Analysis and Mathematical Physics (2022) 12:55
https://doi.org/10.1007/s13324-022-00674-w

Quadrature identities with a background PDE

Layan El Hajj[1] · **Henrik Shahgholian**[2]

Received: 14 February 2022 / Revised: 24 February 2022 / Accepted: 25 February 2022 /
Published online: 18 March 2022

Abstract
Our prime goal with this text is to introduce a nonlinear version of quadrature identities, related to semilinear PDEs, and discuss a few basic properties.

Keywords Nonlinear Quadrature domains · Existence · Uniqueness · Symmetry · Semilinear elliptic PDE

1 Introduction

1.1 Background

For any given (bounded) open set $D \subset \mathbb{R}^n$ $(n \geq 2)$, let $HL^1(D)$, and $SL^1(D)$ denote the set of integrable harmonic, respectively subharmonic functions over D.

It is known (see [13], [15]) that for a given positive function $f \geq c_0 > 0$, and a non-negative bounded function (generally a measure) μ, concentrated enough, one can find a bounded open set $D \Supset \text{supp}(\mu)$ such that the quadrature identity (inequality) holds

$$\int_D h(y)(f(y) - \mu)dy \geq 0, \qquad \forall\, h \in SL^1(D). \tag{1.1}$$

Here, the open set D, with this property, is called a Quadrature Domain.[1] Now, if a QI (for the harmonic class)

H. Shahgholian was supported by Swedish Research Council. We would like to thank Björn Gustafsson for valuable comments. Harold S. Shapiro, in memoriam.

✉ Henrik Shahgholian
henriksh@kth.se

1 American University in Dubai, Dubai, United Arab Emirates

2 KTH Royal institute of Technology, Stockholm, Sweden

[1] The word domain is obviously misused here, since D may not be connected. But the terminology is widely used.

$$\int_D h(y)(f(y)-\mu)dy = 0, \qquad \forall\, h \in HL^1(D), \tag{1.2}$$

is given then, a priori it is not obvious whether the triple (μ, f, D) also constitutes a quadrature identity for the class $SL^1(D)$. This depends on the fact that generally the theory of quadrature domains does not offer us uniqueness results. We refer to [7], [8] for further background.

Consider now a hybrid quadrature identity which is modeled by adding a boundary integral to the left hand side of (1.2)

$$\int_D h(y)(f(y)-\mu)dy + \int_{\partial D} \lambda_0 h(y) d\sigma_y = 0, \qquad \forall h \in H(\overline{D}), \tag{1.3}$$

where $\lambda_0 \geq 0$ is a given constant (or a smooth function) and, to avoid technicalities, we have reduced the class to $H(\overline{D})$, which denotes harmonic functions over $\overline{D}$.

Equation (1.3) in terms of Newtonian kernels $K(x)$ is expressed as

$$U^{f\chi_D - \mu, \lambda_0}(x) = 0 \qquad \forall\, x \in D^c. \tag{1.4}$$

Here for any functions (or bounded measures) $a, b \geq 0$ we have used the notation

$$U^{a,b}(x) = U^{a,b,D}(x) = \int_D a(y)K(x-y)dy + \int_{\partial D} b(y)K(x-y)d\sigma_y,$$

where $K(x) = c_n|x|^{2-n}$, and in $\mathbb{R}^2$ we have $K(x) = c_2 \log|x|$; i.e., the fundamental solution to the Laplace operator normalised such that $\Delta K(x) = -\delta_0(x)$.

By standard approximation theory (see [14]) one can show the equivalence between (1.3) and (1.4).

1.2 Nonlinear quadrature identities

We shall now introduce a nonlinear version of quadrature identities, related to semilinear problems. To do so, let us consider a bounded domain D, and a solution u to the following (background) PDE[2]

$$\Delta u = g(x,u) \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{1.5}$$

where, unless otherwise stated, $g(x,u) = g_2(x,u) - \mu$, with $g_2(x,u) \approx c_0(u_+)^a$ for $u \approx 0$ with $-1 < a < 1$, and $\mu \geq 0$ is a bounded function (or a Radon measure) with compact support. The equation above is in the sense of distributions.

