Analysis and Mathematical Physics
https://doi.org/10.1007/s13324-019-00340-8

# On the order of strong starlikeness and the radii of starlikeness for of some close-to-convex functions

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Received: 10 February 2018 / Revised: 5 July 2019 / Accepted: 5 September 2019


**Abstract**
In this paper we show several sufficient conditions for close-to-convex functions to be strongly starlike of some order. The results continue the line of study from the first author's paper on the order of strong starlikeness of strongly convex functions, (Nunokawa in Proc Japan Acad Ser A 69(7):234–237, 1993). Also it appears an small improvement of a certain classical results of Ch. Pommerenke. As an application, we also derive estimates for the radii of star-likeness for close-to-convex functions.

**Keywords** Analytic functions · Convex functions · Starlike functions · Univalent functions · Strongly starlike

**Mathematics Subject Classification** Primary 30C45 · Secondary 30C80

## 1 Introduction

Let $\mathcal{H}$ denote the class of analytic functions in the unit disc $\mathbb{D} = \{z : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$. The set of all functions $f \in \mathcal{A}$ that are convex univalent in $\mathbb{D}$ we denote by $\mathcal{K}$. The set of all functions $f \in \mathcal{A}$ that are starlike univalent in $\mathbb{D}$ with respect to the origin we denote by $\mathcal{S}^*$. Recall that a set $E \subset \mathbb{C}$ is said to be starlike with respect to a point $w_0 \in E$ if and only if the line segment joining $w_0$ to any other point $w \in E$ lies entirely in $E$. A set $E$ is said to be convex if and only if it is starlike with respect to each of its points, that is if and only if the linear segment joining any

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Published online: 13 September 2019

two points of $E$ lies entirely in $E$. An univalent function $f$ maps $\mathbb{D}$ onto a convex domain $E$ if and only if [11]

$$\mathfrak{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0 \quad \text{for all } z\in\mathbb{D}.$$

Such a function $f$ is said to be convex in $\mathbb{D}$ (or briefly convex). In [8] Sakaguchi proved that if $f\in\mathcal{A}$ and $g\in\mathcal{S}^*$, then

$$\left[\mathfrak{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0,\quad z\in\mathbb{D}\right]\Rightarrow\left[\mathfrak{Re}\left\{\frac{f(z)}{g(z)}\right\}>0,\quad z\in\mathbb{D}\right]. \tag{1.1}$$

This result was also generalized, see [2] and [7]. In [6] Pommerenke established a formula for $\beta=\beta(\alpha)$ such that

$$\left|\mathrm{Arg}\left\{\frac{f'(z)}{g'(z)}\right\}\right|<\frac{\alpha\pi}{2},\quad z\in\mathbb{D}\quad\Rightarrow\quad\left|\mathrm{Arg}\left\{\frac{f(z)}{g(z)}\right\}\right|<\frac{\beta\pi}{2},\quad z\in\mathbb{D},$$

where $f(z)\in\mathcal{K}_\alpha$. This is a generalization of the relation of type (1.1) because is in the class $\mathcal{K}_\alpha$ $\alpha$, $0<\alpha\le 1$ whenever $f(z)\in\mathcal{A}$ and there exist a function $g(z)\in\mathcal{K}$ such that

$$\left|\mathrm{Arg}\left\{\frac{f'(z)}{g'(z)}\right\}\right|<\frac{\alpha\pi}{2},\quad z\in\mathbb{D}. \tag{1.2}$$

Here we understand that $\mathrm{Arg}w$ is a number in $(-\pi,\pi]$. It is known that if $f(z)\in\mathcal{K}_\alpha$, then $f(z)$ is close-to-convex and so $f(z)$ is univalent in $\mathbb{D}$. The class $f(z)\in\mathcal{K}_\alpha$ is called the class of strongly close-to-convex functions of order $\alpha$.

This result has found many applications. Condition (1.2) with $g(z)=f(z)$ becomes

$$\left|\mathrm{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right|<\frac{\pi\alpha}{2},\quad z\in\mathbb{D}, \tag{1.3}$$

and it says that $f(z)$ is a strongly starlike function of order $\alpha$, $0<\alpha\le 1$. The class of strongly starlike functions was introduced in [1,10], we denote this class here by $\mathcal{S}^*(\alpha)$. One can consider functions satisfying condition (1.3) with $0<\alpha<2$ and in this case we will named such functions also strongly starlike of order $\alpha$, $0<\alpha<2$. It is known that if $f(z)$ is strongly starlike of order $\alpha>1$, then $f(z)$ need not to be univalent in $\mathbb{D}$.

