and a homothetic vector field which is not Killing is called *proper homothetic*. The collection of all conformal (respectively, homothetic) vector fields on M is a Lie algebra under the Lie bracket operation, denoted by $C(M)$ (respectively, $H(M)$) and each is finite-dimensional. They are referred to, respectively, as the *conformal* and *homothetic* algebras for (M, g). Clearly $K(M) \subset H(M) \subset C(M)$. In fact $\dim H(M) \leq 11$ and $\dim C(M) \leq 15$ and if (M, g) is not conformally flat $\dim C(M) \leq 7$. For details, see [14].

For the Lie algebras $C(M)$ and $H(M)$ one has, for $m \in M$, equivalents of the subalgebra I_m for $K(M)$. There are also obvious equivalents of homothetic and conformal orbits in M. If the isotropy algebra I_m arising from $K(M)$ is not trivial it imposes restrictions on the algebraic types of $Ricc(m)$ and $C(m)$ and, in particular, $\dim I_m \geq 3$ implies that $C(m) = 0$. If a *proper* homothetic vector field vanishes at m, *each eigenvalue of* $Ricc$ *and* C *vanishes at* m and so $C(m)$ has Petrov type **N** or **O** and $Ricc(m)$, if not zero, has Segre type $\{(211)\}$ or $\{(31)\}$ with eigenvalue zero in each case [14].

5 pp Waves

A non-flat space–time (M, g) which admits a smooth, global, nowhere zero, covariantly constant, *null bivector field* will be called a pp wave. *This definition will be adopted throughout this paper and occasionally referred to as definition* A. [It is remarked here that a different (non-equivalent) definition of a pp wave is given in [28] where the above assumption regarding the null bivector is replaced by (definition B) the existence of a smooth, global, nowhere zero, covariantly constant, *null vector field*. It will be seen later that definition B is implied *locally* by definition A and that for important special cases the two definitions turn out to be (locally) equivalent. The original definition of a pp-wave was given in [7] and will be related to the above ones later.]

To examine this definition a little further suppose (M, g) is a pp wave (definition A) with global, covariantly constant, null bivector F. Then for $m \in M$ there exists a coordinate neighbourhood U of m, a smooth, spacelike vector field x and a smooth, null vector field l orthogonal to x, both on U, so that $F_{ab} = 2l_{[a}x_{b]}$. Then $F_{ac}F^{cb} = -\alpha l_a l_b$ $(\alpha = x^c x_c)$ is covariantly constant on U and hence l is recurrent on U with $l_{a;b} = l_a q_b$ for some smooth covector field q which, after a back substitution, is seen to be locally a gradient on some possibly reduced coordinate neighbourhood of m, $q_a = \psi_{,a}$, where a comma denotes a partial derivative. Then $e^{-\psi} l$ is covariantly constant. Thus (M, g) satisfies, locally, definition B above. The original definition [7] was as in A but was applied only to the vacuum and null fluid cases and insisted only on a covariantly constant bivector field. This bivector field was then shown to be necessarily null in these cases.

If (M, g) is a pp wave, the metric g may be written in a local coordinate domain about any $m \in M$ with coordinates u, v, x, y, as (see [7] section 2-5.3)

$$ds^2 = H(u, x, y)du^2 + 2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 \tag{8}$$

for some smooth function H. Here, the consequent, covariantly constant, null (co)vector field may be taken as $l_a = u_{,a}$ and $Ricc$ is given by [7]

$$R_{ab} = \tilde{R}_{ab} = \frac{1}{2}(\partial^2 H/\partial x^2 + \partial^2 H/\partial y^2) l_a l_b \tag{9}$$

It follows from (2) that $E_{abcd} l^d = 0$. The Ricci identity $2l_{a;[bc]} = l_d R^d{}_{abc}$ then gives $R_{abcd} l^d = 0$ and so, from (1)

$$C_{abcd} l^d = 0 \tag{10}$$

It follows from (9) that $Ricc$ (and $\tilde{Ricc}$ and T), if not zero, are of Segre type $\{(211)\}$ with their (single) eigenvalue zero and null eigenvector l (a null fluid) and from (10) that (M, g) is of Petrov type **N** (with repeated principal null direction spanned by l) or **O** at each $m \in M$ [Such strong results do not follow from definition B].

