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Non-variational weakly coupled elliptic systems

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To the memory of Harold S. Shapiro.
A.S. was Harold's student.
He is forever grateful for all inspiration and encouragement.

Abstract
We establish the existence of a nonnegative fully nontrivial solution to a non-variational weakly coupled competitive elliptic system. We show that this kind of solutions belong to a topological manifold of Nehari-type, and apply a degree-theoretical argument on this manifold to derive existence.

Keywords Weakly coupled elliptic system · Positive solution · Uniform bound · Nehari manifold · Brouwer degree · Synchronized solutions

Mathematics Subject Classification 35J57 · 35J61 · 35B09 · 47H11

1 Introduction and statement of results

In this paper we consider the existence of solutions to the elliptic system

$$\begin{cases} -\Delta u_i = \mu_i u_i^p + \sum\limits_{j\neq i} \lambda_{ij} u_i^{\alpha_{ij}} u_j^{\beta_{ij}}, \\ u_i \geq 0, \ \ u_i \not\equiv 0 \text{ in } \Omega, \\ u_i \in H_0^1(\Omega), \quad i, j = 1, \dots, \ell, \end{cases} \tag{1.1}$$

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where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 2$, $1 < p < \frac{N+2}{N-2}$ if $N \geq 3$, $1 < p < \infty$ if $N = 2$, $\mu_i > 0$, $\lambda_{ij} < 0$, $\alpha_{ij}, \beta_{ij} > 0$ and $\alpha_{ij} + \beta_{ij} < p$ for $i, j = 1, \dots, \ell$, $j \neq i$. This system arises as a model for the steady state distribution of ℓ competing species coexisting in Ω. Here u_i represents the density of the i-th population, μ_i corresponds to the attraction between the species of the same kind, or more generally, $\mu_i u_i^p$ can be replaced by $f_i(u_i)$ and represent internal forces. The parameters $\lambda_{ij}, \lambda_{ji}$ (which may not be equal) correspond to the interaction (repulsion) between different species. In particular, if $\alpha_{ij} = \beta_{ij} = 1$, then the interaction is of the Lotka-Volterra type while $\alpha_{ij} = 1$, $\beta_{ij} = 2$ corresponds to the interaction which appears in the Bose-Einstein condensates. In the latter case one also has $\lambda_{ij} = \lambda_{ji}$ and the system is variational.

In what follows we do not assume $\lambda_{ij} = \lambda_{ji}$ or $\beta_{ij} = \alpha_{ji}$. The system (1.1) is non-variational except for some very special choices of $\lambda_{ij}, \alpha_{ij}$ and β_{ij}. While there is an extensive literature concerning the existence (and multiplicity) of solutions for variational systems like (1.1), there are not so many results in the non-variational case. Here we could mention [1, 6–9] where, however, the right-hand sides are quite different from ours. In particular, in [6–9] the interaction term is of the Lotka-Volterra type (or is a variant of it) while the terms $f_i(u_i)$ are different from $\mu_i u_i^p$. For these f_i one obtains uniform bounds on the solutions when $\lambda_{ij} \to -\infty$. Existence of such bounds allows to study the limiting behaviour of solutions. To be more precise, if $\lambda_{ij,n} \to -\infty$ and $(u_{1,n}, \dots, u_{l,n})$ is a corresponding solution with uniform bound on each component, then one expects that $u_{i,n} \to u_i$ (in an appropriate space) and $u_i(x) \cdot u_j(x) = 0$ a.e. in Ω for all $i \neq j$, i.e. different components separate spatially. This has been studied in the above mentioned papers. In [1, 6] the emphasis is in fact on the properties of limiting configurations, including regularity of free boundaries between the components.

The main result of this paper is the following

Theorem 1.1 *The system* (1.1) *has a solution.*

Existence proofs in the above-mentioned papers do not seem to be applicable here. Our problem can be reformulated as an operator equation in the space $\mathcal{H} := H_0^1(\Omega)^\ell$ and one can use degree theory to obtain a nontrivial solution. However, this could give a semitrivial solution (i.e. $u_i = 0$ for some but not all i). To rule out such solutions we introduce a Nehari-type manifold on which all u are fully nontrivial in the sense that no u_i is identically zero, and then we apply a degree-theoretical argument on this manifold.

We do not know if there always exist solutions for (1.1) which are uniformly bounded, see Problem 5.5. Moreover, as we shall see in Sect. 5, under a suitable choice of exponents and parameters and for $\ell = 2$ there exists a sequence of solutions which are synchronized in the sense that $u_{i,n} = t_{i,n} v_n$ $(i = 1, 2)$ and such that $\|u_{i,n}\| \to \infty$ as $\lambda_{12,n}, \lambda_{21,n} \to -\infty$. So the components neither separate spatially nor are bounded.

Let $u_i^+ := \max\{u_i, 0\}$, $u_i^- := \min\{u_i, 0\}$, and consider the system

$$\begin{cases} -\Delta u_i = \mu_i (u_i^+)^p + \sum\limits_{j \neq i} \lambda_{ij} (u_i^+)^{\alpha_{ij}} (u_j^+)^{\beta_{ij}}, \\ u_i \in H_0^1(\Omega), \quad i, j = 1, \dots, \ell. \end{cases} \tag{1.2}$$

In Proposition 3.4(v) we shall show that any fully nontrivial solution to this system also solves (1.1).

In what follows we shall work with (1.2) and we shall also need the parametrized system

$$\begin{cases} -\Delta u_i = \mu_i (u_i^+)^p + t \sum\limits_{j \neq i} \lambda_{ij} (u_i^+)^{\alpha_{ij}} (u_j^+)^{\beta_{ij}}, \\ u_i \in H_0^1(\Omega), \quad i, j = 1, \ldots, \ell, \quad 0 \leq t \leq 1. \end{cases} \tag{1.3}$$

Note that (1.3) homotopies (1.2) to an uncoupled system. Since

$$t \sum_{j \neq i} |\lambda_{ij}| (u_i^+)^{\alpha_{ij}} (u_j^+)^{\beta_{ij}} \leq C(1 + (u_1^+)^q + \cdots + (u_\ell^+)^q),$$

where $\alpha_{ij} + \beta_{ij} \leq q < p$ for all i, j, the following statement holds true.

Lemma 1.2 *All solutions $u = (u_1, \ldots, u_\ell)$ of (1.3) are uniformly bounded in $L^\infty(\Omega)$ and hence in $H_0^1(\Omega)$. This bound is independent of $t \in [0, 1]$.*

This has been shown, in a much more general setting, in [13] for a single equation and in [11] for two equations. It is easy to see that the argument in [11] extends to an arbitrary number of equations. In both papers a blow-up procedure is used in order to reduce the problem to a Liouville-type result. For the reader's convenience, in Appendix A we shall provide a simple proof of such reduction, adapted to our special case. The assumption $q < p$ is crucial for the validity of this lemma. Indeed, in [10] it has been shown that the conclusion may fail if $q = p$.

The paper is organized as follows. In Sect. 2 we state and prove a lemma for functions in $\mathbb{R}^\ell$. In Sect. 3 we define a Nehari-type manifold $\mathcal{N}$ similar to the one introduced in [5]. We also show that solutions to (1.2) correspond to solutions for an operator equation in an open subset of the product of the unit spheres $\mathcal{S}_i \subset H_0^1(\Omega)$, $1 \leq i \leq \ell$. The idea comes from [4]. To our knowledge, this is the first time a Nehari-type manifold appears in a non-variational setting. Theorem 1.1 is proved in Sect. 4 and synchronized solutions are discussed in Sect. 5. As we have already mentioned, Lemma 1.2 is proved in Appendix A.