We say that $f(z)\in\mathcal{K}_\beta^\alpha$ whenever $f(z)\in\mathcal{A}$ and there exists a real $\alpha$, $0<\alpha\le 1$ and a function $g(z)\in\mathcal{S}^*(\beta)\cap\mathcal{K}$, $0<\beta\le 1$, such that

$$\left|\mathrm{Arg}\left\{\frac{f'(z)}{g'(z)}\right\}\right|<\frac{\alpha\pi}{2},\quad z\in\mathbb{D}. \tag{1.4}$$

**Lemma 1.1** *Let* $f(z)\in\mathcal{K}_\beta^\alpha$ *with* $0<2\alpha+\beta\le 1$, $0<\alpha\le 1$, $0<\beta\le 1$. *Then we have*

$$\left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| < \frac{\pi}{2}(\beta + 2\alpha), \quad z \in \mathbb{D},$$

*or $f(z)$ is strongly starlike function, of order $\beta + 2\alpha$.*

***Proof*** If $f(z) \in \mathcal{K}_\beta^\alpha$, then $f(z)$ is univalent in $\mathbb{D}$ and so

$$\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}$$

exists for all $z \in \mathbb{D}$. It is known the following Pommerenke's result [6, Lemma 1, p. 180]: If $f(z)$ is analytic and $g(z)$ is convex in $\mathbb{D}$, then

$$\begin{aligned} &\exists \alpha \in (0, 1]\ \forall z \in \mathbb{D}: \quad \left|\operatorname{Arg}\frac{f'(z)}{g'(z)}\right| < \frac{\alpha\pi}{2} \\ &\Rightarrow \quad \forall z_1, z_2 \in \mathbb{D}: \quad \left|\operatorname{Arg}\frac{f(z_2) - f(z_1)}{g(z_2) - g(z_1)}\right| < \frac{\alpha\pi}{2}. \end{aligned} \tag{1.5}$$

Applying this Lemma with $z_1 = 0$ gives

$$\begin{aligned} \operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\} &= \operatorname{Arg}\left\{\frac{f'(z)}{g'(z)}\frac{zg'(z)}{g(z)}\frac{g(z)}{f(z)}\right\} \\ &\le \operatorname{Arg}\left\{\frac{f'(z)}{g'(z)}\right\} + \operatorname{Arg}\left\{\frac{zg'(z)}{g(z)}\right\} + \operatorname{Arg}\left\{\frac{g(z)}{f(z)}\right\}, \end{aligned} \tag{1.6}$$

it follows that

$$\begin{aligned} \left|\operatorname{Arg}\left\{\frac{zf'(z)}{f(z)}\right\}\right| &\le \left|\operatorname{Arg}\left\{\frac{f'(z)}{g'(z)}\right\}\right| + \left|\operatorname{Arg}\left\{\frac{zg'(z)}{g(z)}\right\}\right| + \left|\operatorname{Arg}\left\{\frac{g(z)}{f(z)}\right\}\right| \\ &< \frac{\alpha\pi}{2} + \frac{\beta\pi}{2} + \frac{\alpha\pi}{2} \\ &= \frac{\pi}{2}(\beta + 2\alpha). \end{aligned}$$

□

We note that a result of the form related to (1.5) was proved in [5, Theorem 2.1].

**Lemma 1.2** *Let $h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ be in $\mathcal{H}$. If*

$$|\operatorname{Arg}\{h(z)\}| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

*for some $\alpha > 0$, then*

$$|\operatorname{Arg}\{h(z)\}| < \alpha \sin^{-1}\frac{2|z|}{1 + |z|^2}, \quad z \in \mathbb{D}, \tag{1.7}$$

*and*

$$\mathfrak{Re}\{h^{1/\alpha}(z)\} > \frac{1-|z|}{1+|z|}, \quad z \in \mathbb{D}. \tag{1.8}$$