Suppose now that for $\lambda_0 \geq 0$, the following quadrature identity holds

$$\int_D g(y,u(y))h(y)dy + \int_{\partial D} \lambda_0 h(y) d\sigma_y = 0, \qquad \forall h \in H(\overline{D}). \tag{1.6}$$

[2] An equivalent way of definition, of the concept of nonlinear QD, would be the consideration of the Neumann problem $\partial_\nu u = \lambda_0$, on ∂D for $\lambda_0 \geq 0$. Then, $u = U^{g(\cdot,u)\chi_D, \lambda_0} + C_0$, for some C_0.

Using the Newtonian kernel $K(x)$ as in (1.4), we have

$$\Delta U^{g(\cdot,u)\chi_D,\lambda_0} = g(x,u) \quad \text{in } D,$$

and

$$U^{g(\cdot,u)\chi_D,\lambda_0}(x) = 0, \qquad \partial_\nu U^{g(\cdot,u)\chi_D,\lambda_0} = \lambda_0 \quad x \in \partial D, \tag{1.7}$$

where ν is the inward unit normal vector on ∂D.

From (1.5)–(1.7) we conclude $u = U^{g(\cdot,u)\chi_D,\lambda_0}$. Therefore u solves the semilinear free boundary problem

$$\Delta u = g(x,u) \quad \text{in } D, \qquad u = 0, \quad \partial_\nu u = \lambda_0 \quad \text{on } \partial D. \tag{1.8}$$

By Green's theorem the converse also holds true. I.e., if D admits a solution to (1.8), then D is a quadrature domain in the sense of (1.6).

The classical QI (1.2) in light of this new definition has the background PDE

$$\Delta u = f - \mu \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D,$$

where $f - \mu$ plays the role of $g(x, u(x))$, and $\lambda_0 = 0$.

Definition 1 (Nonlinear QD) We say a bounded open set D is a QD for the class $H(\bar{D})$, and with respect to the pair $(\lambda_0, g(x,u))$ if u is a solution to equation (1.5) and the QI (1.6) is satisfied. We call u the background potential, and equation (1.5) the background PDE.

As just discussed the equivalent definition for a nonlinear QD is given by the semilinear free boundary problem (1.8).

Remark 1 It is noteworthy, that even though our definition of nonlinear QD may seem slightly unorthodox, for QD-community, such types of considerations/definitions are common practices in shape optimization problems; see [3] and the references therein. In domain variational approach for shape optimization one gives a domain and a solution to a PDE inside the domain, with boundary values (usually zero), and asks to minimize certain functionals among all such domains. The optimal shape is then a solution to a free boundary problem, where the free boundary condition depends on the functional to be minimized.

Example 1 We mention some examples of $g(x,u)$ that are connected to standard free boundary problems.

a) For $-1 < q < 1$, and μ concentrated enough (say an smooth approximation of Dirac masses) one can take

$$g(x,u) = u^q - \mu, \qquad \lambda_0 = 0, \qquad u \geq 0,$$

which resembles semilinear free boundary problem, that has been well studied; see [2], and [4].

b) The well-known Pompeiu problem[3] with $\mu = 0$, and

$$g(x,u) = -au + 1, \qquad \lambda_0 = 0.$$

c) Schiffer's conjecture[4] (relates also to Pompeiu problem) with $\mu = 0$

$$g(x,u) = -au, \qquad \lambda_0 > 0.$$

In b)-c) u may change sign, and thus the famous conjectures of solutions to b)-c) being spherical, becomes very difficult.

2 Main results

In this section we shall state the main results, that concern the nonlinear quadrature domains. Our intention is primarily the introduction of the concept and statement of some basic facts, that link to already existing results for the standard quadrature domains.

Therefore, rather than getting involved with various technical statements and tools, we shall state and prove existence, in a general framework, and prove standard uniqueness results, along with a symmetry problem for quadrature domains. The latter is also new in the context of standard QD. These results, in the eyes of experts, are obvious but we hope it can tease the appetite of others new to the topic.

It is noteworthy, that the topic of quadrature domains with a Helmholtz PDE background arises in non-scattering phenomena in recent literature. This partly motivates our work here. We also refer to recent works [10], [11], [16] for further background in terms of minimization and partial balayage in potential theory.