In the proof of Theorem 1.1 we shall employ a topological degree argument. Since our operator is not admissible for common infinite-dimensional degree theories, we introduce a sequence of finite-dimensional ("Galerkin-like") approximations and use the Brouwer degree, see (4.6) and (4.9–4.11) below.

2 A lemma on functions in $\mathbb{R}^\ell$

Let $a_i, \alpha_{ij}, \beta_{ij} > 0$, $b_i, d_{ij} \geq 0$, $\alpha_{ij} + \beta_{ij} < p$ for all $i, j = 1, \ldots, \ell$, $j \neq i$. Define $M : (0, \infty)^\ell \to \mathbb{R}^\ell$ as

$$M(s) := (M_1(s), \ldots, M_\ell(s)),$$

where

$$M_i(s) := a_i s_i - b_i s_i^p + \sum_{j\neq i} d_{ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}}, \qquad i, j = 1, \ldots, \ell.$$

Lemma 2.1 *(i) If $b_i = 0$ for some i, then $M(s) \neq 0$ for any $s \in (0,\infty)^\ell$.*

(ii) If $b_i > 0$ for all i, then there exists $s \in (0,\infty)^\ell$ such that $M(s) = 0$. Moreover, if $0 < a \leq a_i \leq \overline{a}$, $0 < b \leq b_i \leq \overline{b}$ and $d_{ij} \leq \overline{d}$ for all i, j, then there exist $0 < r < R$, depending only on $a, \overline{a}, b, \overline{b}, \overline{d}$, such that $s \in (r, R)^\ell$.

(iii) The solution s in (ii) is unique.

(iv) The solution s in (ii) depends continuously on $a_i, b_i > 0$, $d_{ij} \geq 0$.

Proof (i) : If $b_i = 0$ then

$$M_i(s) = a_i s_i + \sum_{j\neq i} d_{ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}} > 0 \qquad \text{for all } s \in (0,\infty)^\ell.$$

(ii) : Let $0 < r < R$ be such that, for every $i, j = 1, \ldots, \ell$,

$$a_i t - b_i t^p > 0 \qquad \text{if } t \in (0, r],$$
$$a_i t - b_i t^p + \sum_{j\neq i} d_{ij} t^{\alpha_{ij}+\beta_{ij}} < 0 \qquad \text{if } t \in [R, \infty)$$

(such R exists because $\alpha_{ij} + \beta_{ij} < p$). If $s = (s_1, \ldots, s_\ell) \in (0,\infty)^\ell$ and $s_i \geq s_j$ for all j, then

$$M_i(s) = a_i s_i - b_i s_i^p + \sum_{j\neq i} d_{ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}} \leq a_i s_i - b_i s_i^p + \sum_{j\neq i} d_{ij} s_i^{\alpha_{ij}+\beta_{ij}}.$$

Therefore, $M_i(s) < 0$ whenever $s_i = \max\{s_1, \ldots, s_\ell\} \geq R$, and $M_i(s) > 0$ if $0 < s_i \leq r$. If $a \leq a_i \leq \overline{a}$, $b \leq b_i \leq \overline{b}$, $d_{ij} \leq \overline{d}$, then

$$a_i t - b_i t^p \geq at - \overline{b} t^p, \qquad a_i t - b_i t^p + \sum_{j\neq i} d_{ij} t^{\alpha_{ij}+\beta_{ij}} \leq \overline{a} t - b t^p + \sum_{j\neq i} \overline{d} t^{\alpha_{ij}+\beta_{ij}},$$

so r, R may be chosen as claimed.

Let

$$G(s) := \rho - s \qquad \text{where} \quad \rho := \tfrac{r+R}{2}(1, \ldots, 1).$$

Then $H(s, \tau) := \tau M(s) + (1-\tau) G(s) \neq 0$ on the boundary of $[r, R]^\ell$ for every $\tau \in [0, 1]$. Hence this is an admissible homotopy for the Brouwer degree (see e.g. [18,Appendix D] for the definition and properties of this degree). So

$$\deg(M, (r, R)^\ell, \rho) = \deg(G, (r, R)^\ell, \rho) = (-1)^\ell$$

and $M(s) = 0$ must have a solution.

(iii) : If $M(s_1^0, \ldots, s_\ell^0) = 0$, then $\widetilde{M}(1, \ldots, 1) = 0$ where

$$\widetilde{M}_i(s) = \widetilde{a}_i s_i - \widetilde{b}_1 s_i^p + \sum_{j \neq i} \widetilde{d}_{ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}}$$

with $\widetilde{a}_i := a_i s_i^0$, $\widetilde{b}_i := b_i (s_i^0)^p$, $\widetilde{d}_{ij} := d_{ij} (s_i^0)^{\alpha_{ij}} (s_j^0)^{\beta_{ij}}$. So we may assume without loss of generality that $M(1, \ldots, 1) = 0$. Then,

$$a_i - b_i + \sum_{j \neq i} d_{ij} = 0.$$

Suppose there is another solution $s = (s_1, \ldots, s_\ell)$. Then, using the previous identity, we get

$$0 = a_i s_i - b_i s_i^p + \sum_{j \neq i} d_{ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}} = a_i s_i - \Big(a_i + \sum_{j \neq i} d_{ij}\Big) s_i^p + \sum_{j \neq i} d_{ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}},$$

and after rearranging the terms,

$$a_i (s_i - s_i^p) = \sum_{j \neq i} d_{ij} (s_i^p - s_i^{\alpha_{ij}} s_j^{\beta_{ij}}).$$

There are two possible cases: If $s_i > 1$ for some i, we may assume without loss of generality that $s_i \geq s_j$ for all j. Then the left-hand side above is negative while the right-hand side is ≥ 0, a contradiction. If, on the other hand, $0 < s_i < 1$ for some i, we may assume $s_i \leq s_j$ for all j. Now the left-hand side is positive and the right-hand side is ≤ 0, a contradiction again.

(iv) : If $a_{n,i}, a_i, b_{n,i}, b_i > 0$, $d_{n,i}, d_i \geq 0$, $a_{n,i} \to a_i$, $b_{n,i} \to b_i$, $d_{n,ij} \to d_{ij}$ then, as in (ii), there exist $0 < r < R$ such that the unique solution s_n to

$$M_{n,i}(s) := a_{n,i} s_i - b_{n,i} s_i^p + \sum_{j \neq i} d_{n,ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}} = 0, \qquad i, j = 1, \ldots, \ell,$$

belongs to $[r, R]^\ell$ for every n. Passing to a subsequence, we have that $s_n \to s \in [r, R]^\ell$ and $M(s) = 0$. $\square$

3 A Nehari-type manifold

Let $\mathcal{H} := H_0^1(\Omega)^\ell$, $u = (u_1, \ldots, u_\ell) \in \mathcal{H}$. As convenient norms in $H_0^1(\Omega)$ and $\mathcal{H}$ we choose

$$\|u_i\| := \left(\int_\Omega |\nabla u_i|^2\right)^{\frac{1}{2}} \quad \text{and} \quad \|u\| := (\|u_1\|^2 + \cdots + \|u_\ell\|^2)^{\frac{1}{2}},$$

and we denote by $\langle\cdot,\cdot\rangle$ the inner product in $H_0^1(\Omega)$. Let

$$I(u) := (I_1(u), \dots, I_\ell(u))$$

where $I_i : H_0^1(\Omega) \to H_0^1(\Omega)$ are given by

$$I_i(u) := u_i - K_i(u) \tag{3.1}$$

and

$$\langle K_i(u), v\rangle := \int_\Omega \mu_i (u_i^+)^p v + \sum_{j\neq i} \lambda_{ij} \int_\Omega (u_i^+)^{\alpha_{ij}} (u_j^+)^{\beta_{ij}} v \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