***Proof*** From the hypothesis $h(z) \neq 0$, so the function

$$h^{1/\alpha}(z), \quad h^{1/\alpha}(0) = 1,$$

is in the class $\mathcal{H}$. Furthermore, we have

$$\mathfrak{Re}\{h^{1/\alpha}(z)\} > 0, \quad z \in \mathbb{D},$$

hence

$$h^{1/\alpha}(z) \prec \frac{1+z}{1-z},$$

and $h^{1/\alpha}(z)$ is contained in the circle with the radius and center

$$R = \frac{2|z|}{1-|z|^2} \quad \text{and} \quad C = \frac{1+|z|^2}{1-|z|^2}$$

respectively. From this, we obtain (1.7) and (1.8). □

If we take $g(z) = z$ then Pommerenke's result (1.5) becomes the following corollary.

**Corollary 1.3** *Let $f(z)$ be in $\mathcal{A}$. If*

$$\left|\mathrm{Arg}\{f'(z)\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

*for some $\alpha$, $0 < \alpha \leq 1$, then*

$$\left|\mathrm{Arg}\left\{\frac{f(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

In the next Corollary we extend $\alpha$ to $1 < \alpha < 2$:

**Corollary 1.4** *Let $f(z)$ be in $\mathcal{A}$. If*

$$\left|\mathrm{Arg}\{f'(z)\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D},$$

*for some $\alpha$, $1 < \alpha \leq 2$, then*

$$\left|\mathrm{Arg}\left\{\frac{f(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

***Proof*** For arbitrary $z \in \mathbb{D}$ and from the hypothesis $1 < \alpha \le 2$, can connect the point $f'(z)$ and $f'(0) = 1$ by a line segment as $\overline{f'(z)f'(0)} = \overline{f'(z)1}$. Applying the same method as in the proof of Pommerenke [6, p. 180], we have

$$\frac{f(z)}{z} = \frac{f(z) - f(0)}{z - 0} = \int_0^1 f'(0 + tz)\mathrm{d}t$$

and so, applying the property of integral mean, we have

$$\left|\operatorname{Arg}\left\{\frac{f(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}.$$

□

To prove the main results, we also need the following generalization of the Nunokawa's Lemma, [3,4].

**Lemma 1.5** [5] *Let $p(z)$ be analytic function in $|z| < 1$ of the form*

$$p(z) = 1 + \sum_{n=k}^{\infty} a_n z^n, \quad a_k \neq 0,$$

*with $p(z) \neq 0$ in $|z| < 1$. If there exists a point $z_0$, $|z_0| < 1$, such that*

$$|\operatorname{Arg}\{p(z)\}| < \frac{\pi\alpha}{2} \quad for \quad |z| < |z_0|$$

*and*

$$|\operatorname{Arg}\{p(z_0)\}| = \frac{\pi\alpha}{2}$$

*for some $\alpha > 0$, then we have*

$$\frac{z_0 p'(z_0)}{p(z_0)} = im\alpha,$$

*where*

$$m \ge \frac{k}{2}\left(a + \frac{1}{a}\right) \ge k \quad when \quad \operatorname{Arg}\{p(z_0)\} = \frac{\pi\alpha}{2} \tag{1.9}$$

*and*

$$m \le -\frac{k}{2}\left(a + \frac{1}{a}\right) \le -k \quad when \quad \operatorname{Arg}\{p(z_0)\} = -\frac{\pi\alpha}{2}, \tag{1.10}$$

*where*

$$\{p(z_0)\}^{1/\alpha} = \pm ia, \quad and \quad a > 0.$$

**Theorem 1.6** *Let $f(z)$ and $g(z)$ be in $\mathcal{A}$. Suppose that*

$$\left|\operatorname{Arg}\left\{\frac{f'(z)}{g'(z)}\right\}\right| < \frac{\alpha\pi}{2} + \tan^{-1}\frac{\alpha\beta\cos\left\{\operatorname{Arg}\frac{g(z)}{zg'(z)}\right\}}{1 - \alpha\beta\sin\left\{\operatorname{Arg}\frac{g(z)}{zg'(z)}\right\}}, \quad z \in \mathbb{D}, \tag{1.11}$$