It is noted that if one supposes only that (M, g) is non-flat and admits a smooth, global, nowhere zero, covariantly constant, null vector field l (definition B) then from the Ricci identity, $R_{abcd} l^d = 0$ and so from (1)

$$C_{abcd} l^d + E_{abcd} l^d + \frac{R}{6} G_{abcd} l^d = 0. \tag{11}$$

Suppose that either (i) (M, g) is of Petrov type **N** (with repeated principal null direction spanned by l) or **O** at each $m \in M$, so that $C_{abcd}l^d = 0$ on M, *and* the Ricci scalar vanishes on M, $R \equiv 0$, *or* (ii) $Ricc$ is either zero or of Segre type $\{(211)\}$ with eigenvalue zero and null eigenvector l, at each $m \in M$ (so that R_{ab} is proportional to $l_a l_b$ on M). Then if (i) holds (11) shows that $E_{abcd}l^d = 0$ on M. It then follows that at each $m \in M$ either $Ricc = 0$ or $Ricc$ has Segre type $\{(211)\}$ with null eigenvector spanned by l [14] and with zero eigenvalue since $R = 0$. Thus R_{ab} is a multiple of $l_a l_b$. If (ii) holds, $R = 0$ and the first of (2) shows that $E_{abcd}l^d = 0$ on M. Then (11) shows that $C_{abcd}l^d = 0$ and so the Petrov type is either **N** (with repeated principal null direction spanned by l) or **O** on M. Thus with the assumption only of a smooth, global, nowhere zero, covariantly constant, null *vector* field one sees that conditions (i) and (ii) are equivalent. Given that either (i) or (ii) (and hence both) hold on M and noting that the combination $Ricc = 0$ and $C = 0$ cannot occur on any non-empty, open subset of M since then $Riem$ would vanish there, one can take advantage of the decompositions of C (whether zero or type **N** with repeated principal null direction spanned by l) and of E (for R_{ab} a multiple (possibly zero) of $l_a l_b$) to write $Riem$ locally as

$$R_{abcd} = \alpha G_{ab}G_{cd} + \beta \overset{*}{G}_{ab}\overset{*}{G}_{cd} + \gamma\left(\overset{*}{G}_{ab}G_{cd} + G_{ab}\overset{*}{G}_{cd}\right) \tag{12}$$

for smooth functions α, β and γ and smooth bivectors $G_{ab} = 2l_{[a}x_{b]}$ and $\overset{*}{G}_{ab} = -2l_{[a}y_{b]}$ and for l, x, y part of a null tetrad l, n, x, y, on some open neighbourhood V of any $m \in M$. Now $G_{ab}l^b = 0$ and covariantly differentiating (using $l^a{}_{;b} = 0$) gives $G_{ab;c}k^c l^b = 0$ for each $k \in T_m M$ and $m \in V$. A Similar result applies to $\overset{*}{G}$. Thus the complex self-dual bivector $B \equiv G + i\overset{*}{G}$ ($\overset{*}{B} = -iB$) satisfies $B_{ab}l^d = 0$ and $B_{ab;c}k^c l^b = 0$ for each $k \in T_m M$ and $m \in V$. It follows that B is (complex) recurrent, $B_{ab;c} = B_{ab}q_c$ for some complex 1-form q on V. Finally from the Ricci identity for B one can calculate

$$2B_{ab;[cd]}(= B_{ab}(q_{c;d} - q_{d;c})) = B_{eb}R^e{}_{acd} + B_{ae}R^e{}_{bcd} = 0 \tag{13}$$

Thus $q_{a;b} = q_{b;a}$ and so q is locally a complex gradient, $q_a = z_{,a}$. It is then easily checked that (cf [7]) $B' \equiv e^{-z}B$ is a covariantly constant (complex) bivector on V and hence its real and imaginary parts are real covariantly constant null bivectors on V. Thus in the important cases of Petrov type **N** or **O** fields with trace-free energy–momentum tensor T or if T is zero or of the null fluid form, over the appropriate regions, the two definitions of a pp wave are locally equivalent there and so (8) holds. (Incidentally, this corrects a slip in [10]). Then for the metric (8) the complex, self-dual Weyl tensor $\overset{+}{C} = C + i\overset{*}{C}$ (recalling that the left and right duals of C are equal and so $\overset{*}{C}$ can be either of them) takes the following form in terms of the covariantly constant, complex bivector B':

$$\overset{+}{C}_{abcd} = Ae^{i\theta}B'_{ab}B'_{cd} \tag{14}$$

for real numbers A and θ with A chosen positive where it is non-zero and hence $\overset{+}{C}$ is complex recurrent, $\overset{+}{C}_{abcd;e} = \overset{+}{C}_{abcd}(Ae^{i\theta})^{-1}(Ae^{i\theta})_{,e}$.