Lemma 3.1 *If $u_n \rightharpoonup u$ weakly in $\mathcal{H}$, then $K_i(u_n) \to K_i(u)$ strongly in $H_0^1(\Omega)$ for each $i = 1, \dots, \ell$.*

Proof Since $p,\ \alpha_{ij}+\beta_{ij} < \frac{N+2}{N-2}$ for $N \geq 3$, after passing to a subsequence $u_{n,i}^+ \to u_i^+$ strongly in $L^{p+1}(\Omega)$ and in $L^{\alpha_{ij}+\beta_{ij}+1}(\Omega)$ for every $j \neq i$. Using Hölder's and Sobolev's inequalities we obtain

$$\begin{aligned}|\langle K_i(u_n) - K_i(u), v\rangle| \leq C\Big(&\big|(u_{n,i}^+)^p - (u_i^+)^p\big|_{\frac{p+1}{p}} \\ &+ \sum_{j\neq i} \big|(u_{n,i}^+)^{\alpha_{ij}} - (u_i^+)^{\alpha_{ij}}\big|_{\frac{\alpha_{ij}+\beta_{ij}+1}{\alpha_{ij}}} + \sum_{j\neq i} \big|(u_{n,j}^+)^{\beta_{ij}} - (u_j^+)^{\beta_{ij}}\big|_{\frac{\alpha_{ij}+\beta_{ij}+1}{\beta_{ij}}}\Big)\|v\|,\end{aligned}$$

where $|\cdot|_r$ denotes the norm in $L^r(\Omega)$. From [18,Theorem A.2] we derive

$$\sup_{v\neq 0} \frac{|\langle K_i(u_n) - K_i(u), v\rangle|}{\|v\|} \longrightarrow 0.$$

Hence, $K_i(u_n) \to K_i(u)$ strongly in $H_0^1(\Omega)$, as claimed. □

We define a Nehari-type set $\mathcal{N}$ by putting

$$\mathcal{N} := \{u \in \mathcal{H} : u_i \neq 0 \ \text{ and } \ \langle I_i(u), u_i\rangle = 0 \ \text{ for all } \ i = 1, \dots, \ell\}.$$

Lemma 3.2 *$\mathcal{N}$ is closed in $\mathcal{H}$.*

Proof Since $\lambda_{ij} < 0$, it follows from the Sobolev inequality that

$$\|u_i\|^2 \leq \mu_i \int_\Omega (u_i^+)^{p+1} \leq C_i \|u_i\|^{p+1}$$

for some $C_i > 0$. Hence there exists $d_0 > 0$ such that, if $(u_1, \dots, u_\ell) \in \mathcal{N}$, then $\|u_i\| \geq d_0$ for all i. This shows that $\mathcal{N}$ is closed in $\mathcal{H}$. □

For $u := (u_1, \dots, u_\ell) \in \mathcal{H}$, $s := (s_1, \dots, s_\ell) \in (0, \infty)^\ell$ and $su := (s_1 u_1, \dots, s_\ell u_\ell)$, we define

$$M_u(s) := (M_{u,1}(s), \dots, M_{u,\ell}(s)),$$

where

$$M_{u,i}(s) := \langle I_i(su), u_i \rangle = a_{u,i} s_i - b_{u,i} s_i^p + \sum_{j \neq i} d_{u,ij} s_i^{\alpha_{ij}} s_j^{\beta_{ij}}$$

and

$$a_{u,i} := \|u_i\|^2, \quad b_{u,i} := \int_\Omega \mu_i (u_i^+)^{p+1}, \quad d_{u,ij} := \int_\Omega (-\lambda_{ij})(u_i^+)^{\alpha_{ij}+1}(u_j^+)^{\beta_{ij}}.$$

Lemma 3.3 *(i) If $a_{u,i} \neq 0$ and $b_{u,i} = 0$ for some i, then $M_u(s) \neq 0$ for any $s \in (0, \infty)^\ell$.*

(ii) If $a_{u,i}, b_{u,i} > 0$ for all i, then there exists a unique $s_u \in (0, \infty)^\ell$ such that $M_u(s_u) = 0$. Moreover, if $0 < a \leq a_{u,i} \leq \overline{a}$, $0 < b \leq b_{u,i} \leq \overline{b}$ and $d_{u,ij} \leq \overline{d}$ for all i, j, then there exist $0 < r < R$, depending only on $a, \overline{a}, b, \overline{b}, \overline{d}$, such that $s_u \in (r, R)^\ell$.

Proof This is an immediate consequence of Lemma 2.1. □

Let

$$\mathscr{S} := \{v \in H_0^1(\Omega) : \|v\| = 1\}, \quad \mathcal{T} := \mathscr{S}^\ell,$$

and

$$\begin{aligned} \mathcal{U} : &= \{u \in \mathcal{T} : s_u \in (0, \infty)^\ell \text{ exists with } M_u(s_u) = 0\} \\ &= \{u \in \mathcal{T} : u_i^+ \neq 0 \text{ for all } i = 1, \dots, \ell\}. \end{aligned} \tag{3.3}$$

The tangent space of $\mathcal{T}$ at u is

$$T_u(\mathcal{T}) := \{(v_1, \dots, v_\ell) \in \mathcal{H} : \langle u_i, v_i \rangle = 0 \text{ for all } i = 1, \dots, \ell\}. \tag{3.4}$$

Proposition 3.4 *(i) $\mathcal{U}$ is a nonempty open subset of $\mathcal{T}$ and $\mathcal{U} \neq \mathcal{T}$.*

(ii) The mapping $m : \mathcal{U} \to \mathcal{N}$ given by $m(u) := s_u u$ is a homeomorphism. In particular, $\mathcal{N}$ is a topological manifold.

(iii) If (u_n) is a sequence in $\mathcal{U}$ such that $u_n \to u \in \partial\mathcal{U}$, then $s_{u_n} \to \infty$ (and hence $\|m(u_n)\| \to \infty$).

(iv) Let $S : \mathcal{U} \to \mathcal{H}$ be given by

$$S(u) := I(s_u u) = s_u u - K(s_u u).$$

Then $S(u) \in T_u(\mathcal{U})$ for every $u \in \mathcal{U}$.

(v) $S(u)=0$ *if and only if* $m(u)=s_u u$ *is a solution for* (1.1).

Proof (i) : That $\mathcal{U}$ is neither empty nor the whole $\mathcal{T}$ is obvious and, since $u\mapsto u_i^+$ is continuous [2,Lemma 2.3], it is easily seen from the second line of (3.3) that $\mathcal{U}$ is open in $\mathcal{T}$.

(ii) : If $u\in\mathcal{U}$, then $s_u u\in\mathcal{N}$ because $\langle I_i(s_u u), s_{u,i}u_i\rangle = s_{u,i}M_{u,i}(s_u)=0$ for all i. So m is well defined. If (u_n) is a sequence in $\mathcal{U}$ and $u_n\to u\in\mathcal{U}$, then $a_{u_n,i}\to a_{u,i}$, $b_{u_n,i}\to b_{u,i}$ and $d_{u_n,ij}\to d_{u,ij}$ for all i,j. By Lemma 2.1(iv), $s_{u_n}\to s_u$. Hence, m is continuous.

If $u\in\mathcal{N}$, then $u_i^+\neq 0$ for all i. Otherwise, $0=\langle I_i(u),u_i\rangle=\|u_i\|^2$, a contradiction. Hence, the inverse of m satisfies

$$m^{-1}(u) := \left(\frac{u_1}{\|u_1\|},\dots,\frac{u_\ell}{\|u_\ell\|}\right)\in\mathcal{U},$$

and it is obviously continuous.

(iii) : Let (u_n) be a sequence in $\mathcal{U}$ such that $u_n\to u\in\partial\mathcal{U}$. If (s_{u_n}) is bounded, then, after passing to a subsequence, $s_{u_n}\to s_*$. Since $\mathcal{N}$ is closed, $s_*u\in\mathcal{N}$ and hence $u\in\mathcal{U}$. This is impossible because $\mathcal{U}$ is open.