To state an existence theory we need to define the corresponding functional, whose minimizers give us possible solutions to the quadrature domain problem, in terms of the PDE (1.8).

Define $G(x,u)$ to be the function satisfying $\frac{\partial G(x,u)}{\partial u} = g(x,u)$ and assume $\lambda_0 \geq 0$,[5] and

$$J(u) = \int_{\mathbb{R}^n} |\nabla u|^2 + 2G(x,u) + \lambda_0^2 \chi_{\{u>0\}}, \tag{2.1}$$

with its admissible class

$$\mathbb{K} = \{u \in W_0^{1,2}(\mathbb{R}^n) : u \geq 0\}.$$

Standing Assumptions (on G, and g): To assure the existence of minimizers, we would need the functional to be bounded from below. Indeed, even simple examples such as $G(x,u) = au + bu^2$ with $|b|$ large may cause the functional not to be bounded

[3] See https://www.scilag.net/problem/G-180522.1

[4] https://www.scilag.net/problem/P-180522.1

[5] Here λ_0 may also be a function of both x and u, but we shall for simplicity assume it is a constant.

from below; see e.g. [11]. Hence the simplest way to avoid this technical point of view would be to assume

$$\int_{\mathbb{R}^n} |\nabla u|^2 + 2G(x,u) \geq -C > -\infty, \qquad \text{for all } u \in \mathbb{K}.$$

This encompasses a large class of functions $G(x,u)$, including classical QD, and also the Helmholtz QD, for small frequencies (see [11]).

We further assume that for some positive constants $0 < c_1 < c_2 < \infty$, and a bounded continuous function $0 < c_0 < c_0(x) < c_0^{-1}$, the function $g = g_2 - \mu$ satisfies

$$g_2(x,t) = c_0(x)t_+^a, \quad -1 < a < 1, \quad t \approx 0,$$
$$c_1 t^a \leq g_2(x,t) \leq c_2 t^a \quad \text{for } t > t_1 > 0, \text{ where } t_1 \approx 0,$$

and

$$\mu \geq 0, \qquad \text{is a bounded function with compact support.}$$

These assumptions, in particular, imply that

$$G(x,u) = G_2(x,u) - \mu u, \qquad \text{where } \partial_u G_2 = g_2. \tag{2.2}$$

The assumption on g_2 (which plays the role of a sink) falls under conditions considered by [2], which forces the support of the minimizer to be bounded, given that μ (the source) has compact support.

We further assume the following concentration condition for μ: For each $z \in \mathrm{supp}(\mu)$, there is a radius $r > 0$, such that

$$u_r(z) > 0, \qquad \text{for any minimizer } u_r \text{ of } J_r \tag{2.3}$$

with

$$J_r(v) = \int_{\mathbb{R}^n} |\nabla v|^2 + 2G_r(x,v) + \lambda_0^2 \chi_{\{v>0\}}, \quad G_r(x,v) = G_2(x,v) - v\mu\chi_{B_r(z)}. \tag{2.4}$$

Theorem 1 *(Existence) For the functional J, under Standing Assumption above and with $\lambda_0 \geq 0$, there is a minimizer u to J solving (1.8), with $D = \{u > 0\}$. Moreover, if $\lambda_0 = 0$, and condition (2.3)-(2.4) is satisfied for some $B_r(z)$ then $z \in \{u > 0\}$. In particular if this holds for all $z \in supp(\mu)$, then $supp(\mu) \subset D$.*

This theorem may be sharpened substantially using Sakai's concentration principle but (probably) would need more restrictions on $g(x,u)$, see [7], [15]. Our assumptions somehow suggests a localization property of concentration of the source versus that of the sink.

When $\partial_u G$ is non-Lipschitz, e.g. $\partial_u G(x,u) \approx u^a$ $(-1 < a < 0)$, or $\lambda_0 > 0$, one would require further restrictions on the measure μ, along with tools from geometric measure theory to obtain results that hint towards $\mathrm{supp}(\mu) \subset D$. This is outside the scope of this paper.

We now state a uniqueness result for the PDE formulation (1.8) of the quadrature domain.