The original definition of a pp wave was for a non-flat, *vacuum* space–time (M, g) [7] and only a global covariantly constant bivector field was assumed. Then it was shown in [7] that this bivector *must* be null. This can be seen directly by a brief excursion into holonomy theory (see [14,20]). Let F be any non-trivial, global, covariantly constant bivector field, $\nabla F = 0$, on any space–time (M, g). Then the holonomy algebra for (M, g) (which is a subalgebra of $o(1, 3)$—see Table 1) is either R_2, R_4 or R_7 (for F spacelike, timelike or non-simple) and R_3 or R_8 (for F null) [11]. Now a non-flat vacuum space-time has holonomy algebra R_8, R_{14} or R_{15} [12,14] and so it follows that the holonomy algebra is R_8 and that F must be null. The question then was how to generalise the original definition to the general case when the vacuum condition is dropped. The definition adopted here retains the covariantly constant *null* bivector field. Also the original definition and metric (in the vacuum case) in [7] had the property that its holonomy algebra is of type R_8. (Actually the definition in [7] allows for a null fluid and then the holonomy algebra is either R_3 or R_8 [12]). Thus the definition A proposed here retains this feature. [In fact it can be shown that if the definition B is adopted together with the insistence that the holonomy algebra is of type R_3 or R_8 then, again, one recovers, locally, definition A with type R_3 necessarily leading to a non-trivial null fluid. This follows since these holonomy algebras necessarily lead to local covariantly constant null bivectors and to an expression like (12) for $Riem$].

6 Plane waves

A pp wave, in the coordinates of (8) (and regarding these now as global coordinates on the manifold M and with $-\infty < x, y, u, v < +\infty$), *and* with the (complex) function $Ae^{i\theta}$ in (14) a function of u only, is called a *plane wave*. Thus $A = A(u)$ and $\theta = \theta(u)$. (In (14) B', and hence $Ae^{i\theta}$, is determined up to a multiplicative complex constant and so this definition is independent of this scaling.) It can then be shown that, after (possibly) a coordinate transformation in (8) which preserves the latter's general form, the function H may be reduced to

$$H(u, x, y) = a(u)x^2 + b(u)y^2 + c(u)xy, \tag{15}$$

for functions a, b, c of u only. It then follows from (9) that $R_{ab} = d(u)l_a l_b$ for some function $d(u)$. Thus the physical features of a plane wave, represented by the Weyl and energy–momentum tensors, are, in this sense, constant on each "wavefront" given by the submanifolds of constant u. Now the pp wave metric (8) admits, in general, a single nowhere-zero Killing (in fact, covariantly constant) vector field, $l = \partial/\partial v$. However, the plane wave metrics represented by (8) and (15) admit at least a 5-dimensional Killing algebra [7,28]. They are of Petrov type **N** or **O** and have energy–momentum tensor either zero or of the null fluid form. The condition on H in (15) for (M, g) to be vacuum is, from (9), that $\partial^2 H/\partial x^2 + \partial^2 H/\partial y^2 = 0$ (that is, $a + b = 0$) and for (M, g) to be conformally flat is that $\partial^2 H/\partial x^2 = \partial^2 H/\partial y^2$ and $\partial^2 H/\partial x \partial y = 0$ (that is, $a = b$ and $c = 0$).