(iv) : Since $\langle I_i(s_u u), u_i\rangle = M_{u,i}(s_u)=0$ for all i, we have that $S(u)\in T_u(\mathcal{T})$ according to (3.4).

(v) : If $u\in\mathcal{U}$ satisfies $S(u)=0$, then $\bar u := s_u u\in\mathcal{N}$ and $\bar u$ is a weak solution to the system (1.2) (see (3.1) and (3.2)). Multiplying the i-th equation in (1.2) by $u_i^- := \min\{\bar u_i, 0\}$ and integrating gives $\int_\Omega |\nabla u_i^-|^2=0$. Hence $u_i^-=0$, i.e., $\bar u_i\geq 0$ for all i. As $\bar u\in\mathcal{N}$, we have that $\bar u_i\neq 0$. This proves that $\bar u$ solves (1.1). The converse is obvious. □

Remark 3.5 If $\alpha_{ij}\geq 1$ for all i and all $j\neq i$, then, as $\bar u_i$ above satisfies the i-th equation in (1.1), we have

$$-\Delta\bar u_i + c(x)\bar u_i \geq 0 \quad \text{where } c(x) := -\sum_{j\neq i}\lambda_{ij}\bar u_i^{\alpha_{ij}-1}\bar u_j^{\beta_{ij}}.$$

Since all u_i are continuous in $\overline{\Omega}$ and $c\geq 0$, it follows from the strong maximum principle (see e.g. [14,Theorem 3.5]) that our solution is strictly positive in Ω in this case.

4 Proof of Theorem 1.1

In this section the sub- or superscript t will be used in order to emphasize that we are concerned with the system (1.3). So, e.g.,

$$I_t(u) := u - K_t(u), \qquad S_t(u) := I_t(s_u^t u) = s_u^t u - K_t(s_u^t u), \tag{4.1}$$

with

$$\langle K_{t,i}(u), v\rangle := \int_\Omega \mu_i (u_i^+)^p v + t \sum_{j\neq i} \lambda_{ij} \int_\Omega (u_i^+)^{\alpha_{ij}} (u_j^+)^{\beta_{ij}} v,$$

and

$$\mathcal{N}_t := \{u \in \mathcal{H} : u_i \neq 0,\ \langle I_{t,i}(u), u_i\rangle = 0 \text{ for all } i = 1, \dots, \ell\}.$$

According to this notation, $\mathcal{N}_1 = \mathcal{N}$. When $t = 1$, we shall sometimes omit the sub- or superscript t.

Consider first the system (1.3) with $t = 0$. In this case the equations are uncoupled, the set

$$\mathcal{N}_0 = \{u \in \mathcal{H} : u_i \neq 0,\ \|u_i\|^2 = \int_\Omega \mu_i (u_i^+)^{p+1} \text{ for all } i = 1, \dots, \ell\}$$

is the product of the usual Nehari manifolds associated to these equations, and the components of $s_u^0 = (s_{u,1}^0 \dots, s_{u,\ell}^0)$ are

$$s_{u,i}^0 = \left(\int_\Omega \mu_i (u_i^+)^{p+1}\right)^{-\frac{1}{p-1}}, \qquad u \in \mathcal{U}.$$

The function I_0 (cf. (4.1)) is the gradient vector field of the functional $\mathcal{J} : \mathcal{H} \to \mathbb{R}$ defined by

$$\mathcal{J}(u) := \frac{1}{2} \sum_{i=1}^{\ell} \|u_i\|^2 - \frac{1}{p+1} \sum_{i=1}^{\ell} \int_\Omega \mu_i (u_i^+)^{p+1}.$$

Note that

$$\mathcal{J}(u) = (\tfrac{1}{2} - \tfrac{1}{p+1})\|u\|^2 \quad \text{if } u \in \mathcal{N}_0.$$

$\mathcal{J}$ has a minimizer $\widetilde{u}_0 = (\widetilde{u}_{0,1}, \dots, \widetilde{u}_{0,\ell})$ on $\mathcal{N}_0$ with $\widetilde{u}_{0,i} > 0$ and $\widetilde{u}_0$ is a solution to the system (1.3) with $t = 0$. Each component $\widetilde{u}_{0,i}$ is a positive least energy solution to the i-th equation of this system. Let $\Psi : \mathcal{U} \to \mathbb{R}$ be given by

$$\begin{aligned}\Psi(u) : &= \mathcal{J}(s_u^0 u) = \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) |s_u^0|^2 \\ &= \left(\tfrac{1}{2} - \tfrac{1}{p+1}\right) \sum_{i=1}^{\ell} \left(\int_\Omega \mu_i (u_i^+)^{p+1}\right)^{-\frac{2}{p-1}}. \end{aligned} \tag{4.2}$$

By [18,Proposition 1.12] one has that $\Psi \in \mathcal{C}^2(\mathcal{U}, \mathbb{R})$. It is easily seen that

$$\Psi'(u)v = \mathcal{J}'(s_u^0 u)[s_u^0 v] = \langle S_0(u), s_u^0 v\rangle \quad \text{for all } u \in \mathcal{U},\ v \in T_u(\mathcal{U}), \tag{4.3}$$

and that u is a critical point of Ψ if and only if $u \in \mathcal{U}$ and $m_0(u) = s_u^0 u$ is a critical point of $\mathcal{J}$, see [4,Theorem 3.3]. Let $u_0 := m_0^{-1}(\widetilde{u}_0)$. Then u_0 is a minimizer for Ψ.

Invoking Lemma 1.2 we may choose $R > 0$ such that all solutions to the systems (1.3) are contained in the open ball $B_R(0) \subset \mathcal{H}$, where R is independent of $t \in [0, 1]$. Then, by Proposition 3.4,

$$\{u \in \mathcal{U} : S_t(u) = 0\} \subset \mathcal{V}_t := m_t^{-1}(B_R(0) \cap \mathcal{N}_t). \tag{4.4}$$

For $a \leq d$ let

$$\Psi^d := \{u \in \mathcal{U} : \Psi(u) \leq d\}, \quad \Psi_a^d := \{u \in \mathcal{U} : a \leq \Psi(u) \leq d\}.$$

It follows from Proposition 3.4(iii) that the set Ψ^d is closed in $\mathcal{T}$ for any $d \in \mathbb{R}$. Note that $\lambda_{ij} < 0$ implies $s_{u,i}^t \geq s_{u,i}^0$ for every $u \in \mathcal{U}$, $t \in [0, 1]$, $i = 1, \dots \ell$. So if $|s_u^t| < R$, then $|s_u^0| < R$; hence $\mathcal{V}_t \subset \mathcal{V}_0$ and, setting $c := (\frac{1}{2} - \frac{1}{p+1})R^2$, we have that the closure of $\mathcal{V}_t$ in $\mathcal{T}$ satisfies

$$\overline{\mathcal{V}}_t \subset \overline{\mathcal{V}}_0 \subset \Psi^c \qquad \forall t \in [0, 1]. \tag{4.5}$$

For each $i = 1, \dots, \ell$ and $k \geq 2$ we choose an ascending sequence $(E_{k,i})$ of linear subspaces of $H_0^1(\Omega)$ such that $\dim E_{k,i} = k$, $u_{0,i} \in E_{2,i}$ (u_0 is the minimizer chosen above) and $\overline{\bigcup_{k\geq 1} E_{k,i}} = H_0^1(\Omega)$. We define

$$E_k := E_{k,1} \times \cdots \times E_{k,\ell} \subset \mathcal{H} \tag{4.6}$$

and denote by P_k the orthogonal projector of $\mathcal{H}$ onto E_k.