Theorem 2 *(Uniqueness) Let u be a non-negative solution to (1.8), or equivalently, let D be a quadrature domain according to (1.6), with background PDE (1.5), and with non-negative potential $U^{g(\cdot,u)\chi_D,\lambda_0}(x) \geq 0$. Suppose further*

$$g_2 \geq 0, \qquad \partial_u g_2(x,u) \geq 0. \tag{2.5}$$

For $\lambda_0 = 0$ assume D is solid,[6] and for $\lambda_0 > 0$ assume D is convex and ∂D is $C^{1,dini}$.

Then the following hold:

(1) D is the largest QD, satisfying the QI (1.6). More precisely, if there is another QD, D_1 with the background potential u_1, then $D_1 \subset D$ and $u_1 \leq u$.
(2) Moreover, in case $\lambda_0 = 0$ we have D is unique (i.e $D_1 = D$), and in case $\lambda_0 > 0$ the domain D is unique among all convex QD, with $C^{1,dini}$ boundary.

Our next result is a symmetry problem, that is enforced through imposing an extra geometric condition on the QI (1.6), for a particular $g(x,u)$. Similar problems, for potentials, have been considered earlier in the literature. See e.g., [6], and for Bessel potentials see [9], and [12], where the authors, under certain mild assumptions, show the only bounded domains whose Bessel potential is constant on the boundary are balls.

To state our rigidity theorem, we need certain assumptions on the function $g_2(t)$, that falls within the range of our earlier work [4].

Further assumptions on g:
For the next theorem we assume $g(x,u) = g_2(u) - g_1(u)\chi_{D_1}$, for some bounded domain D_1. We also assume that for some $-1 < a < 1$, $\kappa_0 > 0$ and $t_1 > 0$ (small) g has the following properties:

$$\begin{cases} a) & C_t := \limsup_{\epsilon > 0} \frac{g(t+\epsilon)-g(t)}{\epsilon} \leq \kappa_0 t^{a-1}, \quad t > 0, \\ b) & g_2(t) = 0 \qquad \text{for } t \leq 0, \\ c) & |g(t)| \leq C_0 \qquad \text{for all } t > t_1, \\ d) & g_i(t) \geq 0 \qquad i = 1,2. \end{cases} \tag{2.6}$$

Finally when $-1 < a < 0$ and $\lambda_0 = 0$, we also assume the following asymptotic expansion for u: At any free boundary point x^0 the weak second-order asymptotic expansion holds

$$u(x) = A_0((x - x^0)\cdot \nu_0)_+^\beta + O(|x - x^0|^{2+\delta_\beta}), \tag{2.7}$$

[6] D is called solid if it is bounded, $int(\bar{D}) = D$, and $\bar{D}^c$ is connected.

for some fixed $A_0 > 0$, a unit normal $\nu_0 = \nu_0(x^0)$, and some $0 < \delta_\beta < \beta - 1$, with $\beta = 2/(1-a)$; see discussion in [4].

Theorem 3 *(Symmetry and Rigidity) Let $\lambda_0 = 0$, and D be a bounded C^1-domain, which is a QD, in the sense of Definition 1, with background PDE (1.5) such that*

$$g(x, u) = g_2(u) - g_1(u)\chi_{D_1}, \qquad D_1 \Subset D, \tag{2.8}$$

where $u \geq 0$, the inclusion D_1 is $C^{1,dini}$ domain, and condition (2.6) is satisfied. Assume also (2.7) is satisfied, on D when $a < 0$. Suppose further that there is a constant $m > 0$ such that

$$U^{g\chi_D, 0}(x) = m, \qquad \textit{for all } \ x \in \partial D_1. \tag{2.9}$$

Then D and D_1 are concentric balls, and $U^{g\chi_D,0}(x)$ is spherically symmetric around the center of D.

We remark that similar results can be obtained when $\lambda_0 > 0$, provided one assume that ∂D is C^2, and u is C^2 in a neighborhood, and up to the the free boundary. For clarity of exposition we have not included this in the statement of the theorem. The reader may easily deduce such a result.

3 Proofs of main theorems

3.1 Proof of Theorem 1

The proof of this theorem follows in the same spirit as that of Theorem 1.4 in [7] (for classical setting), and [11] for the case of Helmholtz operator. We shall sketch in a few lines the steps needed, leaving out the details to readers to check out.