*for some* $\alpha$ *and* $\beta$ *such that* $0 < \alpha$ *and* $0 < \beta < 1$. *Assume also*

$$\left|\frac{g(z)}{zg'(z)}\right| > \beta, \quad z \in \mathbb{D}, \tag{1.12}$$

*then*

$$\left|\mathrm{Arg}\left\{\frac{f(z)}{g(z)}\right\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}. \tag{1.13}$$

***Proof*** Let us write

$$p(z) = \frac{f(z)}{g(z)},$$

then we have

$$\frac{f'(z)}{g'(z)} = p(z)\left(1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right)$$

and

$$\mathrm{Arg}\left\{\frac{f'(z)}{g'(z)}\right\} = \mathrm{Arg}\left\{p(z)\left(1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right)\right\} \tag{1.14}$$

$$= \mathrm{Arg}\,\{p(z)\} + \mathrm{Arg}\left\{1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right\}. \tag{1.15}$$

If there exists $z_0 \in \mathbb{D}$ such that

$$|\mathrm{Arg}\,\{p(z)\}| < \frac{\alpha\pi}{2}, \quad |z| < |z_0|$$

and

$$|\mathrm{Arg}\,\{p(z_0)\}| = \frac{\alpha\pi}{2},$$

then from Lemma 1.5

$$\frac{zp'(z_0)}{p(z_0)} = i\alpha k$$

for some $k \geq 1$ when $\mathrm{Arg}\{p(z_0)\} = \alpha\pi/2$ or for some $k \leq -1$ when $\mathrm{Arg}\{p(z_0)\} = -\alpha\pi/2$. For the case $\mathrm{Arg}\{p(z_0)\} = \alpha\pi/2$, from (1.12), we have

$$\begin{aligned}
&\mathrm{Arg}\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0 g'(z_0)}\right\} = \mathrm{Arg}\left\{1 + i\alpha k\frac{g(z_0)}{z_0 g'(z_0)}\right\} \\
&> \mathrm{Arg}\left[1 + i\alpha\beta k\left(\cos\left\{\mathrm{Arg}\frac{g(z_0)}{z_0 g'(z_0)}\right\} + i\sin\left\{\mathrm{Arg}\frac{g(z_0)}{z_0 g'(z_0)}\right\}\right)\right] \\
&= \tan^{-1}\frac{\alpha\beta k\cos\left\{\mathrm{Arg}\frac{g(z_0)}{z_0 g'(z_0)}\right\}}{1 - \alpha\beta k\sin\left\{\mathrm{Arg}\frac{g(z_0)}{z_0 g'(z_0)}\right\}} \\
&\geq \tan^{-1}\frac{\alpha\beta\cos\left\{\mathrm{Arg}\frac{g(z_0)}{z_0 g'(z_0)}\right\}}{1 - \alpha\beta\sin\left\{\mathrm{Arg}\frac{g(z_0)}{z_0 g'(z_0)}\right\}}.
\end{aligned}$$

This contradicts hypothesis (1.11). For the case $\mathrm{Arg}\, p(z_0) = -\alpha\pi/2$, applying the same method as the above, we obtain a $k \leq -1$ and so

$$\mathrm{Arg}\left\{1 + \frac{z_0 p'(z_0)}{p(z_0)} \frac{g(z_0)}{z_0 g'(z_0)}\right\} \leq -\tan^{-1} \frac{\alpha\beta \cos\left\{\mathrm{Arg} \frac{g(z_0)}{z_0 g'(z_0)}\right\}}{1 - \alpha\beta \sin\left\{\mathrm{Arg} \frac{g(z_0)}{z_0 g'(z_0)}\right\}}.$$

This contradicts hypothesis (1.11) too and so it completes the proof. □

Now, we prove another improvement of Corollary 1.3.

**Theorem 1.7** *Let* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *be in* $\mathcal{A}$. *If*

$$\left|\mathrm{Arg}\{f'(z)\}\right| < \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right), \quad z \in \mathbb{D}, \tag{1.16}$$

*for some* $\alpha$, $0 < \alpha$, *then*

$$\left|\mathrm{Arg}\left\{\frac{f(z)}{z}\right\}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}. \tag{1.17}$$