Lemma 4.1 *Given $d > 0$ there exists $k_d \in \mathbb{N}$ such that*

$$P_k(S_t(u)) \neq 0 \quad \textit{for all } u \in (\Psi^d \setminus \mathcal{V}_t) \cap E_k,\ k \geq k_d,\ t \in [0, 1].$$

Proof Arguing by contradiction, assume that there exist $k_n \to \infty$, $t_n \in [0, 1]$ and $u_n \in (\Psi^d \setminus \mathcal{V}_{t_n}) \cap E_{k_n}$ such that

$$P_{k_n}(S_{t_n}(u_n)) = s_{u_n}^{t_n} u_n - P_{k_n} K_{t_n}(s_{u_n}^{t_n} u_n) = 0 \qquad \forall n \in \mathbb{N}. \tag{4.7}$$

As $u_n \in \Psi^d$, we derive from (4.2) that $\int_\Omega \mu_i (u_{n,i}^+)^{p+1} \geq b$ for some $b > 0$ and all n, i. In the notation of Lemma 3.3, we have $a_{u_n,i} = 1$ and, using the Hölder and the Sobolev inequalities, $b \leq b_{u_n,i} \leq \overline{b}$ and

$$d_{u_n,ij} = t_n \int_\Omega (-\lambda_{ij})(u_{n,i}^+)^{\alpha_{ij}+1}(u_{n,j}^+)^{\beta_{ij}} \leq \overline{d}$$

for some $\overline{b}, \overline{d} > 0$. So Lemma 3.3 asserts that $(s_{u_n,i}^{t_n})$ is bounded and bounded away from 0 for each i. Therefore, after passing to a subsequence, $s_{u_n,i}^{t_n} \to s_i > 0$, $t_n \to t$

and $u_n \rightharpoonup u$ weakly in $\mathcal{H}$. By Lemma 3.1, $K_{t_n}(s_{u_n}^{t_n}u_n) \to K_t(su)$ strongly in $\mathcal{H}$, and we easily deduce that $P_{k_n}K_{t_n}(s_{u_n}^{t_n}u_n) \to K_t(su)$ strongly in $\mathcal{H}$. Now we derive from (4.7) that $s_{u_n}^{t_n}u_n \to su$ strongly in $\mathcal{H}$ and $su - K_t(su) = 0$. Therefore, $su \in \mathcal{N}_t$, $s = s_u^t$ and $S_t(u) = 0$. On the other hand, as $u_n \notin \mathcal{V}_{t_n}$, we have that $\|s_{u_n}^{t_n}u_n\| \geq R$. Hence, $\|s_u^t u\| \geq R$. This is a contradiction. □

Lemma 4.2 *Let c be as in (4.5). Then $\Psi^c \cap E_k$ is contractible in itself for each large enough k.*

Proof Let $\eta : [0, 1] \times \mathcal{U} \to \mathcal{U}$ be given by

$$\eta(\tau, u) := \left(\frac{(1-\tau)u_1 + \tau u_{0,1}}{\|(1-\tau)u_1 + \tau u_{0,1}\|}, \ldots, \frac{(1-\tau)u_\ell + \tau u_{0,\ell}}{\|(1-\tau)u_\ell + \tau u_{0,\ell}\|} \right),$$

where u_0 is the previously chosen minimizer for Ψ on $\mathcal{U}$. Note that η is well defined and maps into $\mathcal{U}$ because $u_{0,i} > 0$ in Ω and $u_i^+ \neq 0$ for all i. Moreover, if $u \in E_k$, then $\eta(\tau, u) \in E_k$ for each $k \geq 2$. So η is a deformation of $\mathcal{U} \cap E_k$ into u_0 and, in particular, of $\Psi^c \cap E_k$ into u_0 in $\mathcal{U} \cap E_k$.

We claim that there exists $\delta_0 > 0$ such that

$$\int_\Omega [((1-\tau)u_i + \tau u_{0,i})^+]^{p+1} \geq \delta_0 \quad \text{for all } \tau \in [0, 1],\ u \in \Psi^c,\ i = 1, \ldots, \ell.$$

Otherwise, there would exist $\tau_n \in [0, 1]$ and $u_n \in \Psi^c$ such that

$$(1-\tau_n)\int_\Omega (u_{n,i}^+)^{p+1} \leq \int_\Omega [((1-\tau_n)u_{n,i} + \tau_n u_{0,i})^+]^{p+1} \to 0 \tag{4.8}$$

(the inequality is satisfied because $u_{0,i} > 0$). From (4.2) we see that there exists $\delta > 0$ such that $\int_\Omega (u_i^+)^{p+1} \geq \delta$ for all $u \in \Psi^c$ and all i. Hence, $\tau_n \to 1$. Since (u_n) is bounded in $\mathcal{H}$, a subsequence of $(u_{n,i})$ converges in $L^{p+1}(\Omega)$. Therefore,

$$\int_\Omega [((1-\tau_n)u_{n,i} + \tau_n u_{0,i})^+]^{p+1} \to \int_\Omega u_{0,i}^{p+1} \geq \delta,$$

a contradiction to (4.8).

So, for every $\tau \in [0, 1],\ u \in \Psi^c,\ i = 1, \ldots, \ell$, we have

$$\begin{aligned} \int_\Omega (\eta_i(\tau, u)^+)^{p+1} &= \int_\Omega \frac{[((1-\tau)u_i + \tau u_{0,i})^+]^{p+1}}{\|(1-\tau)u_i + \tau u_{0,i}\|^{p+1}} \\ &\geq \int_\Omega [((1-\tau)u_i + \tau u_{0,i})^+]^{p+1} \geq \delta_0, \end{aligned}$$

and we deduce from (4.2) that there exists $d > c$ such that

$$\eta(\tau, u) \in \Psi^d \cap E_k \qquad \text{for all } \tau \in [0, 1],\ u \in \Psi^c \cap E_k,\ k \geq 2.$$

Next we show that $\Psi|_{\mathcal{U}\cap E_k}$ does not have a critical value in $[c, d]$ for any large enough k. Indeed, if $u_k \in \Psi_c^d$ is a critical point of $\Psi|_{\mathcal{U}\cap E_k}$, then, according to (4.3),

$$\langle S_0(u_k), s_{u_k}^0 v\rangle = 0 \quad \text{for all } v \in T_{u_k}(\mathcal{U} \cap E_k),$$

i.e., $P_k S_0(u_k) = 0$. Since $u_k \in \Psi_c^d \subset \Psi^d \setminus \mathcal{V}_t$ (see (4.5)), $k < k_d$ according to Lemma 4.1.

Now Proposition 3.4(iii) allows us to use the negative gradient flow of $\Psi|_{\mathcal{U}\cap E_k}$ in the standard way to obtain a retraction $\varrho : \Psi^d \cap E_k \to \Psi^c \cap E_k$; see, e.g., [3,Theorem I.3.2]. Then, $\varrho \circ \eta : [0, 1] \times (\Psi^c \cap E_k) \to \Psi^c \cap E_k$ is a deformation of $\Psi^c \cap E_k$ into a point. □

The following statement is an immediate consequence of Lemma 4.2 and basic properties of homology (see e.g. [12,Sections III.4 and III.5]).

Corollary 4.3 *Denote the q-th singular homology with coefficients in a field $\mathbb{F}$ by $\mathrm{H}_q(\cdot)$. Then $\mathrm{H}_0(\Psi^c \cap E_k) = \mathbb{F}$ and $\mathrm{H}_q(\Psi^c \cap E_k) = 0$ for $q \neq 0$. In particular, the Euler characteristic*

$$\chi(\Psi^c \cap E_k) := \sum_{q\geq 0}(-1)^q \dim_{\mathbb{F}} \mathrm{H}_q(\Psi^c \cap E_k) = 1$$

for every large enough k.