The approach for existence of a minimizer uses a few steps, the first being a comparison argument, showing that smaller λ_0, and g (i.e., larger μ, and smaller g_2) gives rise to larger solutions with larger support; see Lemma 1.1 in [7]. Using this comparison argument, one can compare the minimizer of the functional to another functional, with symmetric ingredients and such that $G^*(x, \cdot) \leq G(x, \cdot)$. Here G^* is a symmetrization in the x variable of G. Next one can use symmetric rearrangement to show that solutions have compact support for the corresponding symmetric functional. This part usually is based on explicit computation. From here one may (using again the comparison argument) consider a restriction of the admissible class to those with compact supports.

Next by lower-semicontinuity, boundedness of $J(u)$ from below, and Fatou type lemma we shall obtain a global minimizer u to the functional, with support $D = \{u > 0\}$ being compact.

When $\lambda_0 = 0$, one easily checks that a minimizer of this functional over $\mathbb{K}$ satisfies equations (1.5) and (1.6).

Now, let z be the point as stated in the theorem, and u_r the minimizer obtained in (2.3)–(2.4). By Lemma 1.1 in [7], we know that $\max(u, u_r)$ is also a minimizer of the function J, and hence $u \geq u_r$. Hence $u(z) > 0$. This completes the proof of the theorem. □

It is noteworthy that in the second statement of Theorem 1, the assumption (2.3)–(2.4) is equivalent to the PDE

$$\exists\, h: \qquad \Delta h = g(x,h) \text{ and } h > 0 \quad \text{in } B_r(z), \qquad h = 0 \quad \text{on } \partial B_r(z), \tag{3.1}$$

admitting a solution h, with $h(z) > 0$. Then $u(z) > 0$. We leave the obvious proof to the readers.

E.g., for classical QD theory, that is when $g(x,h) = 1-\mu$, it suffices that $\mu = A\chi_S$, with $A > 1$. Then for $B_r(z) \subset \text{supp}(\mu)$ we have that the PDE above has a positive solution $h = (A-1)(r^2 - |x-z|^2)/2n$. Hence $B_r(z) \subset \{u > 0\}$, and hence $\text{supp}(\mu) \subset \{u > 0\}$.

3.2 Proof of Theorem 2

Aiming at a contradiction, suppose there exists another QD, D_1 with potential u_1 satisfying

$$\Delta u_1 = g(x, u_1) \ \text{ in } D_1, \quad u_1 = 0 \ \text{ on } \partial D_1$$

and

$$U^{g(\cdot,u_1)\chi_{D_1},\lambda_0}(x) = 0, \qquad x \in D_1^c.$$

The aim is to prove that $D_1 \subset D$. Define $w := u_1 - u + \lambda_0\,(d(x) - d_1)$, where

$$d(x) = \text{distance}(x, D), \qquad d_1 = \sup_{D_1} d(x).$$

Since $u \geq 0$, we obviously have $w \leq 0$ on ∂D_1. When $\lambda_0 > 0$, by the assumption that D is convex, we have $d(x)$ is subharmonic in $\mathbb{R}^n$, and $\Delta d(x) \geq \mathcal{H}^{n-1}_{\partial D}$ (Hausdorff measure on ∂D). The function u (after extending it by zero outside D) also satisfies $\Delta u(x) = g(x,u)\chi_D + \lambda_0 \mathcal{H}^{n-1}_{\partial D}$. When $\lambda_0 = 0$, the distance function is not involved anymore. We thus have in D_1

$$\begin{aligned}\Delta w = g_2(x,u_1)\chi_{D_1} - g_2(x,u)\chi_D - \lambda_0 \mathcal{H}^{n-1}_{\partial D} + \lambda_0 \Delta d(x) \geq c w \chi_{D_1\cap D} \\ + g_2(x,u_1)\chi_{D_1\setminus D},\end{aligned} \tag{3.2}$$

where $c = c(x) = (g_2(x,u_1) - g_2(x,u))/(u_1 - u) \geq 0$ (by the monotonicity assumption). In case $D_1 \setminus D \neq \emptyset$, we have $w = u_1$ in $D_1 \setminus D$ and hence $c(x) = g_2(x,u_1)/u_1 \geq 0$ for $x \in D_1 \setminus D$ that is, $g_2(x,u_1)\chi_{D_1\setminus D} = c(x) w \chi_{D_1\setminus D}$. We thus arrive at

$$\Delta w - c w \geq 0 \qquad \text{in } D_1,$$

and conclude by the maximum principle that $w \leq 0$ in D_1.