***Proof*** Let us write

$$p(z) = \frac{f(z)}{z}, \quad p(0) = 1, \quad z \in \mathbb{D}.$$

Then $p \in \mathcal{H}$ and it is the form

$$p(z) = 1 + \sum_{n=2}^{\infty} a_n z^{n-1}.$$

If there exists a point $z_0$, $|z_0| < 1$ such that

$$|\mathrm{Arg}\,\{p(z)\}| < \pi\alpha/2, \quad \text{in} \quad |z| < |z_0|,$$

and

$$|\mathrm{Arg}\,\{p(z_0)\}| = \pi\alpha/2,$$

for some $\alpha$, $0 < \alpha$, then from [4], we have

$$\frac{z_0 p'(z_0)}{p(z_0)} = ik\alpha,$$

for some $k$, where $k \geq 1$ when $\mathrm{Arg}\{p(z_0)\} = \alpha\pi/2$ while $k \leq -1$ when $\mathrm{Arg}\{p(z_0)\} = -\alpha\pi/2$. From the equality $f(z) = zp(z)$, we have

$$f'(z) = p(z) + zp'(z) = p(z)\left\{1 + \frac{zp'(z)}{p(z)}\right\},$$

and so, for the case $\mathrm{Arg}\{p(z_0)\} = \alpha\pi/2$, we have

$$\begin{aligned}\mathrm{Arg}\left\{f'(z_0)\right\} &= \mathrm{Arg}\left\{p(z_0)\left\{1+\frac{z_0 p'(z_0)}{p(z_0)}\right\}\right\}\\ &= \mathrm{Arg}\left\{p(z_0)\right\} + \mathrm{Arg}\left\{1+\frac{z_0 p'(z_0)}{p(z_0)}\right\}\\ &= \frac{\pi\alpha}{2} + \mathrm{Arg}\left\{1+i\alpha k\right\}\\ &\geq \frac{\pi\alpha}{2} + \tan^{-1}\alpha\\ &= \frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right).\end{aligned}$$

This contradicts the hypothesis (1.16) and for the case $\mathrm{Arg}\{p(z_0)\} = -\alpha\pi/2$, applying the same method as the above, we have

$$\mathrm{Arg}\left\{f'(z_0)\right\} \leq -\frac{\pi}{2}\left(\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right).$$

This also contradicts the hypothesis (1.16) and so we obtain that (1.17) holds true. □

$$\begin{aligned}\mathrm{Arg}\left\{1+\frac{z_0 p'(z_0)}{p(z_0)}\frac{g(z_0)}{z_0 g'(z_0)}\right\} = \mathrm{Arg}\left\{1+i\alpha k\frac{g(z_0)}{z_0 g'(z_0)}\right\} &= \tan^{-1}\frac{\left(\beta\cos\frac{\pi\gamma}{2}\right)k}{1+\left(\beta\sin\frac{\pi\gamma}{2}\right)k}\\ &\geq \tan^{-1}\frac{\beta\cos\frac{\pi\gamma}{2}}{1+\beta\sin\frac{\pi\gamma}{2}}.\end{aligned} \tag{1.18}$$

***Remark 1*** Theorem 1.7 shows that

$$\left|\mathrm{Arg}\{f'(z)\}\right| < \frac{\pi\beta}{2} \quad\Rightarrow\quad \left|\mathrm{Arg}\left\{\frac{f(z)}{z}\right\}\right| < \frac{\pi}{2}\left(\beta - \frac{2}{\pi}\tan^{-1}\alpha\right) < \frac{\pi\beta}{2}$$

in $\mathbb{D}$, where $\beta = \alpha + \frac{2}{\pi}\tan^{-1}\alpha$, for some $\alpha$, $0 < \alpha$.

***Remark 2*** In [6], Pommerenke supposed that $0 < \alpha \leq 1$ but in Theorem 1.7 we supposed $0 < \alpha$ only.

**Theorem 1.8** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $\mathcal{K}_\alpha$ for some $\alpha$, $0 < \alpha \leq 1$. Suppose also that*

$$|\mathrm{Arg}\{f'(z)\}| < \frac{\pi}{2}\left\{\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right\}, \quad z \in \mathbb{D}.$$

*If*

$$2\alpha + \frac{2}{\pi}\tan^{-1}\alpha \leq 1, \tag{1.19}$$

*then $f(z)$ is starlike. If*

$$2\alpha + \frac{2}{\pi}\tan^{-1}\alpha > 1. \tag{1.20}$$

*then $f(z)$ is starlike in $|z| < |z_0|$, where $|z_0|$ is the smallest positive root of the equation*

$$|z|^2 \sin\left(\frac{\pi}{2\left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)}\right) - 2|z| + \sin\left(\frac{\pi}{2\left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)}\right) = 0, \tag{1.21}$$