For u_0 as above, let

$$\sigma_i : \mathscr{S} \setminus \{-u_{0,i}\} \to (\mathbb{R}u_{0,i})^{\perp} =: F_i$$

be the stereographic projection. The product $\sigma = (\sigma_1, \dots, \sigma_\ell)$ of the stereographic projections is a diffeomorphism. So its derivative at u

$$\sigma'(u) : T_u(\mathcal{U}) \to F := F_1 \times \dots \times F_\ell$$

is an isomorphism for every $u \in \mathcal{U}$. Note that, as $u_{0,i} \in E_{2,i}$, we have that $\sigma_i((\mathscr{S} \cap E_k) \setminus \{-u_{0,i}\}) \subset F_i \cap E_k$ for all $k \geq 2$.

Proof of Theorem 1.1 Let $\mathcal{O} := \sigma(\mathcal{V}_0)$ with $\mathcal{V}_0$ as in (4.4). As $u_0 \in \mathcal{V}_0$ we have that $0 \in \mathcal{O}$, and as $\overline{\mathcal{V}}_0 \subset \mathcal{U}$ and $-u_0 \notin \mathcal{U}$, $\mathcal{O}$ is bounded in F. Set $\mathcal{O}_k := \mathcal{O} \cap E_k$ and $F_k := F \cap E_k$. Then $\mathcal{O}_k$ is a bounded open neighborhood of 0 in F_k, and $\overline{\mathcal{O}}_k \subset \sigma(\Psi^c \cap E_k)$ for c as in (4.5).

Fix $k_c \in \mathbb{N}$ as in Lemma 4.1. Recall that

$$S_t(u) = s_u^t u - K_t(s_u^t u) \in T_u(\mathcal{U}) \qquad \forall u \in \mathcal{U}$$

(see Proposition 3.4(iv)). Define $G_{t,k} : \sigma(\mathcal{U} \cap E_k) \to F_k$ by

$$G_{t,k}(w) := (\sigma'(\sigma^{-1}(w)) \circ P_k \circ S_t \circ \sigma^{-1})(w). \tag{4.9}$$

Note that

$$G_{t,k}(w) = 0 \iff P_k(S_t(\sigma^{-1}(w))) = 0.$$

So, if $k \geq k_c$, $w \in \sigma(\mathcal{U} \cap E_k)$ and $G_{t,k}(w) = 0$, Lemma 4.1 asserts that $w \in \mathcal{O}_k$. In particular, $G_{t,k}(w) \neq 0$ for every $w \in \partial\mathcal{O}_k$. From the homotopy and the excision properties of the Brouwer degree we get that

$$\deg(G_{1,k}, \mathcal{O}_k, 0) = \deg(G_{0,k}, \mathcal{O}_k, 0) = \deg(G_{0,k}, \sigma(\Psi^c \cap E_k), 0). \tag{4.10}$$

On the other hand, using (4.3) and (4.9) we get

$$\begin{aligned}
(\Psi \circ \sigma^{-1})'(w)z &= \Psi'(\sigma^{-1}(w))[(\sigma^{-1})'(w)z] \\
&= s^0_{\sigma^{-1}(w)} \langle P_k(S_0(\sigma^{-1}(w))),\ (\sigma^{-1})'(w)z\rangle \\
&= s^0_{\sigma^{-1}(w)} \langle (\sigma^{-1})'(w)(G_{0,k}(w)),\ (\sigma^{-1})'(w)z\rangle \\
&= \frac{4s^0_{\sigma^{-1}(w)}}{(\|w\|^2+1)^2} \langle G_{0,k}(w), z\rangle \qquad \forall w \in \sigma(\mathcal{U} \cap E_k),\ z \in F_k.
\end{aligned}$$

The last identity is obtained by a simple calculation, see e.g. [15,Lemma 3.4]. Since $(\Psi \circ \sigma^{-1})|_{\sigma(\mathcal{U}\cap E_k)}$ is of class $\mathcal{C}^2$ (see (4.2)) and $-(\Psi \circ \sigma^{-1})'(w)$ points into $(\Psi \circ \sigma^{-1})^c$ for all $w \in (\Psi \circ \sigma^{-1})^c_c$, from [3,Theorem II.3.3] and Corollary 4.3 we obtain

$$\begin{aligned}
\deg(G_{0,k}, \sigma(\Psi^c \cap E_k), 0) &= \deg(((\Psi \circ \sigma^{-1})|_{\sigma(\mathcal{U}\cap E_k)})', \sigma(\Psi^c \cap E_k), 0) \\
&= \chi(\sigma(\Psi^c \cap E_k)) = \chi(\Psi^c \cap E_k) = 1.
\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11) gives

$$\deg(G_{1,k}, \mathcal{O}_k, 0) = 1.$$

Hence, for each $k \geq k_c$ there exists $w_k \in \mathcal{O}_k$ such that $G_{k,1}(w_k) = 0$. Then $u_k := \sigma^{-1}(w_k) \in \mathcal{V}_0 \cap E_k \subset \Psi^c \cap E_k$ satisfies $P_k(S(u_k)) = 0$, i.e.,

$$s_{u_k} u_k = P_k K(s_{u_k} u_k). \tag{4.12}$$

As in the proof of Lemma 4.1 (with t_n replaced by 1 and $s^{t_n}_{u_n} u_n$ by $s_{u_k} u_k$) one shows that $(s_{u_k,i})$ is bounded and bounded away from 0 for each i. So passing to a subsequence, $s_{u_k} \to s$ and $u_k \rightharpoonup u$ weakly in $\mathcal{H}$. Taking limits in (4.12) and using Lemma 3.1, we obtain that $s_{u_k} u_k \to su$ strongly in $\mathcal{H}$ and $su = K(su)$. Hence, $su \in \mathcal{N}$, $s = s_u$ and $S(u) = s_u u - K(s_u u) = 0$. So, according to Proposition 3.4(v), $s_u u$ is a solution to (1.1). □

5 Synchronized solutions

A solution $u = (u_1, \ldots, u_\ell)$ to (1.1) is called *synchronized* if $u_i = t_i v$ and $u_j = t_j v$ for some $i \neq j$, $v \in H_0^1(\Omega) \setminus \{0\}$ and $t_1, t_2 > 0$. In this section we consider a system of 2 equations:

$$\begin{cases} -\Delta u_1 = \mu_1 u_1^p + \lambda_{12} u_1^{\alpha_{12}} u_2^{\beta_{12}}, \\ -\Delta u_2 = \mu_2 u_2^p + \lambda_{21} u_2^{\alpha_{21}} u_1^{\beta_{21}}, \\ u_1, u_2 \geq 0 \text{ in } \Omega, \quad u_1, u_2 \in H_0^1(\Omega) \setminus \{0\}. \end{cases} \tag{5.1}$$

Recall that according to our assumptions $\alpha_{12} + \beta_{12} < p$ and $\alpha_{21} + \beta_{21} < p$.