Consider now two cases.

Case $\lambda_0 = 0$: By the conclusion above $w \le 0$ in D_1, and since $\lambda_0 = 0$ we must have $u_1 \le u$ in D_1. Using that D is solid, we should have $D_1 \setminus D = \emptyset$, since otherwise $\Delta u_1 \ge 0$ on $D_1 \setminus D$, and this set contains points $z \in \partial D_1$ (where $u_1(z) = 0$), at the same time that $u_1 \le u \le 0$ on this set. Let r be small enough so that $B_r(z) \subset D^c$. Then u_1 would obtain a local maximum in $B_r(z)$, at z, contradicting the strong maximum principle. This proves case (1) in the statement of the theorem, when $\lambda_0 = 0$.

We consider now case (2) in the statement of the theorem, when $\lambda_0 = 0$. If $D \setminus D_1 \ne \emptyset$ then by positivity and monotonicity of g

$$0 = \int_{D_1} \Delta u_1 = \int_{D_1} g(x, u_1) \le \int_{D_1} g(x, u) < \int_D g(x, u) = \int_D \Delta u = 0, \quad (3.3)$$

which is a contradiction.

Case $\lambda_0 > 0$:

We first observe that $w = -u + \lambda_0(d(x) - d_1)$ on ∂D_1. Obviously $w < 0$ on $\partial D_1 \cap D$, and $w = \lambda_0(d(x) - d_1)$ on $\partial D_1 \setminus D$. Let $z \in \partial D_1$ be any point that realizes $d_1 = d(z)$. Then $w(z) = 0$, and the normal vector ν at ∂D_1, is also normal to the level sets of $d(x)$ at z. Since ∂D_1 is $C^{1,dini}$, by Hopf boundary point lemma we must have

$$\frac{\partial w}{\partial \nu}(z) > 0, \qquad \text{where } \nu \text{ is the unit outward normal vector,}$$

and consequently we arrive at the following contradiction

$$-\lambda_0 = \frac{\partial u_1}{\partial \nu}(z) > -\lambda_0 \frac{\partial d}{\partial \nu}(z) = -\lambda_0. \quad (3.4)$$

Here we have used that $u \equiv 0$ in $D_1 \setminus D$, and that at z we have $\frac{\partial d}{\partial \nu}(z) = |\nabla d(z)| = 1$. The latter depends on the fact that the level surface $d(x) = d_1$ and ∂D_1 have the same normal vector at z. Since D is convex, and hence solid, we conclude that $D_1 \setminus D = \emptyset$, and that $D_1 \subset D$. This proves the first statement in the theorem in the case $\lambda_0 > 0$. The second statement follows straightforwardly, by replacing the role of D_1 and D, whenever D_1 is another convex QD, with a solution $u_1 \ge 0$. $\square$

The proof of our symmetry theorem, is very standard, and as mentioned earlier uses moving plane-techniques. We thus give a sketch of the proof without much details. Readers may consult [4] for further detail on the technical issues, and for an interior symmetry problem.

3.3 Proof of Theorem 3

The function $u(x) := U^{g\chi_D,\lambda_0}(x)$, satisfies

$$\begin{cases} \Delta u = g_2(u) - g_1(u)\chi_{D_1} & \text{in } D, \\ u(x) = m & x \in \partial D_1, \\ u = \partial_\nu u = 0 & \text{on } \partial D, \end{cases} \quad (3.5)$$

We show that u is spherically symmetric, around a center, and the level subsets $\{u > l\}$ are concentric balls.