*which has the form*

$$|z_0| = \tan\left(\frac{\pi}{4\left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)}\right) < 1. \tag{1.22}$$

***Proof*** From Theorem 1.7, we have

$$\left|\operatorname{Arg}\frac{f(z)}{z}\right| < \frac{\pi\alpha}{2}, \quad z \in \mathbb{D}.$$

Therefore, we have

$$\begin{aligned}\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| &\leq |\operatorname{Arg}\{f'(z)\}| + \left|\operatorname{Arg}\frac{f(z)}{z}\right| \\ &< \frac{\pi}{2}\left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right).\end{aligned} \tag{1.23}$$

If $\alpha$ satisfies (1.19), then (1.23) follows that $f(z)$ is a starlike function. If $\alpha$ satisfies (1.20), then applying Lemma 1.2, we have

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < \left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)\sin^{-1}\frac{2|z|}{1+|z|^2} < \frac{\pi}{2}$$

for some small $|z|$. It follows that $f(z)$ is starlike in $|z| < |z_0|$, where $|z_0|$ is positive root of the equation (1.21). Simple calculation shows that

$$|z_0| = \frac{1-\sqrt{1-c^2}}{c}, \quad \text{where,} \quad c = \sin\left(\frac{\pi}{2\left(2\alpha + \frac{2}{\pi}\tan^{-1}\alpha\right)}\right),$$

so we obtain (1.22). □

**Theorem 1.9** *Let $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be in $\mathcal{K}_\alpha$ for $\alpha$, $0 < \alpha \leq 1$. Then $f(z)$ is starlike in*

$$|z| < \tan\frac{\pi}{4(1+2\alpha)} < 1. \tag{1.24}$$

***Proof*** From Lemma 1.1 and hypothesis of the Theorem, we have

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}(1+2\alpha), \quad z \in \mathbb{D}$$

and hence from Lemma 1.2, we have

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < (1+2\alpha)\sin^{-1}\frac{2|z|}{1+|z|^2}, \quad z\in\mathbb{D}.$$

Putting

$$(1+2\alpha)\sin^{-1}\frac{2|z|}{1+|z|^2} < \frac{\pi}{2},$$

we have

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < \frac{\pi}{2}, \quad z\in\mathbb{D},$$

for $|z|$ as in (1.24). □

***Remark 3*** The radius of starlikeness in the class $\mathcal{K}_\alpha$ may be equal or larger than the above value. This is an open question.

We say that $f(z)\in\mathcal{K}_\beta^\alpha$ whenever $f(z)\in\mathcal{A}$ and there exist a real $0<\alpha\le 2$ and $0<\beta\le 1$ and a function $g(z)\in\mathcal{K}$ such that

$$\left|\operatorname{Arg}\frac{f'(z)}{g'(z)}\right| < \frac{\alpha\pi}{2}, \quad z\in\mathbb{D} \tag{1.25}$$

and

$$\left|\operatorname{Arg}\frac{zg'(z)}{g(z)}\right| < \frac{\beta\pi}{2}. \tag{1.26}$$

For $\beta\in(0,1]$ the function

$$g(z) = z\exp\int_0^z \frac{(1+t)^\beta-1}{t}\mathrm{d}t, \quad z\in\mathbb{D},$$

is in $\mathcal{K}$ because

$$\mathfrak{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} = \mathfrak{Re}\left\{(1+z)^\beta+\frac{\beta}{1+z}\right\} > \frac{\beta}{2}, \quad z\in\mathbb{D},$$

and

$$\left|\operatorname{Arg}\frac{zg'(z)}{g(z)}\right| = \left|\operatorname{Arg}\left\{(1+z)^\beta\right\}\right| < \frac{\beta\pi}{2}.$$

Therefore, a function $f(z)$ such that $f(z)\equiv g(z)$ is in the class $\mathcal{K}_\beta^\alpha$ for all $0<\alpha\le 2$. For this class $\mathcal{K}_\beta^\alpha$ holds the following theorem under some restriction on $\alpha$ and $\beta$.