Theorem 5.1 *The system* (5.1) *has a synchronized solution if and only if* $\alpha_{12} + \beta_{12} = \alpha_{21} + \beta_{21} =: q$ *and*

$$\frac{\lambda_{12}}{\lambda_{21}} = \left(\frac{\mu_1}{\mu_2}\right)^{(\alpha_{21}-\beta_{12}-1)/(p-1)}. \tag{5.2}$$

Proof Inserting $u_1 = t_1 v$, $u_2 = t_2 v$ into (5.1) we obtain

$$\begin{cases} -t_1 \Delta v = \mu_1 t_1^p v^p + \lambda_{12} t_1^{\alpha_{12}} t_2^{\beta_{12}} v^{\alpha_{12}+\beta_{12}} \\ -t_2 \Delta v = \mu_2 t_2^p v^p + \lambda_{21} t_2^{\alpha_{21}} t_1^{\beta_{21}} v^{\alpha_{21}+\beta_{21}}. \end{cases}$$

Dividing the first equation by t_1, the second one by t_2 and subtracting gives

$$(\mu_1 t_1^{p-1} - \mu_2 t_2^{p-1}) v^p + (\lambda_{12} t_1^{\alpha_{12}-1} t_2^{\beta_{12}} v^{\alpha_{12}+\beta_{12}} - \lambda_{21} t_2^{\alpha_{21}-1} t_1^{\beta_{21}} v^{\alpha_{21}+\beta_{21}}) = 0.$$

So $\alpha_{12} + \beta_{12} = \alpha_{21} + \beta_{21} = q$,

$$\mu_1 t_1^{p-1} - \mu_2 t_2^{p-1} = 0 \quad \text{and} \quad \lambda_{12} t_1^{\alpha_{12}-1} t_2^{\beta_{12}} = \lambda_{21} t_2^{\alpha_{21}-1} t_1^{\beta_{21}}.$$

Inserting the solution

$$t_2 = \left(\frac{\mu_1}{\mu_2}\right)^{1/(p-1)} t_1$$

of the first equation into the second one gives (5.2).

We have shown that the conditions in Theorem 5.1 are necessary. It remains to show that they are also sufficient. To this aim observe that, if $\alpha_{12} + \beta_{12} = \alpha_{21} + \beta_{21} =: q$, (5.2) holds true, and w satisfies

$$-\Delta w = \mu_1 w^p - a w^q, \qquad w \geq 0,\ w \in H_0^1(\Omega) \setminus \{0\}, \tag{5.3}$$

with

$$a := -\lambda_{12} \left(\frac{\mu_1}{\mu_2}\right)^{\beta_{12}/(p-1)},$$

then $\left(w, \left(\frac{\mu_1}{\mu_2}\right)^{1/(p-1)} w\right)$ solves the system (5.1). Consider the functional

$$\Phi(w) := \frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{a}{q+1}\int_\Omega (w^+)^{q+1} - \frac{\mu_1}{p+1}\int_\Omega (w^+)^{p+1}.$$

By standard arguments (see e.g. [17] or [18]), Φ is of class $\mathcal{C}^1$ and critical points of Φ are solutions to the equation

$$-\Delta w + a(w^+)^q = \mu_1 (w^+)^p. \tag{5.4}$$

We shall complete the proof by showing that Φ has a nontrivial critical point $w \geq 0$. We use the mountain pass theorem (see e.g. [17] or [18]). By easy calculations (as e.g. in [18,Proof of Theorem 1.19]), Φ has the mountain pass geometry. Here it is important that $p > 2$ and $p > q$. Next we show that Φ satisfies the Palais-Smale condition. Let (w_n) be such that $\Phi(w_n) \to c$ and $\Phi'(w_n) \to 0$. Then

$$\begin{aligned} c + 1 + \|w_n\| &\geq \Phi(w_n) - \frac{1}{p+1}\Phi'(w_n)w_n \\ &= \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_\Omega |\nabla w_n|^2 + a\left(\frac{1}{q+1} - \frac{1}{p+1}\right)\int_\Omega (w_n^+)^{q+1} \end{aligned}$$

for all n large enough. Hence (w_n) is bounded, so passing to a subsequence, $w_n \to w$ weakly in $H_0^1(\Omega)$, and strongly in $L^p(\Omega)$ and $L^q(\Omega)$. It follows by a standard argument (see e.g. [18,Proof of Lemma 1.20]) that $w_n \to w$ strongly also in $H_0^1(\Omega)$. Finally, multiplying (5.4) by w^- gives $\int_\Omega |\nabla w^-|^2 = 0$, so $w^- = 0$. The proof is complete. □

Remark 5.2 It is easy to show that if $q = p$, then there are no synchronized solutions for $-\lambda_{ij}$ sufficiently large, as is well known in the variational case, see e.g. [4,Proposition 3.2].

Remark 5.3 Let $\lambda_{ij,n} < 0$, $i \neq j$, and let $u_n = (u_{n,1}, \ldots, u_{n,\ell})$ be a solution to (1.1) with λ_{ij} replaced by $\lambda_{ij,n}$. It is easy to see that, if the sequence (u_n) is bounded in $\mathcal{H}$, the components $u_{n,i}$ *separate spatially* as $\lambda_{ij,n} \to -\infty$. More precisely, after passing to a subsequence, $u_{n,i} \to u_i \neq 0$ weakly in $H_0^1(\Omega)$ and strongly in $L^p(\Omega)$ for each i, and $u_i(x) \cdot u_j(x) = 0$ a.e. in Ω for $i \neq j$. There is an extensive literature on spatial separation of solutions and limiting profiles, under the assumption that the sequence (u_n) is bounded and under different assumptions on the nonlinearities. See e.g. [6, 7, 16] and the references therein.

Obviously, synchronized solutions to (1.1) do not separate spatially. So we cannot expect the sequence (w_n) given by (5.3) to be bounded. Indeed, we have the following

Proposition 5.4 *Let (w_n) be a sequence of solutions to (5.3) with $a = a_n$. If $a_n \to \infty$, then (w_n) is unbounded in $H_0^1(\Omega)$.*

Proof Suppose (w_n) is bounded. Then, passing to a subsequence, $w_n \to w$ weakly in $H_0^1(\Omega)$, strongly in $L^p(\Omega)$ and in $L^q(\Omega)$. Since

$$\int_\Omega |\nabla w_n|^2 + a_n \int_\Omega w_n^q = \mu_1 \int_\Omega w_n^p,$$

we have that $w_n \to 0$ in $L^q(\Omega)$. So $w = 0$ and therefore $w_n \to 0$ strongly in $H_0^1(\Omega)$. This is a contradiction because by the Sobolev inequality,

$$\int_\Omega |\nabla w_n|^2 \le \mu_1 \int_\Omega w_n^p \le C \left(\int_\Omega |\nabla w_n|^2 \right)^{p/2}$$

for some constant C, so $\|w_n\|$ is bounded away from 0. □

It is well known that, when the system (1.1) is variational, least energy solutions are bounded in $\mathcal{H}$, independently of λ_{ij}. We close this section with the following open question.

Problem 5.5 *Given $\lambda_{ij,n} \to -\infty$ for $i \neq j$, does the system (1.1) with λ_{ij} replaced by $\lambda_{ij,n}$ have a solution u_n such that the sequence $(u_{n,i})$ is bounded in $H_0^1(\Omega)$ for all i?*

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A Appendix

In this appendix we prove Lemma 1.2. We employ some arguments which may be found in [11, 13]. First we note that by standard regularity results the solutions u_i of (1.3) are in $\mathcal{C}^2(\Omega) \cap \mathcal{C}(\overline{\Omega})$.

Suppose there exists a sequence of solutions (u_n) with $|u_n|_\infty \to \infty$. Passing to a subsequence, we may assume $|u_{n,i}|_\infty \to \infty$ and $|u_{n,i}|_\infty \ge |u_{n,j}|_\infty$ for some i and all j. There exists $x_n \in \Omega$ such that

$$\max_{x \in \Omega} u_{n,i}(x) = u_{n,i}(x_n).$$

Let $\beta := \frac{2}{p-1}$ and choose ϱ_n so that

$$\varrho_n^{\beta} \, |u_{n,i}|_{\infty} = 1.$$

Then $\varrho_n \to 0$ and passing to a subsequence, $x_n \to x_0 \in \overline{\Omega}$. Let

$$\Omega_n := \{y \in \mathbb{R}^N : \varrho_n y + x_n \in \Omega\}$$

and

$$v_{n,j}(y) := \varrho_n^{\beta} u_{n,j}(\varrho_n y + x_n), \quad j = 1, \ldots, \ell. \tag{A.1}$$

Then,

$$0 \leq v_{n,i} \leq 1, \quad v_{n,i}(0) = 1 \quad \text{and } v_{n,j}|_{\partial\Omega_n} = 0 \text{ for all } j. \tag{A.2}$$

Passing to a subsequence, there are two possible cases and we shall complete the proof by ruling out both of them. Denote the distance from x to a set A by $d(x, A)$.
Case 1. $\frac{d(x_n, \partial\Omega)}{\varrho_n} \to \infty$.