The argument uses standard symmetry argument, by the moving plane technique; see [1], [17]. We shall however follow the lines of arguments in [4]. Since our problem is invariant under rigid motion, we may fix a direction, and prove symmetry in that direction. In doing so we choose the x_1-direction, and consider the following set up

$$\begin{cases} T^\tau = \{x \in \mathbb{R}^n \mid x_1 = \tau\} & \text{(the hyperplane)}, \\ x^\tau = (2\tau - x_1, x_2, x_3, \dots) & \text{(the reflection of } x \text{ at } T^\tau), \\ \Sigma^\tau = \{x \,|x_1 < \tau, x^\tau \in D\} & \\ u^\tau(x) = u(x^\tau) & \text{(the reflection of } u \text{ at } T^\tau). \end{cases} \tag{3.6}$$

Start the moving plane from far away (x_1 = large) until the plane touches the boundary ∂D, for some value $\tau = \tau_1$. Since ∂D is C^1, for $\tau_1 - \epsilon < \tau < \tau_1$, and ϵ small enough, we have that Σ^τ is inside D. For such values of τ, consider $w^\tau := u - u^\tau$ in Σ^τ. When $a \geq 0$, we have $g'(u) \geq 0$, for $u \approx 0$ (see Standing Assumption), and hence on the set $\{w^\tau < 0\} \cap \Sigma^\tau$, we have $\Delta w^\tau \leq 0$, so the minimum principle applied to w^τ on that set gives us a contradiction. Therefore, $\{w^\tau < 0\} \cap \Sigma^\tau = \emptyset$ for $\tau_1 - \epsilon < \tau < \tau_1$. For $a < 0$, by the asymptotic property (2.7) we obviously have $w^\tau > 0$ in Σ^τ.[7]

Next we move the plane further in the negative x_1-direction, as long as $w^\tau > 0$. Let

$$\tau_0 = \inf\{\tau : \; w^t > 0 \text{ in } \Sigma_t, \quad \forall \tau < t < \tau_1\}. \tag{3.7}$$

We first note that

$$(D_1)^{\tau_0} \setminus D_1 = \emptyset. \tag{3.8}$$

To prove this, We first notice that since $g_2 \geq 0$, the function u is a subsolution in $D \setminus D_1$, and hence maximum value for u is attained on ∂D_1. The latter being a $C^{1,dini}$ allows invoking the Hopf's boundary principle to conclude $\partial_\nu u > 0$ on the boundary ∂D_1, implying that $u > m$ in a vicinity of ∂D_1 inside D_1. Here ν is the interior unit normal direction on ∂D_1.

Now, if $(D_1)^{\tau_0} \setminus D_1 \neq \emptyset$, it means that we can find some value $\tau_2 > \tau_0$ such that $(D_1)\tau_2 \setminus D_1 \neq \emptyset$, and this set is very close to ∂D_1. By the discussion just made we have $u < m$, and $u^{\tau_2} > m$ in this set, implying $w^{\tau_2} < 0$ in this set, contradicting the definition of τ_0, and the construction of w^{τ_0}, see (3.7).

To proceed, we may assume $\Sigma^{\tau_0} \not\equiv D \cap \{x_1 < \tau_0\}$, or equivalently $w^{\tau_0} \not\equiv 0$, since otherwise we are done. From here on, the argument follows that of [4], by moving the plane slightly further and call it T^{τ_i} with $\tau_i = \tau_0 - \epsilon_i > 0$ and ϵ_i being very small, so that a small set $D_i = \{w^{\tau_i} < 0\}$ appears. As in [4], we slide back the plane

[7] We bring the readers attention to the fact that maximum principle does not work in standard way as in the literature, when we are close to the free boundary. This is due to the nature of the right hand side $g_2(u) = u^a$ close to $u = 0$, where $a < 0$.

by letting $\epsilon_i \to 0$, so that the limit domain $D_0 = \lim D_i$ consists of points z where $w^{\tau_0}(z) = |\nabla w^{\tau_0}|(z) = 0$.