To describe the symmetries of plane waves it is first noted that for *any* space–time which is *not* conformally flat, $\dim C(M) \leq 7$ [14]. Further, $\dim K(M) \leq 6$ since, otherwise, one would have $\dim I_m \geq 3$ at each $m \in M$ and hence $C \equiv 0$ on M (see the end of Sect. 4). Thus for plane waves in which the Weyl tensor is nowhere zero (and hence of Petrov type **N** everywhere) one has the result $5 \leq \dim K(M) \leq 6$ [7,28]. In addition, a proper homothetic vector field always exists (see, e.g. [14]) whose zeros constitute part of a null geodesic (this will be discussed in more detail later) and so $6 \leq \dim H(M) \leq 7$. Thus for such plane waves one has either $\dim K(M) = 5$ and $\dim H(M) = 6$ (the general situation and where the Killing and homothetic orbits coincide (since the homothetic vector field lies in these orbits) being the 3-dimensional, dimensionally stable, null hypersurfaces of constant u, *or* $\dim K(M) = 6$ and $\dim H(M) = 7$, where the extra Killing vector field cannot lie in the $u =$ constant hypersurfaces (otherwise one would have $\dim I_m = 3$ for each $m \in M$) and so there is a single orbit M. The latter plane waves are called *homogeneous*. For a conformally flat plane wave one again has the above 5-dimensional Killing algebra of vector fields lying in the $u =$ constant hypersurfaces plus another independent, rotation-like, Killing vector field and which also lies in the $u =$ constant hypersurfaces and, in general, there are no other independent Killing vector fields. Thus $\dim I_m = 3$ at each $m \in M$ and the Killing orbits are the 3-dimensional, dimensionally stable and null hypersurfaces of constant u. Again a homothetic vector field exists (whose zeros constitute a null geodesic) and which lies in these orbits. There may again be one more independent Killing vector field (and no others) and which leads to a single Killing orbit (the conformally flat, homogeneous plane waves). Thus in the conformally flat case either $\dim K(M) = 6$ and $\dim H(M) = 7$, or $\dim K(M) = 7$ $\dim H(M) = 8$.

It is remarked that for any Petrov type **N** (non-vacuum) plane wave (M, g) with, say, $R_{ab} = \rho(u)l_a l_b$, for some function $\rho(u)$, the metric $g' = e^{2\sigma(u)}g$ is, for $\sigma(u)$ satisfying $2\ddot{\sigma} - 2\dot{\sigma}^2 = \rho(u)$ with a dot denoting d/du, a vacuum (type **N**) plane wave (see Sect. 7(vi)) conformally related to g and defined on the open subset on which σ is defined. Thus any Petrov type **N** plane wave is locally conformally related to a Petrov type **N** vacuum plane wave.

7 Some special properties of pp and plane waves

The pp and plane wave metrics have many distinctive properties and are often the sole reason why various geometric conjectures, inspired by positive definite metrics, fail in the Lorentz case. A few examples will be given here.

(i) Suppose g and g' are smooth *positive definite* metrics on a 4-dimensional, connected manifold M which are both non-flat, Ricci-flat and conformally related so that $g' = e^{2\sigma}g$ for some smooth real function σ on M. Then σ is a constant function on M. This follows from [16] since now g and g' have the same curvature tensor. However, if g and g' are smooth Lorentz metrics on a 4-dimensional manifold M which are both non-flat, vacuum and conformally related as above then either σ is constant on M or some open subset of M is a pp wave. This is (part of) Brinkmann's theorem [5] (for details see [14,28]).

(ii) Suppose (M, g) is an n-dimensional, connected manifold ($n \geq 2$) with positive definite metric g admitting a proper homothetic vector field X. The vector field X need not admit any zeros but if M is geodesically complete then X necessarily admits a zero [20], say at $m \in M$ that is, $X(m) = 0$. If X is proper homothetic and admits a zero it is necessarily an isolated zero, that is, there exists an open neighbourhood U of m in which m is the only zero of X (see, e.g. [14]). In addition, some neighbourhood of m (which after a possible reduction in size may be chosen to coincide with U) with its induced geometry from g, is *flat* [14,20]. In the case of a space–time, however, the situation is quite different. First, a zero of a proper homothetic vector field may, but need not, be isolated and second, whether isolated or not, some neighbourhood of the zero need not be flat. It was noted above that (non-flat!) plane waves admit proper homothetic vector fields with non-isolated zeros. In fact, quite generally, if (M, g) is a non-flat space–time admitting a proper homothetic vector field with a non-isolated zero at $m \in M$ then some neighbourhood of m is isometric to a plane wave [1,14]. It is remarked that non-flat space–times which are not plane waves but which admit a proper homothetic vector field with an isolated zero also exist [2,14].