Since $\varrho_n y + x_n \in \Omega$ if $|y| < \frac{d(x_n, \partial\Omega)}{\varrho_n}$, for each $R > 0$ there exists n_0 such that $B_R(0) \subset \Omega_n$ whenever $n \geq n_0$. For $y \in B_R(0)$ and $n \geq n_0$ we have

$$\begin{aligned} -\Delta_y v_{n,i} &= \varrho_n^{\beta+2} \Delta_x u_{n,i} = \varrho_n^{\beta+2}\Big(\mu_i u_{n,i}^p + \sum_{j \neq i} \lambda_{ij} u_{n,i}^{\alpha_{ij}} u_{n,j}^{\beta_{ij}}\Big) \\ &= \varrho_n^{\beta+2-\beta p} \mu_i v_{n,i}^p + \sum_{j \neq i} \varrho_n^{\beta+2-\beta(\alpha_{ij}+\beta_{ij})} \lambda_{ij} v_{n,i}^{\alpha_{ij}} v_{n,j}^{\beta_{ij}}. \end{aligned}$$

Since $\beta + 2 - \beta p = 0$ and $\gamma_{ij} := \beta + 2 - \beta(\alpha_{ij} + \beta_{ij}) > 0$, we can re-write this identity as

$$-\Delta v_{n,i} = \mu_i v_{n,i}^p + \sum_{j \neq i} \varrho_n^{\gamma_{ij}} \lambda_{ij} v_{n,i}^{\alpha_{ij}} v_{n,j}^{\beta_{ij}}.$$

By elliptic estimates, $(v_{n,i})$ is bounded in $W^{2,q}(B_R(0))$ for some $q > N$. So passing to a subsequence, $v_{n,i} \to v_i$ weakly in $W^{2,q}(B_R(0))$ and strongly in $\mathcal{C}^1(B_R(0))$. Since $\varrho_n^{\gamma_{ij}} \to 0$, v_i is a nonnegative solution to the equation

$$-\Delta v = \mu_i v^p$$

in $B_R(0)$. Let now $R_m \to \infty$. Then for each m we get a solution $v_{i,m}$ of the above equation in $B_{R_m}(0)$. Passing to subsequences and applying the diagonal procedure, we see that $v_{i,m} \to w$, weakly in $W^{2,q}_{loc}(\mathbb{R}^N)$ and strongly in $\mathcal{C}^1_{loc}(\mathbb{R}^N)$. So $-\Delta w = \mu_i w^p$ in $\mathbb{R}^N$, $w \geq 0$, $w(0) = 1$ according to (A.2), and $w \in \mathcal{C}^2(\mathbb{R}^N)$ by Schauder estimates. Replacing w with cw for a suitable $c > 0$ we may assume $\mu_i = 1$. Hence it follows from [13,Theorem 1.2] that $w = 0$ which rules out Case 1.
Case 2. $\frac{d(x_n,\partial\Omega)}{\varrho_n} \to d \in [0, \infty)$.

It is clear that $x_0 \in \partial\Omega$ and we may assume without loss of generality that $x_0 = 0$ and $\nu = (0, \dots, 0, 1)$ is the unit outer normal to $\partial\Omega$ at x_0. Let

$$\mathbb{H}^N := \{y \in \mathbb{R}^N : y_N < d\} \quad \text{where } y = (y_1, \dots, y_N).$$

We shall need the following result.

Lemma A.1 *(i) Let $A \subset \mathbb{H}^N$ be compact. Then there exists n_0 such that $\varrho_n y + x_n \in \Omega$ for all $n \geq n_0$ and $y \in A$.*
(ii) Let $A \subset \mathbb{R}^N \smallsetminus \overline{\mathbb{H}^N}$ be compact. Then there exists n_0 such that $\varrho_n y + x_n \notin \Omega$ for all $n \geq n_0$ and $y \in A$.

Proof (i) : Since A is compact, there exists $\varepsilon > 0$ such that $y_N < d - 2\varepsilon$ for all $y \in A$. For each n there exists $\widehat{x}_n \in \partial\Omega$ which is closest to x_n, i.e., $d(x_n, \partial\Omega) = |x_n - \widehat{x}_n|$. As $\partial\Omega$ is tangent to the hyperplane $x_N = 0$ at 0,

$$\frac{|x_n - \widehat{x}_n|}{\varrho_n} = \frac{\widehat{x}_{n,N} - x_{n,N}}{\varrho_n} + o(1).$$

Therefore,

$$\frac{x_{n,N} - \widehat{x}_{n,N}}{\varrho_n} < -d + \varepsilon \quad \text{and} \quad y_N + \frac{x_{n,N} - \widehat{x}_{n,N}}{\varrho_n} < -\varepsilon$$

for all $y \in A$ if n is large enough. There exists $C > 0$ such that

$$\left| y + \frac{x_n - \widehat{x}_n}{\varrho_n} \right| \leq C \quad \text{for all } y \in A.$$

Using this, we see that there is n_0 such that, if $n \geq n_0$ and $y \in A$, then $\varrho_n y + x_n - \widehat{x}_n \in \Omega$ and, as $\widehat{x}_n \in \partial\Omega$ and $\partial\Omega$ is tangent to the hyperplane $x_N = 0$ at 0, $\varrho_n y + x_n = \varrho_n y + (x_n - \widehat{x}_n) + \widehat{x}_n \in \Omega$.

(ii) : This time $y_N > d + 2\varepsilon$ for $y \in A$,

$$\frac{x_{n,N} - \widehat{x}_{n,N}}{\varrho_n} > -d - \varepsilon \quad \text{and} \quad y_{n,N} + \frac{x_{n,N} - \widehat{x}_{n,N}}{\varrho_n} > \varepsilon$$

if n is sufficiently large, and the conclusion follows by a similar argument as above. □

Now we can continue with Case 2. Let $\omega_R := B_R(0) \cap \{y \in \mathbb{R}^N : y_N < d - 1/R\}$. Then $\overline{\omega}_R \subset \Omega_n$ for all $n \geq n_0$ by Lemma A.1(i). Let $v_{n,i}$ be given by (A.1) and using (A.2) extend it by 0 outside Ω_n. According to Lemma A.1(ii), if $A \subset \mathbb{R}^N \setminus \overline{\mathbb{H}^N}$ is compact, then $\varrho_n y + x_n \notin \Omega$ for all $y \in A$ and n large enough. So

$$\lim_{n\to\infty} v_{n,i}(x) = 0 \quad \text{for all } x \notin \mathbb{H}^N. \tag{A.3}$$

We can repeat the argument of Case 1 which now gives a nonegative solution to the equation $-\Delta w = \mu_i w^p$ in $\mathbb{H}^N$ such that $w(0) = 1$. By (A.3), $w = 0$ on $\partial\Omega$. As before, $w \in \mathcal{C}^2(\mathbb{H}^N)$, and since the extended functions $v_{n,i}$ are continuous in $\mathbb{R}^N$, $w \in \mathcal{C}^0(\overline{\mathbb{H}^N})$. So $w = 0$ according to [13,Theorem 1.3], a contradiction. Hence also Case 2 is ruled out.

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