The key point is to show $D_0 = \emptyset$. This set, may appear at several places, and one needs to prove each of these possibilities would result in a contradiction. Indeed, one needs to consider three different possibilities:

$$D_0 \cap \Sigma^{\tau_0}, \qquad D_0 \cap T^{\tau_0}, \qquad D_0 \cap \partial D,$$

and prove each of these sets are void. For this, we need to calculate the Laplacian of w^{τ_0}, in order to use the minimum principle for dealing with the above cases, in an exact way as done in [4]. We have

$$\Delta w^{\tau_0} = \left(g_2(u) - g_1(u)\chi_{D_1}\right) - \left(g_2(u^{\tau_0}) - g_1(u^{\tau_0})\chi_{(D_1)^{\tau_0}}\right),$$

which in light of (3.8) can be rephrased as

$$\Delta w^{\tau_0} \leq \left(g_2(u) - g_2(u^{\tau_0})\right) - \left(g_1(u) - g_1(u^{\tau_0})\right) \chi_{D_1}.$$

By assumptions (2.6) we conclude that $\Delta w^{\tau_0} \leq C w^{\tau_0}$, for some c, as in a) in (2.6). Since also $w^{\tau_0} \geq 0$ in Σ^{τ_0}, we have the scene ready for applying the comparison principle in a more elaborated way as done in [4]. Indeed, from here, we may follow exactly the same lines of proofs mutatis-mutandis, as in [4], to arrive at a contradiction. See also [5] for the case of fully nonlinear operators. □

Funding Open access funding provided by Royal Institute of Technology.

Data Availibility Statement All data needed are contained in the manuscript.

Declarations

Conflict of Interest The authors did not receive support from any organization for the submitted work. The authors have no competing interests to declare that are relevant to the content of this article.

References

1. Alexandroff, A.D.: A characteristic property of spheres. Ann. Mat. Pura Appl. **58**, 303–315 (1962)
2. Alt, H.W., Phillips, D.: A free boundary problem for semilinear elliptic equations. J. Reine Angew. Math. **368**, 63–107 (1986)
3. Henrot, A., Mazari, I., Privat, Y.: Shape optimization of a Dirichlet type energy for semilinear elliptic partial differential equations. ESAIM Control Optim. Calc. Var. **27**, 32 (2021)

4. El Hajj, L., Shahgholian, H.: Radial symmetry for an elliptic PDE with a free boundary. Proc. Am. Math. Soc. Ser. B **8**, 311–319 (2021)
5. El Hajj, L., Jeon, S., Shahgholian, H.: Symmetry for a fully nonlinear free boundary problem. In preparation
6. Fraenkel, L.E.: Introduction to maximum principles and symmetry in elliptic problems. Cambridge tracts in mathematics. Cambridge University Press, London (2000)
7. Gustafsson, B., Shahgholian, H.: Existence and geometric properties of solutions of a free boundary problem in potential theory. J. Reine Angew. Math. **473**, 137–179 (1996)
8. Gustafsson, B., Shapiro, H.S.: What is a quadrature domain? Quadrature domains and their applications. Op. Theory Adv. Appl. **156**, 1–25 (2005)
9. Xiaolong, H., Lu, G., Zhu, J.: Characterization of balls in terms of Bessel-potential integral equation. J. Diff. Eq. **252**(2), 1589–1602 (2012)
10. Kow, P., Larson, S., Salo, M., Shahgholian, H.: Quadrature domains for the Helmholtz equation with applications to non-scattering phenomena. In preparation
11. Kow, P., Larson, S., Salo, M., Shahgholian, H.: A minimization problem with free boundary related to Helmholtz operator. In preparation
12. Reichel, W.: Characterization of balls by Riesz-potentials. Ann. Mat. Pura Appl. **188**(2), 235–245 (2009)
13. Sakai, M.: Quadrature domains. lecture notes in mathematics. Springer-Verlag, Berlin-New York (1982)
14. Sakai, M.: Solutions to the obstacle problem as Green potentials. J. Anal. Math. **44**, 97–116 (1986)
15. Sakai, M.: Sharp estimates of the distance from a fixed point to the frontier of a Hele-Shaw flow. Potential Anal. **8**(3), 277–302 (1998)
16. Salo, M., Shahgholian, H.: Free boundary methods and non-scattering phenomena. Res. Math. Sci. **8**(4), 58 (2021). https://doi.org/10.1007/s40687-021-00294-z
17. Serrin, J.: A symmetry problem in potential theory. Arch. Rational Mech. Anal. **43**, 871 (1971)