(iii) For a 4-dimensional manifold with metric g of arbitrary signature and with $m \in M$ one may define the *sectional curvature function* associated with (M, g) at m as the function which maps any *spacelike or timelike* 2-space at m to the real number

$$\frac{R_{abcd} F^{ab} F^{cd}}{(g_{ac} g_{bd} - g_{ad} g_{bc}) F^{ab} F^{cd}}, \tag{16}$$

where F is any simple bivector with blade equal to the 2-space in question and the definition is clearly independent of the bivector F chosen. The quantity (16) is then the *sectional curvature* of the 2-space. Suppose g and g' are smooth, *positive definite* metrics on a 4-dimensional manifold M. Suppose also that the sectional curvature function for g at m is not a constant function (on all 2-spaces at m) for any $m \in M$ and that g and g' have the *same sectional curvature functions* at each $m \in M$. Then $g = g'$ [22]. Now suppose that g and g' are smooth, Lorentz metrics on a space–time manifold M which have the same sectional curvature function for each $m \in M$ and which is not a constant function (on non-null 2-spaces at m) for any $m \in M$. Then either $g = g'$ or there exists an open subset $U \subset M$ on which g and g' are distinct, conformally related, conformally flat plane waves [9,14,21]. In fact, for any given conformally flat plane wave g one can always find, locally, such a distinct conformally related plane wave g'.

(iv) Suppose l is an affinely parametrised, null, geodesic vector field on (M, g) and that (M, g) has energy–momentum tensor either zero or of Segre type $\{(211)\}$. Then interpreting l, physically, as some kind of null beam, one can associate with l the *optical scalars*, *expansion* (denoted by θ), *shear* (σ) and *twist* (ω) and which describe the change in size, shape and orientation of a shadow (made by interposing a small circular disc contained in a wave surface to l in front of the beam) as the shadow is carried along through successive wave surfaces to l by the beam. These optical scalars are then independent of the wave surfaces used and describe physical properties of the beam. This so-called "shadow experiment" has been fully described in [7,25,26]. For such a beam a special case is when $\theta = \sigma = \omega = 0$. But for such special cases there is another optical scalar, the *rotation* of the beam, denoted τ. This extra scalar in a sense describes the change in direction of the beam and vanishes if and only if some scaling of l is covariantly constant [7,25,26] which is equivalent (Sect. 5) to (M, g) being, locally, a pp wave.

(v) Suppose g and g' are Lorentz metrics on a space–time manifold M with respective Levi-Civita connections ∇ and ∇'. Suppose also that (M, g) is a vacuum space–time whose curvature tensor is nowhere zero and (recalling Einstein's geodesic principle) that the non-null geodesic paths of ∇ and ∇' coincide. Then [15] (see also [23]) $\nabla = \nabla'$ on M (and so (M, g') is also vacuum) *and* either $g' = g$ *or* about any point $m \in M$ there is an open coordinate neighbourhood U on which there is defined a non-trivial, null (for both g and g') covariantly constant vector field l^a and $g'_{ab} = a g_{ab} + b l_a l_b$ where a and b are constants. Thus (Sect. 5 and cf [14]) g and g' are each, locally, pp waves.

(vi) Consider a pp wave g in the local coordinates and domain given by U in (8) and let $g' \equiv e^{2\sigma(u)} g$ be a metric on U. Then g' is also, locally, a pp wave (on U). To see this (and in the above notation) note that since $l_a \equiv u_{,a}$ is covariantly constant with respect to g and using $|$ to denote a covariant derivative with respect to (the Levi-Civita connection of) g', one gets $l_{a|b}$ is a multiple of $l_a l_b$ and $R'^a{}_{bcd} u_{,a} = 0$ where $R'^a{}_{bcd}$ are the components of the curvature tensor for g'. (Formulae in [28] section 3.7 are useful here). It follows that l_a may be scaled to be covariantly constant with respect to g' on some open coordinate neighbourhood of any point of U. In addition, the conformal relationship between g and g' shows, since g is a pp wave, that the Ricci tensor for g' is either zero or of the null fluid form and that the Petrov types of g and g' are equal, on U. This completes this proof. Further if g above is a plane wave (with the usual domain above) so also, locally, is g'. To see this, briefly, note that g admits a proper homothetic vector field Z, $\mathcal{L}_Z g = cg$, for some non-zero constant c, whose zeros are not isolated and which is orthogonal to the covariantly constant, null vector field l. A short calculation then shows that $\mathcal{L}_Z g' = cg'$ since $Z^a u_{,a} = 0$ and so Z is a proper homothetic vector field with respect to g' whose zeros are not isolated. The result now follows from Sect. 7(ii) above. Again for plane waves it is clear that the 5- (or 6-) dimensional vector space W of Killing vector fields orthogonal to l (that is, lying in the hypersurface of constant u) for g constitute a Lie subalgebra of $K(M)$ for g since, for X, Y in this subspace with $X.l = Y.l = 0$, the Lie bracket $[X, Y]$ is a Killing vector field orthogonal to l because l is covariantly constant. It is easy to check that the members of W are Killing for g' since each is g-orthogonal to l. If an extra Killing vector field exists for g (and it may not) it is certainly in the conformal algebra for g' (since g and g' are conformally related) but not necessarily Killing for g' and does not lie in the hypersurfaces of constant u. Thus, in this sense, and with the usual "local" clauses, the above conformal change of g to g' converts a homogeneous plane wave into a non-homogeneous plane wave g' (and vice versa [10]). If g (and hence g') is Petrov type **N** no further conformal symmetries may then exist for either g or g' (Sect. 6). Thus one gets an example of a non-conformally flat space–time admitting the maximum dimension (= 7) for $C(M)$ (Sect. 4). In the vacuum case there is, with a stretch of the imagination, a somewhat related result. Suppose that (M, g) is a vacuum space–time whose curvature tensor is nowhere zero on M and which admits a *proper conformal* vector field X. Then each $m \in M$ admits an open neighbourhood on which g restricts to a (vacuum) pp wave. Further, X is unique in the sense that if Y is any other proper conformal vector field on M for g, some linear combination of X and Y is homothetic (or Killing) [14]. Thus, for such space–times, $\dim C(M) \leq \dim H(M) + 1$.