**Theorem 1.10** *Let $f(z)=z+\sum_{n=2}^\infty a_n z^n$ be in $\mathcal{K}_\beta^\alpha$ for some $\alpha>0$ and $0<\beta\le 1$. If $2\alpha+\beta\le 1$, then $f(z)$ is starlike in $\mathbb{D}$, if $2\alpha+\beta>1$, then $f(z)$ is starlike in*

$$|z| < \tan\frac{\pi}{4(\beta+2\alpha)} < 1. \tag{1.27}$$

***Proof*** From (1.25), (1.26) and from (1.5), we have

$$\begin{aligned}\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| &= \left|\operatorname{Arg}\left\{\frac{f'(z)}{g'(z)}\frac{zg'(z)}{g(z)}\frac{g(z)}{f(z)}\right\}\right| \\ &\leq \left|\operatorname{Arg}\left\{\frac{f'(z)}{g'(z)}\right\}\right| + \left|\operatorname{Arg}\left\{\frac{zg'(z)}{g(z)}\right\}\right| + \left|\operatorname{Arg}\left\{\frac{g(z)}{f(z)}\right\}\right| \\ &\leq \frac{\alpha\pi}{2} + \frac{\beta\pi}{2} + \frac{\alpha\pi}{2} \\ &= \frac{\pi}{2}(2\alpha+\beta), \quad z \in \mathbb{D}. \end{aligned} \tag{1.28}$$

If $2\alpha+\beta \leq 1$, then $f(z)$ is starlike in $\mathbb{D}$ because in this case (1.28) is lees than $\pi/2$. For the case $2\alpha+\beta > 1$ we can apply the same method as in the proof of Theorem 1.9 to obtain (1.27). □

**Theorem 1.11** *Let $f(z), g(z)$ be in $\mathcal{A}$. Suppose that*

$$\left|\operatorname{Arg}\frac{f'(z)}{g'(z)}\right| < \frac{\pi}{2}(\alpha-\beta-\gamma), \quad z \in \mathbb{D}, \tag{1.29}$$

*and*

$$\left|\operatorname{Arg}\frac{zf'(z)}{f(z)}\right| < \frac{\gamma\pi}{2}, \quad \left|\operatorname{Arg}\frac{zg'(z)}{g(z)}\right| < \frac{\beta\pi}{2}, \quad z \in \mathbb{D}, \tag{1.30}$$

*where $0 < \alpha-\beta-\gamma$, $0 < \beta$ and $0 < \gamma$. Then we have*

$$\left|\operatorname{Arg}\frac{f(z)}{g(z)}\right| < \frac{\alpha\pi}{2}, \quad z \in \mathbb{D}. \tag{1.31}$$

***Proof*** Putting $p(z) = f(z)/g(z)$, $z \in \mathbb{D}$, gives $p(0) = 1$ and $f(z) = p(z)g(z)$. It follows that

$$\frac{f'(z)}{g'(z)} = p(z)\left(1 + \frac{zp'(z)}{p(z)}\frac{g(z)}{zg'(z)}\right) = p(z)\frac{zf'(z)}{f(z)}\frac{g(z)}{zg'(z)}.$$

If there exists a point $z_0 \in \mathbb{D}$ such that

$$|\operatorname{Arg}\{p(z)\}| < \frac{\alpha\pi}{2} \quad \text{for} \quad |z| < |z_0|,$$

and

$$|\operatorname{Arg}\{p(z_0)\}| = \frac{\alpha\pi}{2},$$

then we have

$$\left|\mathrm{Arg}\frac{f'(z_0)}{g'(z_0)}\right| \geq |\mathrm{Arg}\{p(z_0)\}| - \left|\mathrm{Arg}\frac{z_0 f'(z_0)}{f(z_0)}\right| - \left|\mathrm{Arg}\frac{z_0 g'(z_0)}{g(z_0)}\right|$$
$$> \frac{\alpha\pi}{2} - \frac{\beta\pi}{2} - \frac{\gamma\pi}{2}.$$

This contradicts the hypothesis, and it completes the proof. □

Some related condition for starlikeness can be found from [9].

**Acknowledgements** The authors express their sincerest thanks to the referee for the invaluable suggestions.

## Compliance with ethical standards

**Conflict of interest** The authors declare that they have no conflict interest.



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