8 Isotropies and wave surfaces

In this section another definition of a plane wave will be given. It is slightly more general than that given in Sects. 5 and 6 in that it includes not only those contained in (8) and (15) but also the Petrov type **N** pure radiation fields with non-zero cosmological constant given in [6] (see also [28]), and only these.

Let (M, g) be a space–time which is non-flat and which admits a Killing algebra $K(M)$ such that at each $m \in M$ there exists a *unique* null direction spanned by a null vector $l' \in T_m M$, called the *wave direction*, with the property that the transformations ϕ_{t*} arising from all members of the isotropy algebra I_m from $K(M)$ are *transitive* on the set $W_{l'}(m)$ of wave surfaces to l' at m for each $m \in M$. Thus I_m is not trivial and given two such wave surfaces $W', W'' \in W_{l'}(m)$ some ϕ_{t*} arising from I_m maps W' to W''. This assumption guarantees the (metric) indistinguishability of the wave surfaces to l', that is, the indistinguishability of the instantaneous wave surfaces to l' for all observers at m for each $m \in M$ (Sect. 2). It is also assumed, in order to remove any pathological cases which may arise (but which, however, can be handled without any serious effect on the result), that all Killing orbits associated with $K(M)$ are either 4-dimensional or (proper and) dimensionally stable and that no $m \in M$ satisfies the "constant curvature" condition $R_{abcd}(m) = \frac{R(m)}{6} G_{abcd}(m)$ (see (1)).

It is first noted and easily checked that (i) if $W \subset W_{l'}(m)$ and $k \in W$, there exists $W' \subset W_{l'}(m)$ such that $k \notin W'$ and (ii) if $W \subset W_{l'}(m)$ and $k \in T_m M$ with k orthogonal to each member of W (but *not* proportional to l'), there exists $W' \subset W_{l'}(m)$ such that some member of W' is not orthogonal to k. From these remarks one can show that if $k \in T_m M$ and is *not* proportional to l' the members of I_m cannot *all* fix the *direction* of k. This follows easily by considering separately the cases $k.l' = 0$ and $k.l' \neq 0$. Thus the only possible candidates for I_m are those which either fix exactly one (necessarily null and spanned by l') direction or those which fix no directions, at m. One can now consult Table 1 (taking exponentiations) to see that I_m cannot be one of the types R_2, R_3, R_4, R_5, R_6, R_7, R_{10} or R_{13} and that $R_{15} \equiv o(1, 3)$ is ruled out since it is transitive on the wave surfaces to *all* null directions at m. Thus the only possibilities for I_m are the subalgebras R_8, R_9, R_{11}, R_{12} and R_{14} (with l' replaced by l) and so $2 \leq \dim I_m \leq 4$. Now each of these possibilities *fixes the null direction l* (and l is unique in this respect) and is certainly transitive on $W_l(m)$ (since R_8 is and the others contain R_8 as a subalgebra) and give the solution to this part of the problem.

Now define the subsets $M_k = \{m \in M : \dim I_m = k\}$ of M so that from the last result $M = M_2 \cup M_3 \cup M_4$. Thus from (7) M_k is a union of orbits of dimension ($\dim K(M) - k$) and the rank theorem reveals that M_2 and $M_2 \cup M_3$ are open in M. Now suppose that O is a proper (and by assumption, dimensionally stable) orbit in M and so $1 \leq \dim O \leq 3$. If $\dim O \leq 2$ it is easily (and quite generally) shown [13,14] that for $m \in O$ one gets the contradiction $\dim I_m \leq 1$. So any proper orbit O has dimension 3. Further, since O is dimensionally stable and given that $m \in O$, it follows ([13,14] and see Sect. 4) that I_m is special, with common annihilator normal to the orbit O. Thus, from the above result, I_m is either isomorphic to R_8 or R_{11}, and the common annihilator is (proportional to) the wave direction l (see Table 1); hence the *orbits are null*. For a general $m \in M$ if I_m is isomorphic to either R_9, R_{12} or R_{14}, then (Sect. 4) $C \equiv 0$ and ([14] page 302) (M, g) satisfies the Einstein space condition at m. Thus (Sect. 1) one gets the contradiction that the "constant curvature" condition holds at m. It follows that any orbit is either 3-dimensional and null, or 4-dimensional and that I_m is always isomorphic to R_8 or R_{11} at each $m \in M$. From (7) one finds $\dim K(M) \geq 5$ and $V_0 = \emptyset$. Further, since all the 3-dimensional orbits here are (null and) dimensionally stable, the subset V_3 (and also, necessarily, V_4) is open (and $V_1 = V_2 = \emptyset$). Hence $M = V_3 \cup V_4$ and since M is connected, either $M = V_3$ *or* $M = V_4$.

So it has been proved that either:

(**A**) $M = V_4$ admits a single 4-dimensional orbit and is hence homogeneous and that I_m is either isomorphic to R_8 at each $m \in M$ ($\Rightarrow \dim K(M) = 6$) or I_m is isomorphic to R_{11} at each $m \in M$ ($\Rightarrow \dim K(M) = 7$), *or*

(**B**) $M = V_3$ and each orbit is 3-dimensional and null with normal along the wave direction everywhere and that I_m is either isomorphic to R_8 at each $m \in M$ ($\Rightarrow \dim K(M) = 5$) or I_m is isomorphic to R_{11} at each $m \in M$ ($\Rightarrow \dim K(M) = 6$).

The remainder of the computation is as given in [17,28]. First it can be calculated that in case **B**, $R = 0$, a (local) covariantly constant, null vector field is admitted and, since I_m is of type R_8 or R_{11} at each $m \in M$, the energy–momentum tensor is either zero or that of a null fluid on M [14]. If $\dim K(M) = 5$, I_m is isomorphic to R_8 at each $m \in M$ and either $C(m) = 0$ or the Petrov type is **N**, on M whilst if $\dim K(M) = 6$, I_m is isomorphic to R_{11} at each $m \in M$ and $C \equiv 0$ on M. The resulting space–time (M, g) is locally a non-homogeneous plane wave. In case **A** one has the possibility that R is non-zero (and necessarily constant) and then the metrics of Defrise are obtained locally with $\dim K(M) = 6$ and I_m isomorphic to R_8 and the metrics are homogeneous, Petrov type **N** pure radiation fields with non-zero cosmological constant [6]; they admit no covariantly constant vector fields. Otherwise, for case **A**, $R = 0$, and either I_m is isomorphic to R_8, $\dim K(M) = 6$ and the Petrov type is **N** or **O** on M, *or* I_m is isomorphic to R_{11}, $\dim K(M) = 7$ and the Petrov type is **O** on M. In each case a (local) covariantly constant null vector field is admitted and homogeneous plane waves result locally. For case **A** the splitting into $R = 0$ and $R \neq 0$ is equivalent to the splitting $\tau = 0$ and $\tau \neq 0$ (see Sect. 7(iv) and [28]). Thus the alternative definition given above regarding transitivity on wave surfaces reproduces locally all plane waves satisfying (8) and (15) together with the special cases given in [6], and only these. (It is clear that the plane waves (8) and (15) have this wave surface transitivity property).

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